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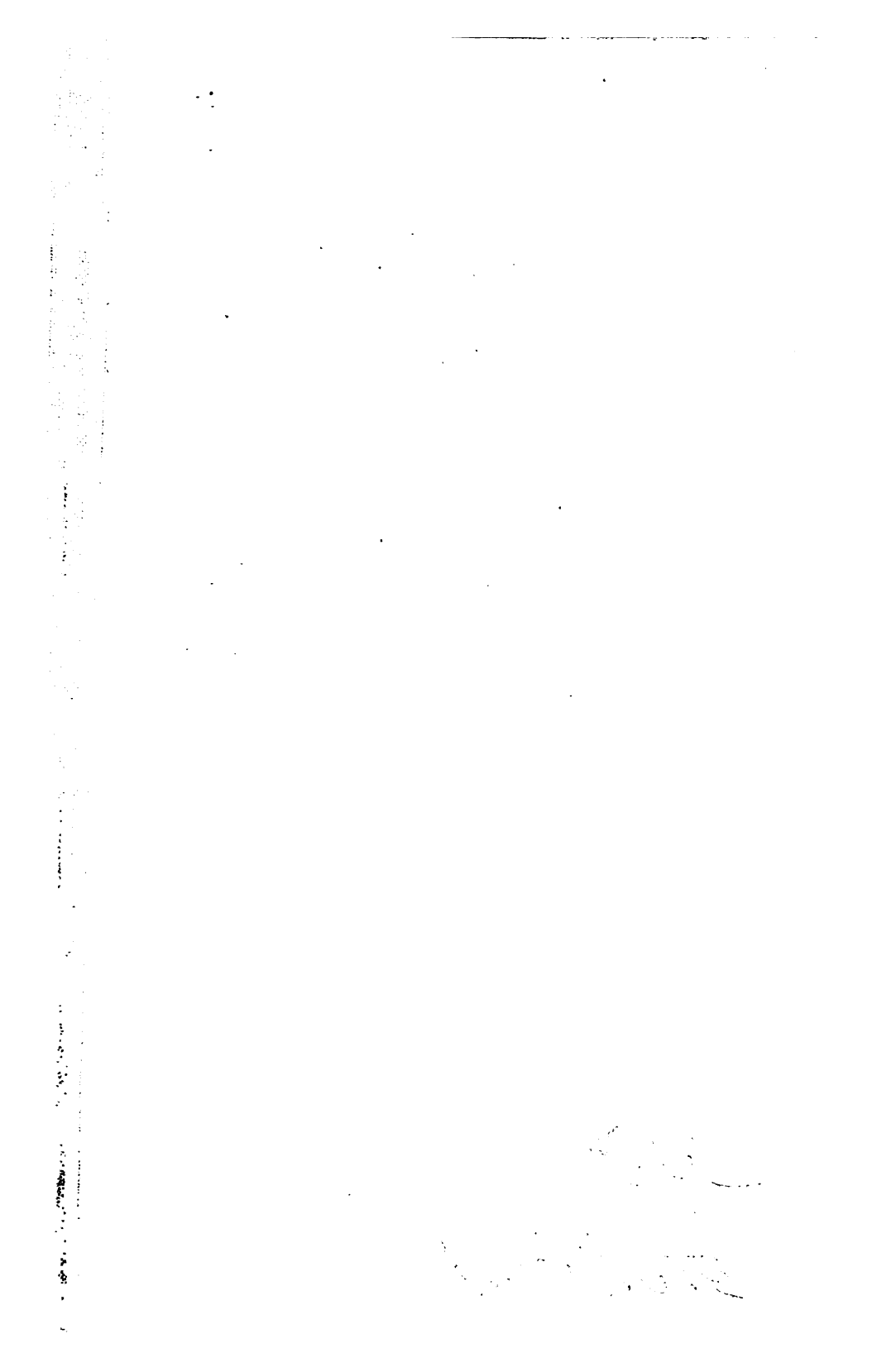
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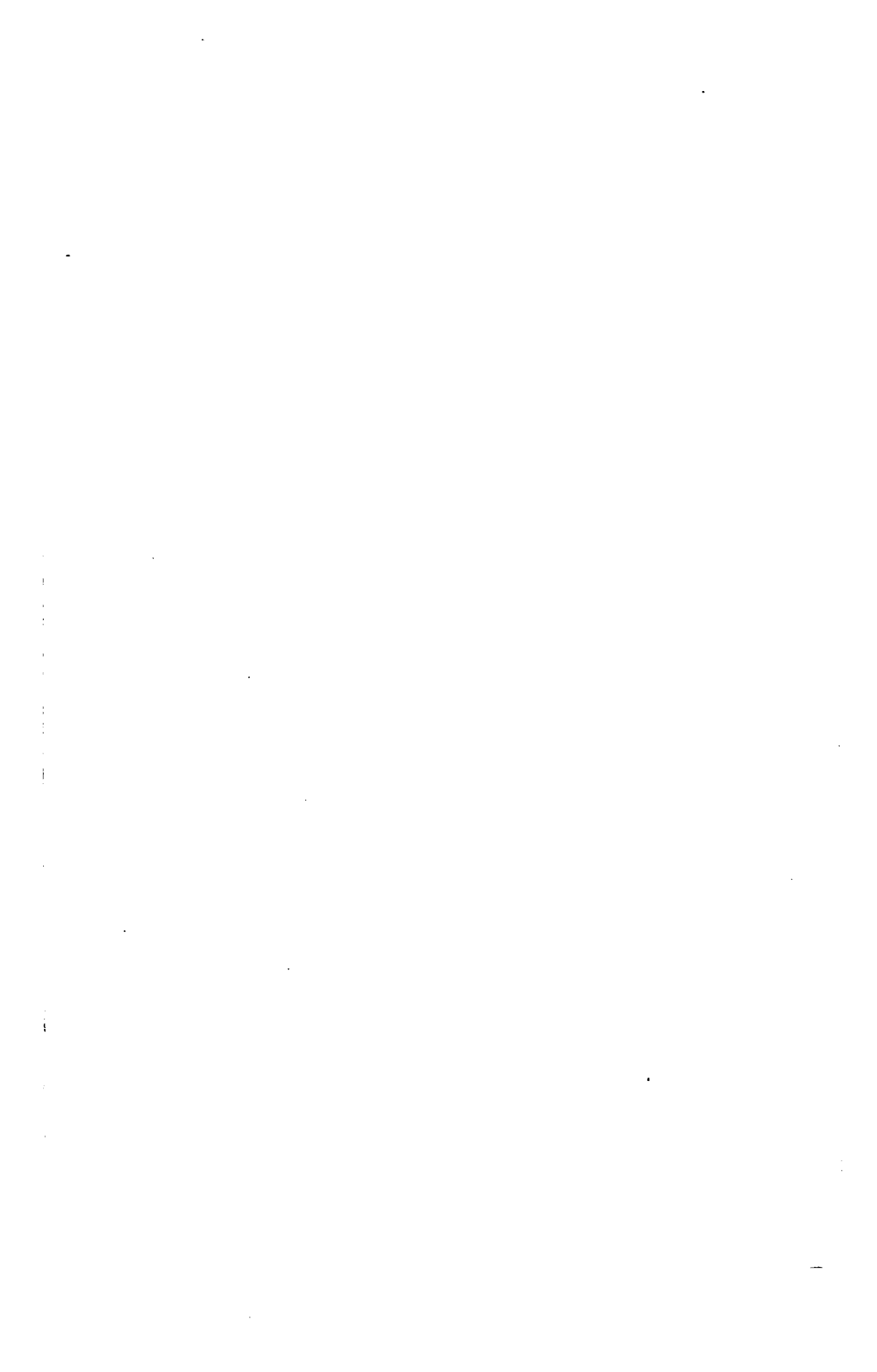
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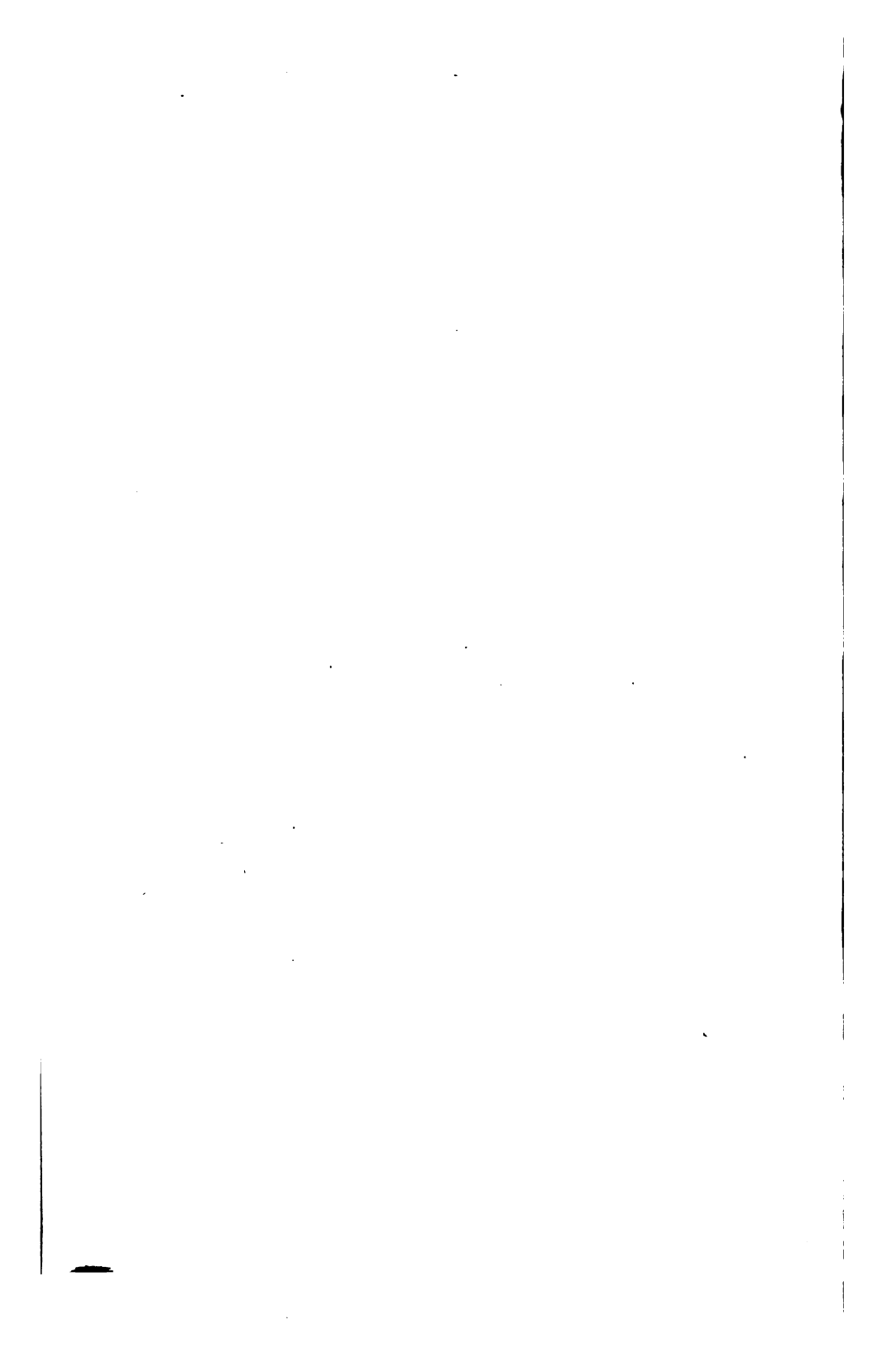
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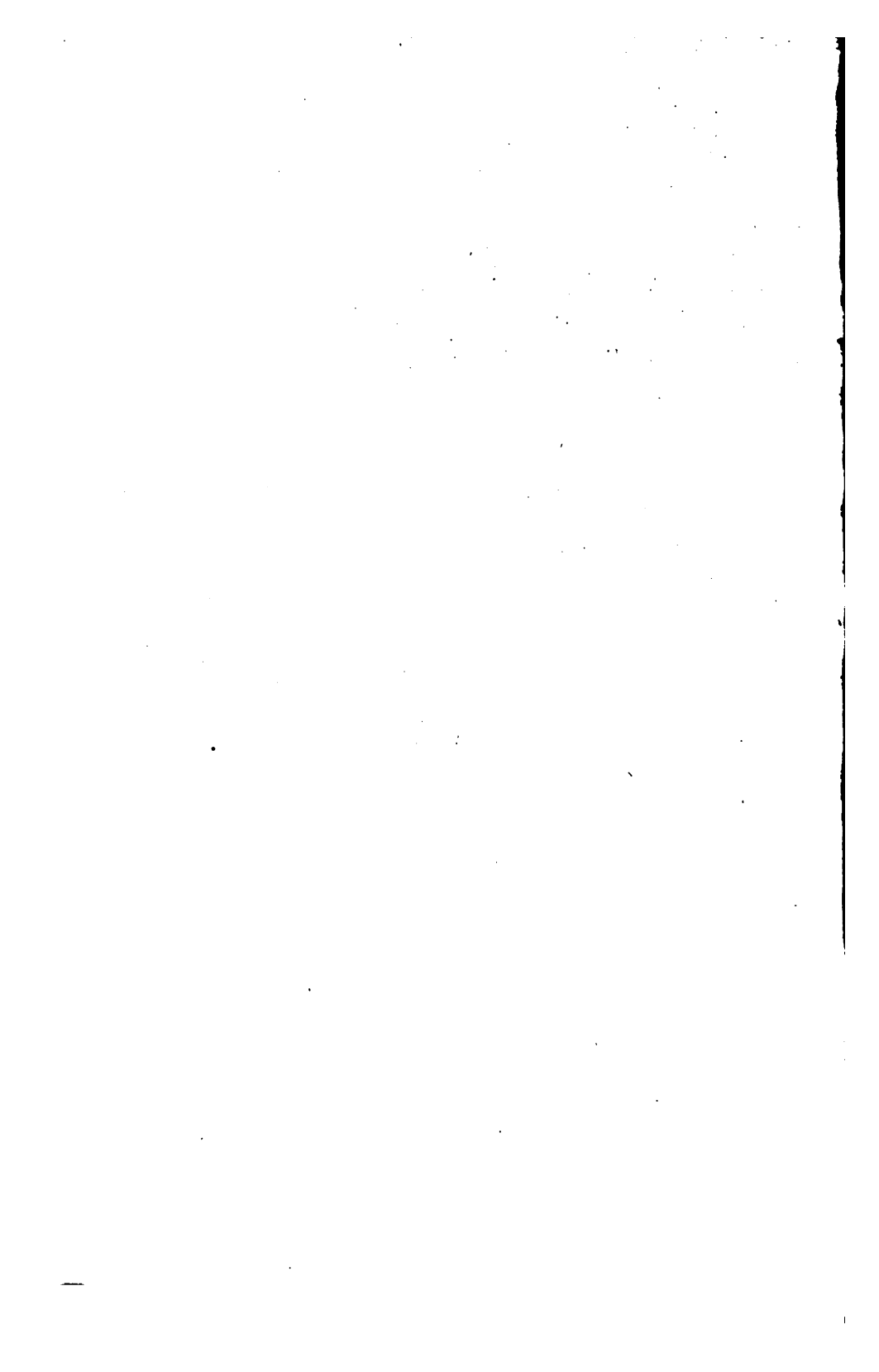






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**JOURNAL**  
**OF THE**  
**INSTITUTE OF ACTUARIES**  
**AND**  
**ASSURANCE MAGAZINE.**

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"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—Bacon.

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**VOL. XV.**

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JOURNAL  
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*Railway Debenture Stock considered as a Security for the Investment of the Funds of a Life Assurance Society. By JOHN COLES, Fellow of the Statistical Society and Fellow of the Institute of Actuaries.*

[Read before the Institute, 21st December, 1868.]

TWO valuable contributions to what may perhaps be called the "Finance of Life Assurance" appear in the *Assurance Magazine*; both by eminent Members of this Institute. The first in 1858 by our present President "On the Investments of the Funds of Assurance Companies;" the second in 1862 by Mr. A. H. Bailey, "On the Principles on which the Funds of Life Assurance Societies should be invested."

As these gentlemen have made no special allusion to Railway Debentures,\* which now rank among the assets of so many Life Assurance Companies, I have ventured on the following remarks, though not without considerable hesitation, being well aware that the Members of this Institute generally are fully conversant with the subject. Yet the discussion which usually follows the reading of a paper of this kind may prove interesting to many, since the position of Railway Debentures\* must be regarded as important, not only in reference to the investment of the funds of a Life

\* Bonds and Stock.

Office, but to the public generally, who have invested upwards of one hundred millions in these securities.

The history of the infancy and growth of our great railway system is far beyond the proposed limits of this paper, as is also the almost equally wide question regarding the immense additions made from time to time to Railway Capital, particularly for "Extension Lines;" many of which have never yielded any adequate return: the latter point has but little in it to interest a debenture holder; it undoubtedly concerns however the holders of ordinary stock, and those who hold preference stocks contingent on the profits of a single year. The attention of those interested in this matter has now been thoroughly awakened, and we may rest assured that the future Capital expenditure, and general policy of Railway Companies will be much more narrowly watched than heretofore.

Like most subjects discussed in this room the one under consideration presents new features from time to time, arising from legislation or the unwritten law, which may be designated as Public Opinion. For example, legislation has lately enabled the Great Eastern Railway Company to bridge over a financial difficulty by authorizing the issue of a so-called "B Debenture Stock" which stands neither with debenture stock proper, nor with preference stock, but between the two—next public opinion has induced Railway Companies to commence in earnest the issue of debenture stock to pay off bonds falling due, thus converting an unfunded into a funded debt. To these subjects I shall again have occasion to refer.

A Debenture Bond, as is well known, is a promise to pay the amount advanced at an agreed rate of interest, usually at the end of a specified period, varying in practice from three to ten years. The special Act by which a Railway is incorporated sets forth the amount of Capital to be raised in Shares or Stock, and determines the limit of its borrowing power. The two general principles of limitation are—1st. That no Debentures shall be issued until the whole capital has been subscribed and a portion (usually one-half) actually paid. 2nd. That the amount borrowed on Debentures shall not exceed one-third of the Share Capital.

The usual form of a debenture bond is that of a mortgage deed, assigning as security for the loan "the undertaking tolls and sums of money arising under the special Act of Incorporation." The bond is transferable by deed only, duly stamped, the consideration money being stated therein, and the transfer registered

at the Office of the Company within thirty days after its date (8 & 9 Vict., cap. 16).

Prior to the judgment of Lord Cairns in the early part of 1867 very incorrect ideas prevailed as to the legal position of the holders of debenture bonds.

That decision in the case *Gardner v. London, Chatham and Dover Railway*, clearly shewed that the holders of these bonds have simply the first claim on the tolls or net earnings of a railway without any power whatever to seize the fee-simple of the land on which the railway is constructed (or even the surplus lands or unpaid calls unless the bond be specially so drawn). In fact they cannot interfere with the carrying capabilities of what is regarded, when opened, as a common public highway, which any one has the right to use who will conform to the established regulations, and pay the Act of Parliament tolls.

In using the word "Debentures" I refer to both Debenture Bonds and Debenture Stock, which may be regarded respectively as the Unfunded and Funded debt of a Company; the rights and remedies in both securities, are very much alike: as a holder of either, could, through the Court of Chancery, appoint a "Receiver" in case of default; the Bondholder for Capital and Interest, and the Stockholder for Interest. The "Receiver" may be appointed under the 53rd and 54th Sections of the Company's Clauses Consolidation Act 1845, to "receive tolls, or sums liable to the payment of principal and interest," and he would act on behalf of all, as the Court of Chancery, guarding the public interests, will not allow an individual debenture holder to seize and appropriate the property to the payment of his own particular claim.

In the event of a "Receiver" being appointed there can be but little doubt that debenture bonds would rank according to the dates of the "Acts" under which they were issued, and all the debenture stockholders may have to rank after the bonds, unless they are specially protected, as in the case of the issue of One Million Debenture stock by the London and South-Western Company in 1862 when the "Act" authorizing the issue contained the following provisions—"That the dividends on the Debenture stock shall be guaranteed dividends, and shall have preference to and priority over all principal money and interest secured by mortgage of the Company from time to time after the passing of this Act, issued subsequently to the issue of the respective Debenture stock, and the effect of this provision shall be noticed in all the Mortgages of the Company so issued."

The litigation respecting priorities of rights in the "London, Chatham and Dover" suits will throw much light on the question and no doubt establish precedents as guides to the future.

The over issue of Debentures has it is to be hoped been effectually guarded against by the 29 and 30 Vict., cap. 108, which requires each Company to publish half-yearly, full particulars of its Debenture issue, stating how much has been authorized to be raised, and the sum that has been actually raised to the end of each half-year, specifying also the Act or Acts of Parliament under the powers of which the Company has contracted any Mortgage or Bond debt, or has issued any Debenture stock, also the amount remaining to be borrowed under such Act or Acts. Moreover the following Declaration has now to be appended to every Debenture Bond, and to every Certificate of Debenture Stock :—

"We the undersigned being two of the Directors of the Company specially authorized and appointed for this purpose, and I the undersigned Registered Officer of the Company do hereby declare (each for himself) that [the within written Mortgage Deed is issued or] this Certificate is re-issued under the borrowing powers of the Company as registered on the . . . day of . . . . . and is not in excess of the amount there stated as remaining to be borrowed, and of the amount of Bonds and Debenture stock included in that statement."

The above must be signed by two of the Directors and the Secretary, or Registered Officer of the Company.

Money borrowed by Railway Companies has generally been obtained with the idea of re-borrowing when the Bonds mature, without ever contemplating the final repayment of the Capital.

This re-borrowing has, in times of monetary pressure, proved to almost every Company a matter of anxiety and difficulty, so much so indeed as on one or two occasions to have placed great and important Companies in a condition almost of peril; which it may be hoped after the sharp lessons of 1866 and 1867, they will never be placed in again. If the Financial Crisis of that period proved one thing more clearly than another, it was that Debenture Bonds are altogether unsuited to Railway Companies. To mortgage a year's net tolls for maturing Bonds often equal to three or four times the amount of such tolls is obviously wrong, and the only remedy is the issue of Debenture Stock, the Interest on which a Company can readily meet.

In considering the question of safety of a Company's Debentures, the marketable value of the ordinary stock or the dividend

paid thereon may be almost disregarded, as a large amount of preference stock may follow the debentures, and prevent any portion of the net revenue reaching the ordinary stock; for example, the first mortgagee of a freehold estate would be indifferent as to the income left to the mortgagor if he knew that the estate yielded sufficient to meet interest on a second mortgage for a large amount.

We may in like manner regard Debentures as the first mortgage, preference stocks as the second mortgage, and ordinary stock as the mortgagor.

The annexed Table No. 1 (abstracted from the last official accounts) will more fully illustrate this point, as it will be found on reference thereto, that after paying debenture stock and bonds, the "Great Western," with an aggregate capital of 46 millions, has to meet the interest on £17,300,317 preference stocks, before anything can reach the ordinary stock; whereas the "London and North-Western," with a total capital of 52 millions, has only £7,391,118 preference stocks to provide for.

On further inspection of this Table it may be remarked that the amount raised by Debentures appears to exceed in some instances the legal issue of one third of the remaining stock issued; but this is explained by the fact, that companies generally, except perhaps the Caledonian, do not state the capital of the leased lines which they work, though they avail themselves of the borrowing powers which those lines possessed prior to the lease.

In the year 1867 the total number of railway travellers, exclusive of season ticket holders, was 287 millions—one penny in the pound for Income Tax yields the Exchequer about one million and a half sterling, and in like manner one penny a passenger yields annually more than a million sterling to our Railways. It would perhaps be difficult to furnish a stronger illustration of the wonderful power and resources of these Companies.

At the end of last year the whole of the Railways in the United Kingdom had raised—by Debenture Stock and Bonds, £126,029,676; by Preferential Stocks, £143,209,357; and by ordinary Stock, £233,023,854. In the year 1867 all the Railways earned a net income of £19,631,047 (or 50 per cent of the gross receipts); therefore assuming the average interest paid on Debentures to have been  $4\frac{1}{2}$  per cent, the sum required to pay the Debenture Stock and Bond holders would have been about £5,670,000, thus leaving a surplus net income of about £14,000,000 for the preferential and ordinary stocks.

TABLE NO. 1.—*Railway Capital, 1868.*

Name of Company.	Debenture Bonds.	Debenture Stocks.	Preferential Stocks.	Ordinary Stocks.	Total.
Bristol and Exeter ..	891,200		1,882,729	2,022,460	4,796,389
Caledonian .....	4,828,028	541,591	11,508,124	4,734,434	21,612,177
Dublin and Belfast Junction .....	232,860	13,070		873,500	1,119,430
Dublin, Wicklow and Wexford .....	437,742		811,630	500,000	1,749,372
Glasgow and South-Western .....	1,453,351	51,893	1,405,452	3,756,437	6,667,133
Great Eastern .....	*5,418,402	1,301,257	8,150,127	†9,201,021	24,070,807
Great Southern and Western (Ireland) .	403,920	164,914	1,329,100	4,111,889	6,009,823
Great Western ....	*12,964,108	3,727,343	17,300,317	12,548,481	46,540,249
Great Northern ....	1,332,458	2,853,729	6,364,829	7,788,697	18,339,713
Lancashire and Yorkshire .....	5,048,435	504,021	4,271,798	12,693,594	22,517,848
London and South-Western .....	3,296,734	1,533,122	4,415,146	7,772,886	17,017,888
London and Brighton	2,787,072	1,305,744	7,226,034	6,839,882	18,158,732
London and North-Western .....	12,770,291	2,335,523	7,391,118	29,879,138	52,376,070
Manchester, Sheffield and Lincolnshire ..	*3,607,885	236,246	4,123,908	4,392,553	12,360,592
Metropolitan .....	*1,680,084		815,387	3,699,712	6,195,183
Midland .....	5,737,895	1,122,647	13,389,793	12,344,013	32,594,348
Midland and Great Western (Ireland) .	1,245,486	29,440	264,413	2,157,175	3,696,514
North-Eastern .....	10,253,444	554,209	12,870,016	16,378,958	40,056,627
North London .....	*779,441	7,100	700,000	1,682,080	3,168,621
North Staffordshire ..	1,541,286		1,592,630	3,230,140	6,364,056
North British .....	*5,428,048	567,826	8,894,108	4,184,420	19,074,402
South-Eastern .....	*4,292,361	710,782	5,951,842	7,637,049	18,592,034
South Devon .....	529,225	130,860	576,010	1,498,300	2,734,395
Taff Vale .....	398,183	38,817	285,000	1,016,920	1,738,920
	87,357,939	17,730,134	121,519,511	160,943,739	387,551,323
	105,088,073				

I have confined the annexed Tables to the larger Lines, as the smaller ones scarcely come within the scope of many of my remarks, being more liable to fluctuations and other disadvantages, but it will be noticed that the 24 Companies enumerated in the Tables had at the end of last June raised on Debenture Stock and Bonds £105,088,073 out of the 126 millions shown by the most recent Parliamentary return to have been raised by the whole of the Railways in the United Kingdom at the end of last year.

Tables 2 and 3 show that the 24 Companies in question earned a net income in one year to June last of £16,746,625; and out of this sum they had to pay only £4,658,605 for interest on debenture stock and bonds (amounting to £105,088,073) or about  $4\frac{1}{2}$  per cent, the average rate assumed above for all the railways.

\* Includes temporary loans.

† Includes East Anglian No. 2 Stock.

I therefore venture to affirm that if the issue of Debenture Stock be proceeded with, and the Bonds thereby extinguished, the future of railways may be regarded with the greatest satisfaction, and the annual charge to meet the interest thereon will continue to be so small when compared with the total Net Revenue, that the security of this Stock, especially that issued by the leading English lines, will at no distant date rank next only to Consols.

In arriving at the net revenue available for debenture interest, there is I think but one important question which may be considered at all doubtful—that is the legal position of Leased Lines as compared with Debentures. Their position must depend on the terms of the leases; but in most instances the Special Acts for leases give the lessors considerable power—as a common illustration, Act 28 Vict. C. c., authorizing the lease of the Blackwall Railway to the Great Eastern, empowers the lessors to “levy rent by distress and sale of goods and chattels” of the Great Eastern Company.

TABLE No. 2.—*Railway Net Revenue (after payment of working expenses and the Rents of Leased Lines).*

Name of Company.	Half year ending Decem- ber, 1867.	Half year ending June, 1868.	Total Net Revenue one year to June, 1868.
Bristol and Exeter .....	105,311	97,834	203,145
Caledonian .....	a 383,841	370,643	754,484
Dublin and Belfast Junction .....	23,657	22,753	46,410
Dublin, Wicklow and Wexford .....	25,883	25,182	51,065
Glasgow and South-Western .....	b 157,093	151,333	308,426
Great Eastern .....	399,767	342,460	742,227
" Southern and Western (Ireland) ..	138,904	133,062	271,966
" Western .....	899,715	903,545	1,803,260
" Northern .....	531,643	412,888	944,531
Lancashire and Yorkshire .....	637,633	660,951	1,298,584
London and South-Western .....	416,779	365,545	782,324
" and Brighton .....	302,721	243,578	546,299
" and North-Western .....	1,501,946	1,285,601	2,787,547
Manchester and Sheffield .....	252,880	213,021	465,901
Metropolitan .....	c 141,410	d 149,584	290,994
Midland .....	726,961	702,804	1,429,765
Midland and Great Western (Ireland) ..	55,073	52,614	107,687
North-Eastern .....	1,022,913	898,733	1,921,646
" London .....	82,317	86,169	168,486
" Staffordshire .....	129,006	115,936	244,942
" British .....	288,504	288,353	576,857
South-Eastern .....	410,988	349,840	760,828
" Devon .....	56,657	43,764	100,421
Taff Vale .....	71,762	67,068	138,830
Totals .....	8,763,364	7,983,261	16,746,625

a. Not including £12,375 from Premium account.

b. Not including £3,000 from Premium account.

c. Includes £57,000 from the Contractor.

d. " " "

TABLE No. 3.—*Debenture Interest (Bonds and Stock).*

Name of Company.	Debenture Interest paid. Half Year to Dec., 1867.	Debenture Interest paid. Half Year to June, 1868.	Total paid for one Year to June, 1868.
Bristol and Exeter .....	19,100	20,616	39,716
Caledonian .....	118,104	123,243	241,347
Dublin and Belfast Junction .....	5,915	5,825	11,740
Dublin, Wicklow and Wexford .....	14,401	9,494	23,895
Glasgow and South-Western .....	34,234	34,313	68,547
Great Eastern .....	141,280	140,601	281,881
Great Southern and Western (Ireland) .....	13,089	13,462	26,551
Great Western .....	381,483	390,845	772,328
Great Northern .....	96,172	93,938	190,110
Lancashire and Yorkshire .....	118,731	119,710	238,441
London and South-Western .....	98,226	100,467	198,693
"    and Brighton .....	101,463	94,908	196,371
"    and North-Western .....	323,449	324,400	647,849
Manchester and Sheffield .....	82,780	88,105	170,885
Metropolitan .....	(a) 10,109	(b) 15,734	25,843
Midland .....	147,529	148,442	295,971
Midland and Great Western (Ireland) .....	19,134	19,358	38,492
North-Eastern .....	243,155	239,193	482,348
"    London .....	17,308	17,300	34,608
"    Staffordshire .....	35,596	36,596	72,192
"    British .....	152,686	156,946	309,632
South-Eastern .....	116,395	117,344	233,739
"    Devon .....	16,865	20,421	37,286
Taff Vale .....	10,046	10,090	20,136
Totals .....	2,317,250	2,341,351	4,658,601

(a) £10,109 Interest on Loans charged to Capital in addition.

(b) £24,486 Do. Do. Do.

Perhaps the most decided legislation on this subject—though it has a special, and by no means a general application—is contained in the London, Chatham and Dover Railway (Arrangement) Act, 1867, which sets forth in Clause 31, that “the income received “by the Board shall be applied to the following purposes in the “order specified: In payment of all working and incidental expenses and of all Rates and Tithes and of all annual Rent-charges “payable by the Company, &c. . . . . and also of Rents or “sums in the nature of Rent from time to time payable to the “Sittingbourne and Sheerness Railway Company and the Mid Kent “Railway Company &c.”

I am therefore of opinion that rents of leased lines must as a rule be paid before debentures, and I have made the necessary deductions prior to setting forth in Table No. 2 the net revenue available for debenture interest.

The Tables will prove how really secure are the Debentures of many Companies which have been lately regarded with doubt

and distrust, such as the "Great Western," "Great Eastern," "Brighton," &c.

As the conversion of bonds into stock will doubtless continue, we may, when this operation is completed, consider that Company generally the strongest, which has the largest Net Revenue after paying Debenture Interest. The whole of the Companies may indeed for comparison be placed as I have arranged the largest of them in the Table No. 4, that is, in the order of the highest surplus of Net Revenue after payment of Debenture Interest, which appears to be a good test of each Company's capability to meet the interest on a funded debt, supposing the unfunded debt has ceased to exist. We must not however disregard the percentage column in this Table, which in many instances modifies the effect of a large surplus, by showing the proportion it bears to the debenture debt.\* The case of a leased line may, in some instances, be an exception to this rule, some of these Companies still issuing their own debentures; and since they derive a fixed rental revenue, their position cannot be so liable to fluctuation as independent Companies which have about the same net income. In dealing with a leased line it is however always desirable to have regard to the position of the lessee and the "Act of Parliament" which authorized the lease. In like manner lines under a "working agreement" such as the "Staines, Wokingham and Woking," and the "Salisbury and Yeovil" require special consideration.

I have omitted to tabulate any figures regarding the "London, Chatham and Dover," which helped so much to swell the tide of distrust in 1866-7. It is however impossible at present to correctly apportion the capital, on account of the conflicting claims of the Debentures, the Common Fund Stock, the Western Extension Rent-charge, &c.

The Act of 1844 (cap. 85) authorizes the Lords of the Treasury—subject to the consent of Parliament—to purchase after 1865 any Railway constructed subsequent to the passing of that Act. Such a result would doubtless be very beneficial to Railway Securities, and the question has recently been much discussed, and no doubt will be more so owing to the purchase of the Telegraph Companies by the Government.

The most recent and by far most valuable railway legislation is Act 31 and 32 Vic., c. 119, which has reference to the publica-

\* In only one instance in the Table will it be found that the Debenture Interest absorbs as much as 50 per cent of the amount available to meet it.

tion of uniform accounts by all the railways. This Act comes into operation at the end of this year, and contains clear and explicit schedules of the forms of accounts to be used in future. The only Company which has already adopted the new plan is the "Lancashire and Yorkshire."

Those who wish to analyze railway accounts will hereafter derive very great advantages from this Act. Uniformity in the method of presenting accounts may, I venture to think, be applied with equal advantage to other Public Companies, not excepting Assurance Companies.

TABLE NO. 4.—*Railway Surplus Net Revenue after paying Interest on Debenture Bonds and Stock.*

Name of Company.	Year ending June, 1868.	Percentage of Surplus Income on Debenture Debt.
1. London and North Western .....	2,139,698	14·16
2. North Eastern .....	1,439,298	13·31
3. Midland .....	1,133,794	16·52
4. Lancashire and Yorkshire .....	1,060,143	19·09
5. Great Western .....	1,030,932	6·17
6. Great Northern .....	754,421	18·01
7. London and South-Western .....	583,631*	12·08
8. South-Eastern .....	527,089	10·53
9. Caledonian .....	513,137	9·55
10. Great Eastern .....	460,346	6·85
11. London and Brighton .....	349,928	8·54
12. Manchester and Sheffield .....	295,016	7·67
13. North British .....	267,225	4·45
14. Metropolitan .....	265,151	15·18
15. Great Southern and Western (Ireland)...	245,415	43·14
16. Glasgow and South-Western .....	239,879	15·27
17. North Staffordshire .....	172,750	11·20
18. Bristol and Exeter .....	163,429	18·33
19. North London .....	153,878	17·02
20. Taff Vale .....	118,694	27·16
21. Midland and Great Western (Ireland) ..	69,195	5·42
22. South Devon .....	63,135	9·56
23. Dublin and Belfast Junction .....	34,670	14·09
24. Dublin, Wicklow and Wexford .....	27,170	6·20
	*	
Total.....	12,088,024	
The corresponding Total for the year } ending June, 1866, was..... }	12,475,175†	

\* The following Railways also earned Surplus net incomes exceeding £25,000—Dublin and Drogheda, Furness, Highland, Monmouthshire.

† Including "Scottish North-Eastern," now incorporated with the "Caledonian;" and the "Vale of Neath," now part of the "Great Western."

In the foregoing remarks, and especially in setting forth the comparative strength of the various Companies in Table No. 4, for the guidance of those interested, I have had in view the probability that at no distant period the holders of Debenture Bonds will not have the option of renewing their Bonds, but will instead have the offer of Debenture Stock. Even at this moment some Companies which offer 4 per cent only on the renewal of Bonds, are willing to give  $4\frac{1}{2}$  per cent on Debenture Stock; and the latter seems by far the most advantageous, as the extra half-per-cent appears more than sufficient to meet any depreciation which can possibly arise in the market value of the Stock. Already the public appreciation of the value of Debenture Stocks is shown by the fact that all the recent issues, over 4 per cent, by the leading lines, are at a premium.

For the investment of the funds of a Life Assurance Company, Debenture Stocks of the leading English Lines such as the first ten or twelve in Table No. 4 appear to offer the following advantages:—

- 1st. They afford undoubted security for the due payment of interest.
- 2nd. They yield a fair average rate of interest (say from 4 to  $4\frac{1}{2}$  per cent).
- 3rd. They are readily marketable.
- 4th. They are as a whole less liable to fluctuation than government securities.

It is however quite necessary that gentlemen having in their hands the investment of the funds of Assurance and other Companies should watch the course of railway legislation, especially in reference to the issue of stock bearing the title of Debenture Stock, but which may really have no claim whatever to that designation, as commonly understood; for example, the Great Eastern “B” Debenture stock, to which I have already referred, is merely a pre-preference stock dignified by the title of Debenture stock. The Company is authorized to issue three millions of this Stock by Act of last Session 30 and 31 Vic., cap. ccviii., subject to the priority of “all Mortgage and Debenture Stock granted or issued, “or which may hereafter be granted or issued, under the powers “of any Act relating to the Company, . . . . . and all stock, “including the East Anglian stocks which may be entitled to a “specific charge or lien in any part of the undertaking.”

There can be no doubt that loans on such security as Life Interests, which promote assurances, thus bringing two descriptions of profit, are the most desirable for a Life Office; and these

loans are of a nature not competed for by the general public. Unfortunately the supply of such securities is very limited, and when others have to be considered I think Railway Debenture Stock should rank among the foremost. I do not propose to enter upon the wide question of the relative merits of other than railway investments for the peculiar requirements of a Life Assurance Company, especially as the two gentlemen to whom I have already referred, have so ably investigated this subject. I have proposed rather to confine myself to the elucidation of a single form of investment, by no means the least interesting and important; and in this respect have followed the example set by Mr. Sprague in the valuable paper read by him at the last meeting.

I may however be permitted so far to depart from my purpose as to draw attention to a subject with which my own avocations render me familiar—I mean the system of short loans adopted by banks and discount Companies, and I do so the more because it is a system which does not, perhaps, receive from actuaries so much attention as it deserves.

These loans are from day to day, or week to week, on such securities as Consols, India bonds, Colonial and Foreign bonds; and by this means capital waiting permanent investment, or which would otherwise remain unproductive, may be utilized, and usually a fair rate of interest obtained.

It is not necessary to insist in this room on the importance to an Assurance Company of having no idle money. We all know that compound interest is to its capital what steam is to an engine. The power of the former is almost beyond computation, and an extra quarter or half per cent may be easily obtained in able hands. In fact the financial arrangements of a Company appear to me of such paramount importance, that rising actuaries cannot, I venture to say, survey too carefully the whole field of sound securities; and this field is constantly extending its boundaries.

NOTE.—It is due to the members to state that, the writer having published in 1866 “A few remarks on Railway Debentures,” it has been found impossible to avoid a certain amount of repetition in the foregoing remarks.

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The following account of the discussion which followed the reading of the paper, is abridged from the *Insurance Record*.

The PRESIDENT complimented Mr. Coles on his paper. The question of investment was also before them at their last meeting, but its treatment then depended upon those mathematical calculations in which an Actuary was

engaged. The present subject called into exercise other qualities equally necessary to an Actuary's reputation, viz., prudence and sagacity. He admitted the great value of the property upon which the proposed investments were secured, and the importance of having every field of employment fairly considered. He hoped to elicit opinions upon these points, where they were likely to differ from Mr. Coles. For instance, in what respect did Debenture Stock differ from the Government Funds? It was now generally recognized that it was desirable for a Life Office to tie up as little as possible in Government Securities; for although the interest was safe, yet the return of the capital in full was very uncertain. Debenture Stock has this disadvantage, in addition to some others peculiar to itself. It would doubtless follow the workings of the money market, and be affected by the same causes which disturb the public funds. Again, a new system of intercommunication might sensibly influence railway property. Looking to the security of the capital, he preferred the system of Bonds for short terms, as the sums lent could be recovered, and if desirable re-invested elsewhere.

Mr. SPRAGUE thought that Mr. Coles's paper came at a suitable time; for although it was several years since Railway Companies began to issue Debenture Stock, it was only since the late panic they had set to work in earnest to convert their Debenture Bonds into Debenture Stock. He was disappointed that the question of the relative merits, to an Assurance Company, of Bonds and Stock had not been gone into more fully. The objection to the Stock was that the recovery of the full capital was uncertain; but Mr. Coles considered it had the advantage over Consols, not only of a higher rate of interest, but a less liability to fluctuation. Railway Companies would have got out their Debenture Stocks much faster had they issued them at the average rate of  $4\frac{1}{2}$  per cent, instead of 4 per cent only. He thought that the conversion of Bonds into Stock was due to the Companies' rough experience in the panic, and not to the pressure of public opinion. No doubt Debenture Bonds were unsuitable to Railway Companies, but the very same reasons made them peculiarly eligible for Assurance Offices. It had been a great advantage to them to be able to place out their money at a remunerative rate, with the certainty of recovering it in full in a few years, or of realizing in the mean time at only a trifling loss. He was much interested by Mr. Coles's account of the system of short-term loans, a class of business not yet generally taken up by Assurance Companies, but which seemed well worthy of consideration. It was at present the practice of some Offices to place large sums of money on deposit with the Joint Stock Banks until suitable permanent investments offered; and if better interest with equal security could be got on these short loans, they ought certainly to be adopted. On the other hand, simplicity of finance had great advantages not to be overlooked.

Mr. NEWBATT thought it might be taken as a great fact that a large amount of the funds of Assurance Companies is at the present time invested on the security of Railway Companies in one form or another. Bonds no doubt preponderate, and when there is a choice, with good reason. The conversion of Bonds into Stock was now being proceeded with, and it was for the interest of Assurance Companies that they should well understand what is the alternative which will shortly be presented to them. One Railway Company of considerable magnitude has already closed its Debenture Bond account; and since many are doubtless interested in Bonds which

have emanated from that Company, they must be prepared to decide what they will do with the capital thrown on their hands, since they cannot renew in the shape of Bonds. With reference to the admitted advantages which Bonds possess over Stocks, there were two somewhat modifying considerations. In the first place, Stocks need not be bought at par; it is quite open to buy them at a discount, like other Funds. Secondly, Assurance Companies have no imperative need of realizing at a particular juncture, for they know within narrow limits what money they will be called on to provide at a given period, and if the market be unfavourable for the security in question they need only bide their time. Nevertheless, there was an element of security in the Bonds, lacking in the Stock. Formerly all the great Railway Companies endeavoured to issue their Stock at the insufficient rate of 4 per cent, consequently it has hitherto been at a discount, and may always yet be so; that is, in effect, the want of strict perfection in the security was compensated for by an increased rate of interest. This higher rate of interest did not necessarily imply that the security was unsound, but that it was of a kind not suited to the requirements of many, or not generally understood. In support of this, he would instance the rates obtained in old times upon loans by way of Annuity—6, 7, or even 8 per cent, instead of  $4\frac{1}{2}$  with which they are often content now. No doubt similar reasons have hitherto operated in depressing Railway Securities. The recent judgment of Lord Cairns that the holders of Debentures are only entitled to the first fruits of the tree, and not to the tree itself: that they do not rank as judgment creditors, but have to go in and take their share of what the tolls provide; so that in effect, the capital of Debenture Bonds ranks with the interest on other Debenture Bonds not yet matured, and also on that of the Debenture Stock, has had the effect, speaking broadly, of bringing both Bonds and Stocks into the same category as regards their security, the chief distinction being the chance, in the latter case, of not getting back the capital in its entirety. The same imperfection however existed with regard to other securities, whether ordinary mortgages or not, while here there were counterbalancing profits. Therefore, investments in Debenture Stock, if made with skill and caution and at proper rates of interest, were scarcely, if at all, inferior to Debenture Bonds. He did not share the views of the President as to the possibility of the present system of communication by Railway being superseded. That question might be safely left to our successors.

Mr. HODGE expressed his sense of the value of Mr. Coles's paper, and thought that the statements enabled all to form an opinion as to the advantages and disadvantages of investing in Railway Securities. One point, however, had been left out of consideration, viz., that in lending on Railway Debentures or Stock we are dealing with a mortgage which is not like an ordinary mortgage. In the latter case, we had the security of the whole revenue from the property, but in the former, a large portion of the income pledged was primarily chargeable with expenses—to the extent, on the average, of 50 per cent. A falling off in revenue in this case becomes very important, for unless there be a proportionate reduction of expenses, there is a corresponding failure in the security. Railways were not of the nature of properties, but commercial undertakings. We had a charge only on the produce. There were great advantages attaching to Debenture Bonds, because the money being repayable at a fixed period could be withdrawn if the undertaking were not going on satisfactorily. But, with Debenture Stock, the

power of redemption lay with the mortgagor, and the position of the lender was different. The system of Bonds was, in his opinion, superior, even for the Railway Companies themselves; that of Stocks had been a failure, for the amount converted at present was only one-sixth of the whole debt. The public preferred Bonds, which Bonds could always be issued at a lower rate, and on an average of years would cost less than the same amount converted into Stock—say  $\frac{1}{2}$  per cent. As an example, the London and North-Western Railway had been unable to get their Debenture Stock out at 4 per cent, and had to offer 5 per cent, at the very time they were borrowing on Bonds at  $3\frac{1}{2}$  and  $3\frac{3}{4}$  per cent. A larger rate of interest was necessary when no period was fixed for repayment of the principal, as was the case with the public loans raised by this country in the time of the French wars. A saving of £4,000,000 a year would have been made, had we adopted a different system of borrowing than by selling perpetual annuities. So he thought the Railways were laying on themselves a heavy charge in perpetuity by the issue of Debenture Stock at a higher rate of interest. No one will buy Debenture Stocks at a premium, because they can be created to almost any extent—there being 87 millions ready to be thrown upon the market. He did not think that Debenture Stocks were advisable securities for an Assurance Company, and should strongly protest against them. Bonds, however, might be made a useful and serviceable mode of investment. He did not think that the short loan system could be made available for the purposes of Assurance Companies, for the rates of interest would be low. For the six years ending in 1866 or '67 he had found the average rate of interest allowed by the joint stock banks on deposits was  $3\frac{1}{2}$  per cent. Perhaps 1 per cent more might have been got on short loans; but that was not the business of Insurance Companies, but rather of Bankers, and required a great amount of discrimination and care.

Mr. BAILEY felt sure that the members would all be glad to read Mr. Coles's paper, and that their gratification would be none the less when they remembered that its author was educated as an actuary, and had passed the Institute Examinations. This paper showed how excellent a training he had had in financial affairs. In a paper of his own, he (Mr. Bailey) had laid down, as the first canon of life assurance finance, that the principal must be secure. As the Offices enter into contracts to pay fixed sums at particular epochs, the fluctuations which must occur, should be in the rate of interest and not in the principal. Consequently Stocks of all descriptions, including those of the British or Foreign Governments and Railway Debenture Stocks, should be avoided. He would illustrate by reference to the accounts of a large Assurance Office, which held on 31st December, 1859, £1,315,000 Consols and £2,845,000 Reduced and New Three per Cents, which they valued at the price of the day—96 for Consols and  $94\frac{1}{2}$  for the other Stocks. In the eight years ending with 1867, they had occasion to sell £1,220,000 Consols at £92. 8s. 5d. on the average, and £1,260,000 other Stocks at £88. 17s. 10d. on the average; the actual loss on these sales being no less than £115,877, which would be a deduction from the profits of the current decennial period. As far as he had observed, there was less fluctuation in the price of Government Securities than in any other. He did know why Mr. Coles should affirm the contrary with regard to Railway Debenture Stocks. However, he could not agree with Mr. Hodge in the unsuitability of Debenture Stocks for other purposes—different securities being suited to different purposes. For private trusts, a steady

income was often of more importance than the security of the capital. In the case of marriage settlements, a fixed income was requisite for the life of the husband and wife, while the children were quite content to divide the capital without inquiring very accurately into the original cost of the investment. The present complicated system of Railway finance had arisen from mismanagement. Otherwise, no difficulty would have arisen in renewing Debentures as they fell in. Mortgages on land were easily renewed. He was not sure that the Railway Companies would succeed in their attempt to substitute Debenture Stocks for Debenture Bonds. But if they did succeed, the Assurance Companies would have to look out for other investments.

Mr. EMMENS presumed that Mr. Coles's position was, that Debenture Stocks were characterized by extreme security. It had, however, been pointed out that they were much more insecure than many descriptions of property now adopted as mediums of investment. Railways, being commercial enterprises were subject to many fluctuations; and were liable to be superseded in their turn, as progress was made in mechanical science. It was, therefore, incumbent upon Assurance Offices to see that they have security for 30 or 40 years to come. The course of legislation had been to diminish the value of corporate property. He approved of the system of short loans, if it could be worked with security. His own Company had been able to secure 2 per cent more upon its floating balances, than by the old method of depositing at the Bankers.

Mr. CUTCLIFFE stated that his Society had £630,000 invested in Debenture Bonds of first class Railways at an average rate of interest of  $4\frac{1}{2}$  per cent paid to the day. Bonds were better in regard to security than Stocks, but the latter yielded a better return of interest. He thought that Mr. Coles's object was to point out which were the best securities to take, for his Tables showed which Railway Companies had the largest margins. Debenture Bonds of the great Railways were, in his opinion, the best securities for the surplus funds of an Assurance Company.

Mr. HAGGARD thought that there was no fear as to the soundness of the security offered by the Railways. About £5,000,000 would meet the interest upon the present debt, while there was a surplus of £12,000,000 produced by 280,000,000 passengers—and an additional penny per passenger would add £1,000,000 to the receipts. The late financial crisis had seriously affected the railway tolls—to the extent of about 13,000,000 passengers. The pendulum of the railways beats to the pulse of the world at large, and whatever disturbed the commercial kingdom was felt by the railways also. The financial difficulties of the railways arose from their inability to pay off the matured bonds—for manifestly with only a surplus of £14,000,000 they could not meet the £25,000,000 demanded. At the market price of 105, a 5 per cent bond running for 5 years would only pay 4 per cent, while Debenture Stock on the same terms would pay  $4\frac{1}{2}$  per cent in perpetuity. The Debenture Stock of good Railways was already at par, and that of the Great Northern at a premium. Since the revenue of the Railways was doubling and trebling, he thought that these Debenture Stocks would be largely taken up, and that in the course of the next two or three years the Bonds would be converted to a greater extent than was imagined.

Mr. LEMON thought that Mr. Cutcliffe had rightly interpreted the object of the paper, which was to point out which were the best Railways for

investment, since the existing Bonds would in the course of events be converted into Stock. The value of the paper lay in the percentage Table, which showed the margin each Company offered for security. A security commanding  $4\frac{1}{2}$  or 5 per cent would always fetch its fair value and would not fluctuate much.

Mr. GALSWORTHY thought that Mr. Lemon had overlooked the fact, that an Assurance Company must look to its power of realizing capital, as well as mere security and interest. Mr. Bailey had instanced a loss of £115,000 upon the realization of a fluctuating security—the same would apply to many of the Debenture Stocks. This fact alone would suggest caution. The high rate of interest referred to by Mr. Cutcliffe, as having been made by investments in these Stocks, was doubtless due to their being purchased at a price for which they could not now be obtained. If this rise in value be permanent, they would not pay so well as they had done. He did not think that Debenture Stocks were desirable investments for an Assurance Company. If it were once admitted that they were, there would be no knowing where to draw the line; and there would always be the risk of getting hold of some of the bad securities as well as of the good.

The PRESIDENT was of opinion that the question was, as to the advisability of Assurance Companies investing in Debenture Stock in preference to Debentures. Mr. Coles's Tables were of the utmost value in enabling us to form a conclusion as to which were the right stocks to invest in, if they be purchased at all. There were, however, many drawbacks to such investments. Amongst the principal of which, was the uncertainty of Railways maintaining their position in the future, if any great discovery should be made to cheapen the cost of communication. The next aspect of the case—that of security—deserved serious consideration. In a paper laid before the British Association in 1838 (and the relative positions of preference and debenture debt was much the same now), he had shown that the total capital of the Railways was £315,000,000—of which £78,000,000 consisted of loans, and £58,000,000 of preference stock. These latter bore interest at about £4. 17s. 4d. per cent, and the ordinary stock £3. 12s. per cent, and the loans above £4 $\frac{1}{2}$  per cent. He did not, therefore, think that the Railway Companies would be able to exchange very rapidly their Bonds for Stock at anything like 4 per cent interest. No doubt, by taking advantage of favourable markets, they might in a number of years partially achieve their object. He preferred the system of short loans for the purposes of Assurance Companies. The recent legislation supplied some additional security by giving a considerable check upon railway accounts and preventing an over issue of Debentures. Improvement was going on in the character of Railway Stocks which would enable the public who want regular interest to purchase them. But after all, Assurance Companies must look to those classes of securities which for the main portion of their investments will preserve their capital, as well as afford them a fair rate of interest.

Mr. COLES, in reply, after acknowledging the reception with which his paper had met, explained that he had assumed that the Assurance Companies held very largely in Debenture Bonds, and that they would not have the option of renewing them. The Railways, finding that 4 per cent interest did not facilitate the issue of the Debenture Stocks, had increased the rate. A slight increase turned the scale, and the whole of the Bonds would, he thought, be converted into Stock, and the Assurance Companies

would have the alternative of taking this or nothing. His object was, therefore, to point out that certain Railways are stronger than others, and to supply the means of ascertaining the relative strength of the Companies. Thus, in the case of the North-Eastern, if the alternative of 4 per cent Bonds or  $4\frac{1}{2}$  per cent Stock be presented, the latter would be preferable, since it would not depreciate  $\frac{1}{2}$  per cent per annum. This Stock, though issued at par, was now at 104. A  $4\frac{1}{2}$  per cent Stock gave an advantage of  $1\frac{1}{4}$  per cent over Consols, and was certainly the better finance. He thought that the Stock was equal in security to the Bonds, and that the latter were only an incubus to the Company and depressed the Stock. If the Bonds were cleared off, we should have a species of Railway Consols. The argument as to Railways vanishing, was equally applicable to gas and other undertakings.

In answer to some observations he proceeded to say — The London and North-Western Railway owe under Bonds £12,000,000, and are liable to be called upon to repay those Bonds; if they run on an average for 3 years, they are liable for £4,000,000 a year, and the whole of the surplus, after paying debenture interest, is only £2,000,000. If such a year as 1866 should occur again, they cannot meet these Bonds except by borrowing at an extravagantly high rate. They can now get stock out at  $4\frac{1}{2}$ , and easily at  $4\frac{1}{2}$  per cent, and the stock runs at once to a premium. The conversion of Debenture Bonds into Stock was proceeding rapidly, and would, he thought, in a few years be complete.

*Extracts from the Opening Address of the President of Section F (Economic Science and Statistics), of the British Association for the Advancement of Science, at the Thirty-Eighth Meeting, at Norwich, August, 1868. By SAMUEL BROWN, Esq., F.S.S., President of the Institute of Actuaries.*

#### *Insurance.*

LEAVING subjects still so full of doubts and difficulties (technical education, capital and wages, strikes and trades-unions), we turn to one which, though established upon laws of nature equally recondite, has been pushed into practice with an energy and success highly creditable to this country. Vital statistics are now assuming a form which enable the most complicated problems of human life to be dealt with as if they were certain and simple events. Yet little more than a century has elapsed since the Attorney and Solicitor-General of that day, when reporting on the application for a royal charter to the first society formed on scientific principles for the assurance of life, objected to it on the ground that its success must depend on calculations taken on tables of life and death, whereby the chance of mortality is attempted to be reduced to a certain standard. "This is a mere

"speculation," they observe, "never yet tried in practice, and consequently subject, like all other experiments, to various chances in the execution." Further, considering that the general tables include both healthy and unhealthy lives, they thought that "the register of life and death ought to be confined, if possible, for the sake of exactness, to such persons only as are the objects of insurance." It was argued, to show the small probability of success for the society, that the Royal Exchange Assurance, in forty years, had taken only £10,915 in premiums, of which the profits amounting to only £2,651, must have been nearly exhausted in the charges of management. They would hardly expect a more considerable capital to arise from lower premiums, and the hazard of loss will be increased in proportion as the dealing will be more extensive. The petition was dismissed, but the society (the Equitable) was formed, and in spite of the gloomy prognostications at its birth, had afterwards, at one time, nearly £20,000,000 of assurances on lives in force together. About six years back, from an estimate made on a large proportion of the companies in the United Kingdom, it was computed that about £372,000,000 were assured upon lives; and at the rate of progress made of late years, it is probable that this is now increased to £400,000,000. But a still more remarkable extension of this kind of business has, within a few years, taken place within the United States of America. At the close of 1866, thirty-nine companies doing business in the State of New York, had about 305,390 life policies in force for upwards of £173,000,000, which at the end of 1867 had increased to 403,841 policies for £233,400,000. A single company, the Mutual of New York, had issued in that one year 19,406 policies for £12,450,000, taking in new premiums upwards of £801,000. The assurances in that company alone amounted to nearly £39,000,000, being almost three times the amount at present existing in our largest office in this country.

Considering the great pecuniary interests at stake, the discovery of the true law of mortality is of deep importance. The theory of that eminent mathematician, Benjamin Gompertz, F.R.S., that there is in the human frame a power to oppose destruction, which loses equal proportions in equal times, and consequently that the intensity of mortality is represented by a series in geometrical progression, was founded upon actual observations of numerous tables. Such a theory is congruous with many natural effects, as for instance, the exhaustions of the receiver of an air-pump by strokes repeated at equal intervals of time, and it has

been confirmed for long periods of life in most of the tables of original observations which have been made. There is a marked difference, however, in the constants of mortality in three principal periods of life—childhood, manhood, and old age. There are also exceptional or climacteric years of life, and from the variety of circumstances which may affect health or life, such as occupation, locality, habits, exposure to disease, &c., so many peculiarities in each class of observations, which themselves may have been incorrectly or imperfectly collected, that it is very difficult to detect in any what may be called the true or universal law of mortality.

A very important inquiry is now being made, under the auspices of the Institute of Actuaries, into the actual experience of some of the older companies. The facts relating to 147,000 persons admitted after medical examination as healthy lives, of whom about 24,000 had died, afford ample materials for many interesting deductions in vital statistics, and are especially valuable for the accuracy of the records. A similar investigation is being made in Germany and in the United States of America, conducted by scientific institutions of a like character, and by some of the leading mathematicians of those countries.

It is to be regretted that no authentic information as to the total amounts insured in various branches is known in this country. In fire insurance a rough estimate, deduced from the duty, would give about £1,445,000,000 in 1867, including nearly £80,000,000 on farming stock. The total amount of marine insurance business is equally uncertain, though it must be increasing with the rapid strides of commercial enterprise. Mr. Morrice Black, in his analysis of the accounts of marine insurance companies established since 1859, gives the premiums received by eleven of these companies in 1867 as nearly £2,800,000, insuring about £227,000,000; but this does not include the five oldest and largest companies, nor the great amount underwritten by the members of Lloyd's. The total value of shipping and commercial products liable to risk in voyages between different ports would be the more interesting, as the Statistical Committee of Lloyd's have for the last two years, by the aid of their honorary secretary, Mr. Jeula, collected and compared some of the most striking results of the accidents and losses in which, as a great maritime nation, we take so great a share. In 10,587 vessels in 1866, to which accidents of some kind happened, and 11,424 vessels in 1867, the accidents of different kinds, whether sailing vessels or steamers (the latter being about 10 per cent of the total), show a remarkable regularity in the percentage,

and indicate the value of a more extended inquiry. The tables comprise losses and casualties in all parts of the world, and are divided into thirty-one geographical sections, for the voyages between the ports in each section. From these and the annual wreck registers of the Board of Trade for the dangers nearer our own coasts, and fuller statistics from our own and foreign countries, on a uniform plan, of the amount and value of shipping and cargoes passing from port to port in different trades, we may expect a great increase in our general knowledge on these subjects.

At the Statistical Congresses held in Paris in 1855, and at Berlin in 1863, it was recommended to collect the statistics of all branches of insurance, in such a form that the progress of nations therein might be compared. The heads of the Government Statistical Departments of several countries are now engaged in maturing a plan for comprehending these and other commercial statistics in a full report.

#### *Statistical Congress.*

The International Statistical Congress, after four years since its last meeting in Berlin, was invited by the Government of Italy to meet in Florence in the early part of October last year. The statesmen, learned professors, and social reformers of Italy formed a larger body of natives, 632, with 85 foreigners, than even the Congress in London in 1860. The King had nominated his eldest son, Prince Humbert of Savoy, as general President, and his Excellency the Minister of Agriculture, Industry, and Commerce (S. de Blasiis), took a most active part in the proceedings. The effect of these congresses, of which six have been held since 1853, is seen in the great improvement in the form of collecting Government statistics of all kinds. By uniformity of methods and principles, not only may the relative progress of nations be compared, but the phenomena of political and social economy examined under different conditions and varying aspects. Thus the law of the production and distribution of wealth, of the growth or decay of population, the causes of early or late marriages, the effects of emigration on the country left or adopted, the best means for the prevention or suppression of crime, the evil effects of great standing armies, interfering with the production, or causing wasteful consumption in a country, and numerous other questions of the highest interest to society at large, may be traced on the broad map of a continent, instead of the narrow limits of a single country.

At each successive meeting the able and earnest men, who as

Government officials really have influence to carry out the resolutions agreed to, have discussed nearly all the subjects on which national prosperity depends. The summaries of the previous reports by Dr. Engel, of Berlin, and Dr. Maestri, of Florence, are full of matter for reflection, laying down the principles on which every possible statistical question ought to be studied.

An attempt is now being made by some of the leading members, animated and guided by M. Quetelet, who may be considered the founder of these congresses, to collect on an uniform plan the comparative statistics of all nations under the principal heads. The first volume, on population, was presented by M. Quetelet at the last congress, and is an admirable *resumé* of hundreds of scattered official volumes. It is to be followed by other compilations in different branches of statistics, showing that these meetings have not ended in idle discussions, but will assist the philosopher in his studies as well as the practical man in his efforts to promote the social improvement of the people.

On the suggestion of M. Quetelet a section will be formed in the next congress especially to examine statistical subjects in their direct relation with the theory of probabilities. The laws of the recurrence of events will thus be eliminated, and many questions which are now obscure may, by due value being given to the weight of observations, be reduced to correct or approximate theory.

Such congresses and meetings like the present enable all those who are engaged in a common pursuit, to benefit by the labour and skill, by the depth of thought, or wide experience of their fellow-workers in the same field. They bring the kindly feelings of friendship to the aid of scientific investigations; they allow no man to feel that he can repose, as if he had done enough for society, for he finds in the brief interval since we last met, what new questions have agitated the world, and how he must bestir himself to keep pace with the intellectual progress of the age; they point the way to unexplored tracts of knowledge; they utilise the smallest contributions of thought whilst filling the mind with suggestions of the extent and variety of the problems which still remain to be solved in the social condition of man. The domain of statistics is so vast, that we should welcome any new labourers who will cultivate the new fields of research constantly opening up. If not strictly a science, it is at least a method of investigation which requires to be conducted in a scientific manner, and in an earnest spirit of search for truth. Even undeniable facts may be so collected as to serve the cause of prejudice, to perpetuate error,

or to conceal the laws which they should reveal. In social science and political economy, statistics may be considered the collection of experiments, by the results of which we observe the hidden workings of the laws which regulate the social condition of man and his progress in civilisation. The growth and decay of population, the freedom of capital, and the rights of labour, the duty of voluntary or enforced education, the extent of Government interference in labour or manufactures, the competition of prices, the true principles of commerce, the most effectual means of suppression or prevention of crime, the theory of taxation and national loans, and multitudes of similar questions are all governed by subtle laws affecting the free will of man, checked and kept in place by similar action in others, of which we catch a glimpse sometimes by their irregular action in enforced or abnormal conditions, and sometimes by our having discovered and acted in harmony with the natural law which governs them. But as society is perpetually changing what we have discovered and thought to be truth, seems frequently inadequate to account for the new phenomena presented. It is only by extending our observations from the narrow sphere of a single country or a single class to all countries and all classes, by an uniform collection of statistics as is now being done by all the Governments of Europe, by noting differences as well as analogies, and confessing and correcting errors, and comparing the operations of the same causes under various conditions of interference, that we shall throw light on the many unsolved problems of social and political economy which modern civilisation presents.

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*Government Life Annuities.*

THE following information has been extracted from returns presented to the House of Commons shortly after the commencement of each session, and from the "Finance Accounts" published annually soon after the 31st March.

The greater portion of the annuities is granted under the act 10 Geo. 4, c. 24, the provisions of which suppose the purchaser to be a proprietor of Stock which he desires to transform from a perpetual annuity into one depending on life; power is given, however, to the Commissioners for the Reduction of the National Debt to receive cash equivalent in value, at the market price of the day, to the Stock required.

The "Stock Transferred" and "Money Paid" are shewn

separately, the latter being generally greater in amount than the former; but the sum paid for annuities purchased under the acts 16 & 17 Vic. c. 45, and 27 & 28 Vic. c. 43, commonly called Savings Bank Life Annuities, is included in the amount under the latter heading.

The deferred annuities have been, since the passing of the act 27 & 28 Vic. c. 46, carried to a separate account.

*The Amount of Stock Transferred, and Money Paid, to the Commissioners for the Reduction of the National Debt, for Life Annuities under the Acts 10 Geo. 4, c. 24, 16 & 17 Vic. c. 45, and 27 & 28 Vic. c. 43; and the amount of Annuities granted for the same during the undermentioned periods.*

Year (ending 6th January).	Stock transferred to Commissioners.	Money paid to Commissioners.	Immediate Annuity granted.	Deferred Annuity granted.
	£	£	£	£
1853-54	266,886	450,360	60,643	1411
1854-55	190,502	344,625	46,903	1028
1855-56	243,441	319,984	48,326	1111
1856-57	277,018	361,741	56,591	1300
1857-58	272,533	323,258	54,564	1135
1858-59	341,418	467,234	69,416	811
1859-60	375,535	346,604	61,967	907
1860-61	271,004	398,450	59,874	1137
1861-62	250,571	350,007	54,557	1278
1862-63	278,155	390,501	58,124	644
1863-64	308,742	325,017	56,113	863
1864-65	181,144	227,758	36,846	551
1865-66	347,053	323,918	64,601	90
1866-67	222,439	268,772	43,403	16
1867-68	311,787	363,674	59,630	..

*The Amount of Life Annuities, granted under the above Acts, chargeable on the Consolidated Fund at 31st March in each of the undermentioned years.*

At 31st March.	Annuity chargeable.	At 31st March.	Annuity chargeable.
	£		£
1855	1,036,978	1862	1,025,707
1856	1,031,122	1863	1,017,669
1857	1,030,856	1864	1,008,543
1858	1,035,271	1865	983,941
1859	1,050,945	1866	990,727
1860	1,053,418	1867	973,519
1861	1,032,959	1868	973,548

We are indebted to Mr. Finlaison for the above particulars, which will we think be interesting to several of our readers. Probably it is not generally known that about £1,000,000 of the

public revenue is disbursed every year in the form of life annuities; and that about £600,000 on the average is annually invested in that mode with the Government. The amount of these investments fluctuates considerably; and curiously enough the fluctuations are a very fair index of the state of the money market at the time. Thus, in the 15 years under review, it will be observed that the greatest amount of annuities was granted in 1858, the year immediately succeeding the suspension of the Bank Charter Act in November 1857; and the least in 1864, a period of reckless speculation, the reaction from which is clearly discernible in the largely increased transactions of the following year.

*German Life Assurance Institute.*

IN accordance with the announcement we made in our last Number, we now extract from the proceedings of the German Life Assurance Institute the two following papers.—Ed. J. I. A.

I.—*4th Feb. 1868. DR. ZILLMER on the Arithmometer, or calculating machine invented by M. Thomas (of Colmar).*

Ever since men began to compute, endeavours have been made to facilitate calculations, or to perform them altogether, by means of mechanical contrivances. The history of these would afford to its author abundant and interesting materials. The hands and fingers offer the first and readiest help in computing. We can not only count on the fingers, but even perform complicated calculations. Take the following as an example: If we call the little finger of each hand 6, the next one 7 &c.; we may in a simple manner reduce the multiplication of two numbers between 6 and 10 inclusive, to the multiplication of two numbers under 5, (the multiplication by 10 being assumed). If we have to multiply two numbers 7 and 8 for example, we must hold the finger 7 of one hand against the finger 8 of the other, count the fingers which are held together and all those below them on both hands, and multiply the number by 10; then count for each hand separately the fingers above those which are held together, multiply these numbers together, and finally add the two products. In the above example, the number of the fingers which are held together and of those under them, is 5; and the number of the fingers above them, is on one hand 2, and on the other 3, from which it follows that

$$7 \times 8 = 5 \times 10 + (2 \times 3) = 56.*$$

Coming now to Thomas's calculating machine,† we shall see that it can perform the first four rules of arithmetic with any numbers we choose,

\* The proof of the general truth of this rule is very easily deduced from the formula:

$$ab = (a + b - 10)10 + (10 - a)(10 - b).$$

† For its construction see the ingenious treatise "Die Thomas'sche Rechen-maschine, von F. Reuleaux, Freiberg 1862."

whether integers or decimal fractions. From the fact that it is applicable to all numbers, the machine is of universal use; and possesses in consequence a great advantage, not only over Pascal's machine, which was based upon the special subdivisions of coin current in France in his time, but over the so-called Swedish calculating machine, which, having been invented by Babbage, was completed by the two Swedes, Schentz and his son. The last being based on the principle of interpolation and the calculus of differences, only performs special calculations; but in these, its performances are almost incredible.

In practice the employment of Thomas's computing machine is principally of advantage, where we have to do with multiplications and divisions of large numbers, and where, without it, we should have to employ Tables of Logarithms; and in such cases the machine finishes the work in so short a time, that even a practised computer can scarcely find out the required logarithms within the same time. If it be a question of finding products, of which the sum is wanted, but the separate products are not required, then the machine performs the summation at the same time as the new multiplication,—nothing more being required for this purpose than to let the result of the previous computation remain. If we have computed for instance  $3 \times 4$ , the machine shows 12 as the result; if we let this number stand and compute  $5 \times 6$ , then the machine shows not 30, but  $30 + 12 = 42$ . Here we already save the addition of the separate products, but the employment of the machine is most advantageous when the products to be found and added together have all a common factor. This happens very frequently in computations connected with life insurance; at least, many formulæ admit of being suitably transformed for the purpose. Let us introduce, by way of example, the reserve for an ordinary life insurance. If we express this by  $V_{x|n}$ , where  $n$  denotes the number of years since the completion of the insurance, then we have

$$V_{x|1} = \frac{a_x - a_{x+1}}{1 + a_x} *$$

$$V_{x|2} = \frac{a_x - a_{x+1}}{1 + a_x} + \frac{a_{x+1} - a_{x+2}}{1 + a_x}$$

$$V_{x|3} = \frac{a_x - a_{x+1}}{1 + a_x} + \frac{a_{x+1} - a_{x+2}}{1 + a_x} + \frac{a_{x+2} - a_{x+3}}{1 + a_x},$$

or

$$V_{x|1} = \frac{a_x - a_{x+1}}{1 + a_x}$$

$$V_{x|2} = V_{x|1} + \frac{a_{x+1} - a_{x+2}}{1 + a_x}$$

$$V_{x|3} = V_{x|2} + \frac{a_{x+2} - a_{x+3}}{1 + a_x}$$

&c. &c.

\* We have taken the liberty of transferring Dr. Zillmer's formulæ into the notation commonly used in this country. He writes the above formula thus

$$Res(1) = \frac{R(x) - R(x+1)}{R(x)}$$

$R(x)$  being the annuity-due  $= 1 + a_x$  or  $= a_x - Ed$ . J. I. A.

It follows from these formulæ that we have only to multiply the constant factor  $\frac{1}{1+a_x}$  by the successive annuity-differences, without expunging the result of the previous multiplication, in order to obtain the values of the policy for different durations.

We have another example, where computation by logarithms is inconvenient, in the calculation of the premiums for an Endowment with returnable premiums. If the return of premiums is based upon the net premiums, we can put the formula for the premium in the form:

$$\frac{1}{n + (1-v) \cdot \frac{S_x - S_{x+n} - nN_{x+n}}{D_{x+n}}} *$$

where  $x$  is the age at entry, and  $n$  the number of years at the expiration of which the endowment is to be payable.

If we have to calculate by this formula the premiums for endowments payable at a given age for different ages at entry, we should calculate the second term of the denominator by itself. Here the factor

$$(1-v) \frac{1}{D_{x+n}}$$

is constant for all ages at entry; and the difference between the variable factors for two successive ages at entry, as for example,  $x$  and  $x+1$ , is

$$S_x - S_{x+1} - N_{x+n}$$

or

$$N_x - N_{x+n}.$$

If we first compute the values of this difference for all ages at entry, the calculation goes on very easily. If, for example, the endowment becomes payable at the age 24, we form in the first place the factor

$$\frac{1-v}{D_{24}}$$

and multiply it successively and without expunging the results, first by  $D_{23}$ , then by  $D_{22} + D_{23}$ , then by  $D_{21} + D_{22} + D_{23}$ , and so on.

To the separate results we add successively 1, 2, 3, &c.; and take the reciprocals of the sums. This last operation can be effected by the machine itself, namely by dividing 1 by the number in question. It is more convenient however to have a table of reciprocals, as for example the "Tables of the Reciprocals of Numbers from 1 to 100,000, by Lieut.-Col. W. H. Oakes. London, 1865."

Such a table is generally of great service, by enabling us immediately to replace each division by a multiplication; and with the machine, as with ordinary computations, multiplication is more convenient than division.

\* Here

$$S_x = S_{x-1} \text{ (G. Davies).}$$

$$N_x = N_{x-1} \text{ (do.).}$$

II.—11th March, 1868. DR. AUGUST WIEGAND delivered an address on the "*Antagonism between Theory and Practice*," of which the following is an abridgement:—

He stated that he did not intend to dwell upon the opposition of the over-confident practical man to theory; but rather to treat of the opposition which must of necessity sometimes arise between a perfectly true theory and a perfectly true practice. This, no doubt, might appear strange language for an actuary to employ; but he would go further, and say that in the business of life insurance, there are cases in which theory may bring an office to ruin, and only a prudent practice can save it. This prudent practice, which thus diverges from theory, may be best described by a phrase dating from March 1848, which is now a well-recognised rule in diplomacy. We must "take circumstances into account."

He then continued:—It is time I should give you an illustration. Follow me then to the sick-bed of an incurable consumptive patient: you will not be surprised if this consumptive man, who is insured in my office for a considerable sum, should say to me: "My dear sir, I should soon get well, if I only had the means to enable me to spend next winter in a warm climate: my physician is decidedly of that opinion, and I believe him. If your Office would buy up my policy and not give me too small a sum for it, I could then carry out my resolution. I insured six years ago when 30 years of age for 4,000 Thalers—what can you afford to give me?" Now, am I to answer this question as an actuary, or as a practical man? Shall I let my answer be dictated by strict theoretical rule, or shall I *take circumstances into account*? If I adhere to theory, I must answer him thus: "For the surrender of a policy, my Office allows full three-fourths of the reserve, which, for your insurance, would amount to 200 Thalers exactly"; and if he replied that he could not with 200 Thalers spend a winter in Nice, or Madeira, or even in Montreux, I might perhaps venture to go as far as the extreme theoretical limit, and offer him in addition the remaining one-fourth of the reserve. I should say: "That, however, is the highest sum my Office can offer; for, out of the premiums paid by you, we have nothing left but the policy-reserve: a part of the premiums has gone in expenses of management, a second part in payment of claims by death, and a third part, the smallest, has been carried to the credit of the guarantee Fund, which every Office must have in order to compensate for variations in the mortality of the lives insured." If I spoke strictly as an actuary, this must have been my answer; and it would have been quite right. That, however, which is right in theory, is, in this instance, a practical absurdity. The odds are a hundred to one that any one of you—without troubling himself the least about theory—would whisper to the sick man: "Don't accept those terms, I will give you 500 Thalers with pleasure"—probably another among you would offer 600, and a third even 700 Thalers. Why? Because this is a case where you must take circumstances into account. And what are the circumstances in question? The man dies, in spite of Nice and Madeira; and before the year is out, the sum insured has to be paid: and 4,000 Thalers for 700, or even 800, if it must be so, is still a profitable transaction. These are the actual circumstances, and therefore the principle holds good "If you must choose between two evils, choose the lesser." Where the money is to

come from, whether it has been already provided or not out of the premiums, is no concern of mine. The policy must be bought, even if the owner will not let it go for less than 1,000 Thalers.

I now pass on to the subject which I have specially proposed to myself to investigate to-night, viz. : "the proper limits to a free disposing power " on the part of the insured over their property."

In the above example, theory may have seemed somewhat niggardly; but there are cases where theory would throw away money in a reckless manner, if not prevented by a prudent practice.

In theory, it is an axiom that the sum which is set aside out of the premiums as a reserve for any insurance, represents the credit of the policyholder in the assets of the Company. This axiom is indisputable. If, however, any one should infer from it that the policyholder can under all circumstances freely dispose of his share of the assets, that would certainly be in accordance with theory, but by no means with a wise practice. In the case of the surrender of a policy, theory generally gives more than practice can sanction. Those who surrender their policies will evidently be mostly persons in sound health. But their withdrawal lowers the average state of health of those lives whose policies remain in force. For this reason, every Company must retain a part of the reserve as a compensation for the deterioration of the remaining risks. That there are exceptions in this respect, we have seen in the case of the man suffering from consumption; such cases however are very rare.

There are other branches of insurance business, in which Offices cannot allow any surrender value at all for a policy, or lend money upon it, without depriving themselves of the guarantees of their own stability. To this class belong all policies on one life, securing immediate or deferred annuities. The reason is clear. Such policies are only effected by a man who hopes to live a long time. Should such a change take place in his health, that his hopes of a long life are materially shaken, he cannot do better than surrender his policy. Let an Office once entertain proposals of this sort, and it would no longer be a question of its mortality agreeing with the law; for it would retain on its books only long-lived annuitants, whose annuities it could not pay out of the contributions previously received.

In the case of reversionary annuities instances may be imagined, in which the surrender of the policy is admissible; in the case, for instance, where the life insured should prove by medical evidence that his health is failing, and that the reversioner continues in the same state of health as when the policy was effected. Under such circumstances, however, proposals to surrender the policy will scarcely ever be made to the Office. For even if the husband whose health is impaired, wished to sell the policy, not having any presentiment of the true state of his health, his more prudent wife would certainly oppose it. Why then give a rule for such rare cases?

Another case, where no surrender value can be given, is that of an Endowment without return of premiums. The reasons are just the same as those I have just given in regard to annuity policies. If the premiums are to be returned in the event of death, then the surrender is admissible; but the surrender value must be determined, not by the reserve, but solely by the bare amount of the premiums to be returned: in fact, just as if the policy were cancelled by the death of the life insured.

In all these cases, the assured possesses in theory the same full power of disposing of his policy-reserve as in all other kinds of insurance; but in practice, he cannot and must not have it. Is not however such a practice, a denial of the rights of the assured? Certainly not, for the assured retain the right of disposing of their interests, within certain limits, which are essential for the safety of the Office.

The only cases in which it is a matter of necessity that the assured should have the right of disposing of his policy-reserve are the two following: firstly, when the assured falls into difficulties and is unable to pay the premiums; and secondly, when, owing to unforeseen events, the object of the insurance has ceased. With regard to the first, every Office will most willingly re-purchase the policy, if the insurance is payable at death. If however it is an endowment, every Office will, with equal readiness, convert the existing insurance into one which requires no further payment of premiums, and secures to the assured notwithstanding, either a fixed sum or an annuity.

If the object of the insurance has ceased; for example, if the wife and children, who were to be provided for, have died before the husband, the Offices will readily exchange the insurance for a deferred annuity or an endowment.

From this you will perceive that the Offices are very far indeed from enriching themselves out of the premiums which the insured have perhaps saved with difficulty. But an absolutely free power of disposal over that which, according to theory, is the undeniable property of the assured, the Offices cannot guarantee without exposing themselves to a risk that would set all theory at defiance. Theory is still right, notwithstanding; but only for the whole collectively, never for a special case. In the same way as a man of the age of thirty has not the smallest guarantee that he will reach the age of 64, although the law of mortality allows him that expectation, every theoretical principle loses its meaning, if we seek to apply it to a particular case. Here we must not depend on calculations, but take the special circumstances into account.

There are in the world but few absolute truths: most of them are only relatively true. For that very reason therefore, theory and practice must not wish to be both in the right in each particular case, they must rather become reconciled. Theory is fatal without a judicious practice, but a practice which ignores theory, is simple destruction.

These remarks of Dr. Wiegand's appear to us eminently suggestive; but we cannot agree with him in thinking that a perfectly true theory can ever be in opposition to a perfectly true practice. In the instances he has quoted, it appears to us that practice is in advance of theory; and that the theory, far from being perfectly true, is decidedly imperfect. No doubt the theory is true for the whole of the lives insured collectively; and treating them all as *average lives*, makes a proper reserve for the whole. But when we come to particular cases, we know, as is very forcibly illustrated by Dr. Wiegand, that some of the lives insured are much *below the average* in their prospect of longevity; and that, on the other hand,

the majority of the lives insured are still such as for distinction we should call *select lives*—i. e., such as would without difficulty be insured at the ordinary rate. We learn then, that since a larger reserve is certainly required for the *under-average* lives, a smaller reserve than the average is sufficient for the *above-average* or the *select* lives. But theory is not yet in a position to say what that smaller reserve should be. It is to be hoped that the valuable statistics as to the mortality among insured lives collected by the Council of the Institute of Actuaries, and shortly to be published, will enable this problem to be correctly solved, and throw much light upon many other important questions of a cognate character.—ED. J. I. A.

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*Thirteenth Annual Report of the Insurance Commissioner of the Commonwealth of Massachusetts. January 1, 1868. Part II. Life and Accident Insurance.*

IN the *Journal* for April last, we gave a *resumé* of the Eighth Annual Report of the Superintendent of the Insurance Department of New York, and in that for July, a detailed account of the Mutual Life Insurance Company of the same State. We believe that these contributions to our pages were not unacceptable to our readers, as serving to throw much new light on the condition of Assurance business on the other side of the Atlantic. It was seen that, in regard to Assurance as in everything else, our American cousins were not merely our rivals, but were fast becoming our teachers; that,—a new people, unencumbered with tradition and “the sacred dust of ages,”—their energetic and fruitful minds were striking out new ideas and methods to which we in this country, which had hitherto been regarded as the home if not the birth-place of Assurance, were entire strangers. These ideas and methods were not simply fantastic devices, the mere extravagancies of free thought, but were the product of sober reason and practical sense, and were directed, not only to the development of the Assurance principle, but also to the regulation of Assurance practice.

It is in view of this latter question—that of Assurance Government—that the able State paper which we now have under consideration, has for us at this moment a special interest. It has, indeed, another element of attractiveness, about which we shall probably say something. This is to be found in an elaborate essay by the learned Commissioner on the comparative merits of the “Per-

centage" and "Contribution" plans of dividing surplus,—an essay based on the opinions, printed in an appendix, which he obtained from 17 leading American Actuaries, or other competent authorities, in response to his circular letter inviting their discussion of this question. Amongst these, we find the familiar names of Sheppard Homans, David Parks Fackler, Elizur Wright and Levi W. Meech. It is to the courtesy of the last-named gentleman that we are indebted for the volume before us.

There is now going on amongst us a considerable amount of discussion on the subject of Government interference in the concerns of Assurance Offices. Most of those who are taking part in it, are treating the question as a novelty, to be slowly admitted in principle, and almost impossible to be carried out in practice. On the question of principle, it is not necessary in these columns to make any new declaration of faith. These pages have never, on this question, given out an uncertain sound. As lately as October last the following words were used in reference to it,\*—"On the publication of accounts and the legislation needed for Assurance Offices, . . . we would make but this remark, that never more than now has interference on the part of the State been needed, in order to enforce such a measure of publicity as would enable persons of ordinary intelligence to judge of the solvency of the Companies to which they have committed their interests. The state of things existing with regard to Life Assurance Companies has no parallel. The public have by the nature of things to repose in them great and, in theory at least, perpetual confidence; and they, in turn, too often treat the public with the most studied and persistent secrecy. None but Companies that fear the light would do other than welcome an impartial parliamentary action which, whilst leaving them unfettered in the conduct of their business, would compel them to state its results with such clearness and precision as would at least rob dishonesty of its congenial darkness." We cannot strengthen this language. It now remains to see how this principle, which in America has long since passed out of the stage of irresponsible discussion, is there carried into practical effect, not by a "paternal Government," claiming the right to treat its citizens as soldiers of a national army, but by a people politically free, beyond all previous example, to do every man for himself that which seemeth good in his own eyes.

\* Vol. 14, p. 413.

After stating, with reference to the 47 offices doing business in Massachusetts, that "the number of new Policies issued during the calendar year 1867 was 145,000, and the amount insured \$420,000,000, against 119,000 Policies issued and \$353,000,000 insured during the twelve months preceding November, 1866,"—and after showing that, at the close of 1867, there were outstanding in these Companies, 432,441 Policies, insuring \$1,234,630,474 (£246,926,095 sterling), yielding an Income of \$62,513,378, and having Assets amounting to \$130,485,501,—the Commissioner makes the following pregnant reflections:—

"The statement of these figures is sufficient, not only to illustrate the magnitude of these interests, but to force upon the mind an enquiry of profound interest to every policy-holder. Whether these companies are after all funding *enough* to meet their enormous future obligations, or whether,—between the ambitious struggle to pay large dividends to the assured, on the one hand, and the temptation to pay large commissions to agents, large salaries and perquisites to officers, and large royalties to stockholders, on the other,—the bottom of the fund may not be reached at some day, more or less distant, with a deficiency of a few hundred millions of dollars unprovided for? This is by no means an impertinent question. It is one that every policy-holder, and every one solicited to become a policy-holder, has a right to ask. Shrewd men are asking it every day, and not a few, who might decide more wisely, are deciding it against the companies. Happily, in this country, there are as yet no precedents which justify such forebodings in regard to the future of life insurance. But almost everybody knows that in England, where the system has had time to show its weak as well as its strong points, neither managers of high and titled names, and blood removed but a few degrees from royalty, nor actuaries, the most profoundly versed in the mysteries of the science, have saved scores of life insurance companies from bankruptcy, and thousands of confiding policy-holders from just such cruel disappointment. . . . .

[We entirely demur to this statement as regards the policyholders. Notwithstanding the numerous cases in which Life Insurance Companies have failed, with great loss to the shareholders, we believe we are strictly correct in stating that, hitherto, there have been very few instances indeed, and those of but trifling magnitude, in which the claims of the policyholders have not been so far provided for, that the policies have been transferred to some other Office on the existing terms.]

"A frequent application of a decisive test to the sufficiency of the reserve, is of the utmost importance to the company, and, if well applied, serves at the same time to assure the public of its soundness, or give warning that it is insecure and untrustworthy. An attempt has been made, in obedience to the law of this Commonwealth, which recognizes its importance to an effective supervision of these interests, to apply this test in all fidelity, to each of the forty-seven institutions which, by location or choice, are subject to its application."

In the "sufficiency of the reserve," the whole question is, of course, bound up. In the only two States of the Union in which laws have been passed in reference to Life Assurance "looking towards a valuation of policies"—New York and Massachusetts—the "standard of *legal soundness*," as it is called by the Commissioner, or the "*low water mark*," above which, "with a fair margin, is the line of safety," as it has been elsewhere happily termed, differs, and a recent attempt to establish uniformity has failed. That adopted in Massachusetts, and the consequence of its application, are thus enacted by the Law of that State:—

"When the actual funds of any life insurance company doing business in this Commonwealth are not of a net cash value equal to its liabilities, counting (as such) the net value of its policies according to the 'Combined Experience,' or 'Actuaries' rate of mortality, with interest at four per centum per annum, it shall be the duty of the Insurance Commissioner to give notice to such company and its agents, to discontinue issuing new policies within this Commonwealth until such time as its funds have become equal to its liabilities, valuing its policies as aforesaid."

In interpreting this Statute, "the question is," says the Commissioner, "whether the guarantee capital is to be treated as a liability, so that, if the capital is impaired, the company cannot do business in this State. It seems very clear that it should not. The Act was designed for the protection of the public against companies unsound as regards the security due to the policyholders. The capital, as regards policyholders, is not in any proper sense a liability, and the stockholders it leaves to look out for their own interests. By reference to the Detailed Statements of Assets and Liabilities, in the statistical part of the Report, it will be seen that the standing of each company

“ having a guarantee capital, is summed up in two ways. In obtaining the first balance, viz., *surplus as regards policy-holders*, the capital is not counted as a liability. In getting the second balance, viz., *surplus over capital, or impairment of capital* (as the case may be,) it is. The latter balance addresses itself to the shareholders, as regards the value of their stock as an investment; the former to the insured, as regards their security. It is only when the surplus as regards policy-holders shrinks into a deficiency as regards them, or, in other words, when the whole capital and something more is wanting, that the company must cease to take new policies within the conservative limits of Massachusetts.”

Under this law, and guided by this interpretation of it, the Commissioner makes his valuation of the liabilities, not quinquennially, as in the State of New York, but annually.

The results of the Valuation for 1867, are shown in very voluminous tables, occupying some 60 pages. “ The obligations of the companies are classified under the four general heads of Whole-life policies, Endowment policies, Simple Term policies, and Annuities, with separate tables for each. The Whole-life and Endowment policies are further classified by the year of issue, and according to the number of annual premiums stipulated to be paid on the policy.” The policies were valued “ by the aid of the Tables published by the distinguished actuary, Professor Elizur Wright, and of recognized authority throughout the insurance world,” whilst the method pursued in making the valuation, was, “ with some minor exceptions, that known as the *seriatim*, the reserve on each of the 430,000 policies being computed separately.” The gigantic labour of the valuation, as well as the precise and voluminous character of the returns that must have been required from the Offices in order to obtain it, will readily be understood.

Not merely on account of their intrinsic interest, but also because they may hereafter serve as models for ourselves, we have taken into our columns certain of the Tables, including the final one, showing the “surplus as regards policy-holders” and the “surplus or impairment of capital.” Further description of them is needless, though it may be well to add that the facts as regards valuations are exhibited for each Office in like form.

## WHOLE-LIFE POLICIES

Of Forty-seven Life Insurance Companies doing business in Massachusetts, outstanding December 31, 1867, with their Net Value at that Date, classified according to the Year of Issue, and the Plan or Number of Premiums payable, each Year ending December 31, inclusive.

## ALL THE COMPANIES COMBINED.

YEAR.	NUMBER AND CLASSIFICATION OF POLICIES.					Amount Insured.	Reversionary Dividends or Bonus Additions.	Whole Amount Insured.	Net Value, December 31, 1867.	Ratio.
	Ordinary.	Ten-Pre- mium.	Five- Pre- mium.	Paid-up.	Excep- tional.					
1830.	1					\$2,000 00	..	\$2,000 00	\$1,251 13	62.56
1834.	2					5,000 00	..	5,000 00	3,139 53	62.79
1835.	1					5,000 00	..	5,000 00	2,250 05	45.00
1837.	1					1,500 00	..	1,500 00	795 40	53.03
1838.	1					3,000 00	..	3,000 00	1,304 48	43.08
1839.	1					1,000 00	..	1,000 00	579 59	57.96
1840.	1					2,000 00	..	2,000 00	1,300 66	65.03
1843.	92					\$11,850 00	\$249,370 73	561,220 73	278,514 33	49.63
1844.	188					576,750 00	286,769 84	863,519 84	391,239 38	45.31
1845.	518					1,540,200 00	456,475 66	1,996,675 66	856,027 72	42.87
1846.	997					2,666,165 00	450,323 74	3,116,488 74	1,221,252 09	39.19
1847.	1,292					3,129,190 00	645,117 02	3,774,307 02	1,397,325 03	37.02
1848.	1,507					3,751,393 00	691,531 51	4,442,924 51	1,580,843 13	34.91
1849.	2,233					5,449,575 00	780,643 30	6,180,217 30	2,002,737 94	32.41
1850.	2,637					6,020,220 00	600,546 18	6,520,766 18	1,997,927 38	30.64
1851.	2,093					4,382,105 00	360,983 52	5,243,088 52	1,456,298 73	23.94
1852.	1,651					4,021,240 00	525,840 47	4,547,080 47	1,469,824 71	28.03
1853.	1,688					4,082,673 00	527,129 04	4,609,802 04	1,252,858 75	27.55
1854.	2,026					5,352,470 00	729,383 19	6,081,853 19	1,193,529 92	25.89
1855.	2,069					5,318,990 00	995,924 90	6,314,914 90	1,456,298 73	23.94
1856.	2,695					7,390,840 00	997,321 24	8,388,161 24	1,619,616 81	23.42
1857.	2,612					7,427,610 00	847,432 98	8,275,042 98	1,589,774 49	19.21
1858.	3,666					11,101,570 00	870,012 96	11,971,582 96	2,045,731 14	17.09
1859.	5,031					14,549,439 00	906,219 12	15,455,658 12	2,336,144 22	13.05
1860.	6,463					18,151,164 87	907,371 43	19,058,536 30	2,583,320 63	13.55
1861.	5,847					16,418,147 48	733,093 45	17,151,240 93	2,136,380 42	12.40
1862.	10,118					27,397,821 06	743,893 43	28,141,714 49	3,106,090 35	11.04
1863.	18,541					55,877,960 92	809,168 47	56,687,129 39	5,702,868 26	10.06
1864.	25,818					95,412,065 14	910,136 66	96,322,201 80	9,080,768 81	9.43
1865.	30,800					141,408,405 80	952,704 84	142,361,110 64	11,266,134 28	7.91
1866.	50,120					228,011,063 08	1,310,363 98	229,321,427 06	11,532,839 49	5.03
1867.	69,789					296,245,138 13	229,883 73	296,475,021 86	7,948,593 28	2.68
Totals.	250,439	74,875	911	5,806	4,226	\$967,113,546 48	\$17,367,640 89	\$984,481,186 87	\$77,778,128 65	7.90

## ENDOWMENT-ASSURANCE POLICIES

(Including a few simple Endowments,) of Forty-seven Life Insurance Companies doing business in Massachusetts, outstanding December 31, 1867, with their Net Value at that Date, classified according to the Year of Issue, and the Plan or Number of Premiums payable, each Year ending December 31, inclusive.

## ALL THE COMPANIES COMBINED.

YEAR.	NUMBER AND CLASSIFICATION OF POLICIES.						Amount Insured.	Reversionary Dividends or Bonus Additions.	Whole Amount Insured.	Net Value, December 31st, 1867.	Ratio.
	Ordinary.	Ten-Premium.	Five-Prem.	Paid-up.	Excep-tional.	Total.					
1849, . . .	3	..	..	..	..	3	\$4,600 00	..	\$4,600 00	\$2,773 64	60.16
1850, . . .	4	..	..	..	..	4	7,000 00	..	7,000 00	5,347 60	76.39
1851, . . .	1	..	..	..	..	1	1,000 00	..	1,000 00	771 00	77.10
1852, . . .	1	..	..	1	..	2	3,000 00	..	3,000 00	2,002 84	66.76
1853, . . .	5	..	..	..	..	5	21,000 00	..	21,000 00	14,632 52	69.63
1854, . . .	2	..	..	..	..	2	7,500 00	\$3,318 78	10,818 78	7,587 95	70.30
1855, . . .	33	..	..	..	..	33	91,500 00	21,493 08	112,993 08	54,941 87	48.74
1856, . . .	50	..	..	..	..	50	161,500 00	36,967 13	198,467 13	118,391 20	59.51
1857, . . .	80	..	..	..	..	80	248,050 00	53,159 91	301,209 91	168,517 78	55.95
1858, . . .	87	..	..	..	..	87	202,250 00	21,920 68	224,170 68	81,524 76	36.37
1859, . . .	185	..	..	2	1	188	444,529 02	45,037 58	489,566 60	175,301 70	35.81
1860, . . .	222	..	..	..	..	222	639,450 00	62,983 97	702,433 97	194,993 52	27.70
1861, . . .	494	6	..	3	..	503	1,271,205 40	70,537 89	1,341,743 29	332,636 10	24.79
1862, . . .	1,553	22	..	13	..	1,588	4,277,083 71	135,307 80	4,412,391 51	1,040,924 22	23.59
1863, . . .	3,451	148	4	24	3	3,630	9,648,529 04	142,723 02	9,691,252 06	1,807,697 03	18.61
1865, . . .	8,045	1,789	3	166	20	10,023	26,202,505 03	198,833 64	26,401,338 67	4,144,579 07	15.70
1866, . . .	19,313	7,477	35	86	240	27,151	72,496,888 37	305,777 97	72,802,666 34	7,562,827 27	10.39
1867, . . .	36,237	10,722	77	137	821	47,994	119,068,090 51	85,571 40	119,153,661 91	6,267,688 15	5.27
<b>Totals, .</b>	<b>69,766</b>	<b>20,164</b>	<b>119</b>	<b>432</b>	<b>1,085</b>	<b>91,566</b>	<b>\$234,695,681 08</b>	<b>\$1,183,632 85</b>	<b>\$235,879,313 93</b>	<b>\$21,983,188 22</b>	<b>9.32</b>

## Summary of the Income, Expenditures, Assets, Liabilities, and Balances of the several Companies, December 31, 1867.

NAME OF COMPANY.	Location.	Guarantee Capital.	Gross Income.	Gross Expenditures.	Gross Assets.	Gross Liabilities.	Surplus as regards Policy-holders.	Surplus or Impairment of Capital.
<b>MASACHUSETTS COS.</b>								
Berkshire, . . . . .	Pittsfield, . . . . .	\$40,000	\$461,412+	\$276,409+	\$869,398	\$800,130	\$69,268	+ \$29,268
John Hancock Mutual, . . . . .	Boston, . . . . .	100,000	514,746	194,591	884,708	627,454	257,254	+ 157,254
Massachusetts Hospital, . . . . .	Boston, . . . . .	500,000	49,141	137,855	798,688	126,000	672,688	+ 172,688
Massachusetts Mutual, . . . . .	Springfield, . . . . .	..	948,215+	478,562+	1,858,244	1,582,582	275,662	..
New England Mutual, . . . . .	Boston, . . . . .	..	2,230,078	1,244,962	6,230,942	5,567,978	652,964	..
State Mutual, . . . . .	Worcester, . . . . .	..	173,262	95,605	832,067	773,813	58,254	..
<b>Totals, . . . . .</b>	<b>. . . . .</b>	<b>\$640,000</b>	<b>\$4,366,874</b>	<b>\$2,417,984</b>	<b>\$11,464,047</b>	<b>\$9,477,957</b>	<b>\$1,986,090</b>	<b>+ \$359,210</b>
<b>COS. OF OTHER STATES.</b>								
<i>Anna</i> , . . . . .	Hartford, . . . . .	\$150,000	\$5,158,562	\$2,019,669	\$7,599,609	\$6,121,687	\$1,477,922	+ \$1,327,922
American Popular, . . . . .	New York, . . . . .	100,000	108,945+	63,518+	292,941	148,457	74,484	- 25,516
Atlantic Mutual, . . . . .	Albany, . . . . .	110,000	142,254	32,950	266,939	157,514	109,425	- 575
Brooklyn, . . . . .	Brooklyn, . . . . .	125,000	539,446+	251,697+	688,480	527,106	161,374	+ 36,374
Charter Oak, . . . . .	Hartford, . . . . .	200,000	2,375,694	1,351,870	3,709,081	3,100,801	608,280	+ 408,280
Connecticut General, . . . . .	Hartford, . . . . .	251,000*	125,009	64,218	355,994*	132,893	223,101	- 27,899
Connecticut Mutual, . . . . .	Hartford, . . . . .	..	7,230,006	2,965,838	17,669,029	12,593,655	5,075,374	..
Continental, . . . . .	Hartford, . . . . .	300,000	432,565	125,444	781,234	401,913	379,321	+ 79,321
Continental, . . . . .	New York, . . . . .	100,000	884,250+	224,712+	996,158	747,481	158,677	+ 58,677
Economical Mutual, . . . . .	Providence, . . . . .	100,000*	113,334	70,974	274,108*	155,106	119,002	+ 19,002
Equitable, . . . . .	New York, . . . . .	100,000	3,252,923	1,565,863	5,103,481	4,745,819	358,162	+ 258,162
Excelsior, . . . . .	New York, . . . . .	125,000	30,756	27,941	148,892	33,088	115,804	- 9,196
Germania, . . . . .	New York, . . . . .	200,000	1,016,830	448,909	1,872,863	1,561,718	311,145	+ 111,145
Globe Mutual, . . . . .	New York, . . . . .	100,000	603,698	271,164	1,158,097	1,026,717	131,380	+ 31,380

Great Western, . . . . .	New York, . . . . .	\$115,000	\$98,723	\$64,312	\$261,701	\$135,609	\$126,082	+\$11,092
Guardian Mutual, . . . . .	New York, . . . . .	125,000	708,962	410,869	1,040,546	391,378	149,168	+24,168
Hahnemann, . . . . .	Cleveland, . . . . .	200,000	72,057	66,324	261,551	98,861	162,699	-37,310
Hartford Life and Annuity, . . . . .	Hartford, . . . . .	300,000	81,440	67,915	349,225	50,730	298,495	-1,505
Home, . . . . .	Brooklyn, . . . . .	125,000	1,043,553†	577,689†	1,643,106	1,413,480	229,626	+104,626
Knickerbocker, . . . . .	New York, . . . . .	100,000	2,077,183	946,270	3,020,601	2,659,727	360,874	+260,874
Manhattan, . . . . .	New York, . . . . .	100,000	1,684,476†	1,245,895†	4,305,113	3,619,843	685,270	+585,270
Metropolitan, . . . . .	New York, . . . . .	200,000	27,909	7,964	313,194	90,865	222,329	+22,329
Mutual, . . . . .	New York, . . . . .	..	10,770,891†	5,290,917†	23,995,058	22,500,781	1,494,277	..
Mutual Benefit, . . . . .	Newark, . . . . .	..	6,038,164	2,728,596	14,391,259	12,187,277	2,203,982	..
National, . . . . .	Montpelier, . . . . .	..	213,748	80,100	664,104	471,978	192,126	..
National, . . . . .	New York, . . . . .	130,000	134,164	100,240	282,308	204,913	77,395	-52,605
New Jersey Mutual, . . . . .	Newark, . . . . .	125,000	102,981	63,221	306,466	169,892	136,574	+11,574
New York, . . . . .	New York, . . . . .	..	3,595,922	1,543,412	9,159,754	8,120,569	1,039,185	..
New York State, . . . . .	Syracuse, . . . . .	120,000	60,836	36,443	180,090	76,785	103,305	-16,695
North America, . . . . .	New York, . . . . .	100,000	1,362,879	610,173	2,610,414	2,340,851	69,563	-30,437
Northwestern Mutual, . . . . .	Milwaukee, . . . . .	..	1,709,315	627,349	3,147,165	2,647,211	499,954	+388,226
Phoenix Mutual, . . . . .	Hartford, . . . . .	100,000	1,329,372†	457,886†	2,218,344	1,730,118	488,226	+9,736
Provident Life and Trust, . . . . .	Philadelphia, . . . . .	150,000	174,338	86,022	336,989	177,253	159,736	+152,532
Security, . . . . .	New York, . . . . .	110,000	1,016,661†	424,704†	1,270,854	1,008,322	262,532	+2,181
Standard, . . . . .	New York, . . . . .	125,000	10,115	5,948	144,495	17,314	127,181	+43,995
Traveler's, . . . . .	Hartford, . . . . .	500,000*	167,514	67,481	940,939	396,944	543,995	..
Union Mutual, . . . . .	Augusta, Me., . . . . .	..	1,741,716†	652,388†	2,993,506*	2,484,543	508,963	+416,720
United States, . . . . .	New York, . . . . .	100,000	750,973	302,954	2,470,792	1,954,072	516,720	-96,886
Washington, . . . . .	New York, . . . . .	125,000	503,218	288,543	946,724	918,560	28,164	-100,318
Widows' and Orphans' Benefit	New York, . . . . .	200,000	507,094	286,834	761,318	661,636	99,682	+18,452
World Mutual, . . . . .	New York, . . . . .	200,000	88,036	65,178	248,931	67,333	181,548	..
Totals, . . . . .	..	\$5,311,000	\$58,146,504	\$26,636,394	\$119,021,453	\$98,750,350	\$20,271,103	+ \$3,946,242
Grand Totals, . . . . .	..	\$5,951,000	\$62,513,378	\$29,054,378	\$130,485,500	\$108,228,307	\$22,257,193	+ \$4,305,452

\* For fourteen months, from November 1, 1866.

\* Not including Stockholders' notes.

It will be seen that, as regards the policyholders, all the companies are solvent; but that, as regards the shareholders, there are several in respect of which the *minus* sign points out that the business, as estimated by the legal standard, has so far resulted in loss to the shareholders. Well would it be for many shareholders in this country, if they could, in like manner, receive a timely warning from an authority not to be disputed, that their trusted managers, who are transacting a large business with extremely beneficial results to themselves, are proceeding on unsound principles, which must ultimately, if not corrected in time, terminate in ruin—ruin the more disastrous as it is the longer deferred.

It will doubtless have been present to the minds of our readers, that this balance of Assets over Liabilities appears to have been struck without any enquiry having been made as to the character of the Assets—a question of hardly less importance than that of their amount. Were this so in reality, the whole investigation would have been a farce, or something worse. But it is not. The details of the Assets are given in an appendix, and have not, therefore, hitherto fallen to be noticed by us; but quite sufficiently early in the Report, the Commissioner asks the pertinent question, “What are Assets?”, and the answer he gives is well worth our attention.

“It would,” he says, “be entirely idle to establish a standard of reserve, and compute the liabilities of a company in accordance with it, unless we went one step further, and required the company to respond in safe and legitimate assets.” He does not seem to have the power to pronounce judgment as to what assets are safe and legitimate. “All that is required is, that the specific character of the asset should be distinctly indicated, so that the public may be able to judge of its value as a security or investment.” Hence very detailed returns are required from the Companies. “The returns have generally been made with entire perspicuity and straightforwardness. Occasionally, however, some very plausible generic description has been found, upon sifting down, to cover up something that shunned the light. “‘Personal property,’ ‘leger [*sic*] balances,’ ‘book accounts,’ ‘notes’ ‘receivable,’ &c., which may in terms cover anything and everything, have come to be regarded as decidedly suspicious. In some cases, items of near \$100,000 have been returned in this way, conveying no data for even a shrewd guess of what they consisted, and imposing the necessity of further interrogation. “It is proper to say that every *lucus a non lucendo* of this sort will be hereafter rejected without further question.”

Though without authority to say what assets shall, and what

shall not be regarded as legitimate, the Commissioner discusses this question at great length, and with much force, in order that the assured may be able to determine it for themselves. "About some items," he says, "no doubt can exist, and no question is made. Real estate is allowed to be a good investment," and "mortgages of real estate are among the very best investments,—all things considered, probably the safest and best that can be made," whilst "Securities of the United States ought to be an investment of the most unquestionable character. . . . And so the various State and other public and corporate stocks and bonds, if put in at their actual cash market value, and loans amply secured by any of these as collateral, and cash on hand and in bank, are allowed to be legitimate assets."

It is satisfactory to learn that "all of the foregoing make up about two-thirds of the gross assets of the companies combined, and about four-fifths of their aggregate reserve fund," the proportions varying, of course, in different Companies.

Other assets he pronounces to be legitimate, though their right to be so considered has not gone unquestioned. Of these, are "*Accrued interest*, not due or collected"; "*Unpaid and deferred premiums*," which "fell due during the year ending on the day which is the date of valuation," on policies then in force; and "*Premium notes and loans*" (the American equivalent to the English *Half-credit premiums* in arrear), provided the premium note, or other credit on any policy, did not exceed the then net value of the policy. The extent to which this system is resorted to in America, may be judged of by the fact that the premiums in arrear vary from *nil* to 58·81 per cent of the net assets of the respective Companies, and are, in the aggregate, equal to 28·01 per cent of the total net assets of all the Companies combined. Loans on personal security, and all kinds of stocks and bonds of doubtful or speculative value, "ought," says the Commissioner, "to be entirely abstained from." This condemnation of loans on personal security seems to us rather too sweeping; for the experience of many Offices in this country is that, when transacted with proper care, they are safe and profitable investments, and in every way eligible for Life Insurance Companies. There is one item commonly to be found among the assets of American Companies, against which the Commissioner especially sets his face. We refer to what are called "*Commuted commissions*." "In all cases they have been rejected as assets against the net premium reserve." His reasons for such rejection are stated in the following terms:—"Commissions form

"an important part of the expenses which the margin or loading was intended to provide for, and whether paid in advance, or from year to year, the money paid on account of them is an expenditure, and not an investment. It may be a very judicious expenditure. Whether it is, or is not, wise for a company to burden all the future premiums on a policy with a commission or annuity in favour of the agent who gets the policy, it may be very prudent for a company which is under such a liability to get rid of it. But discharging a liability does not make an asset. . . . It may also be questioned on broader grounds, whether the practice of paying or buying commissions in advance, is one that ought to be encouraged. With some of the companies, it is confessedly a concession to the apparent necessity of securing effective agents by a large immediate inducement, and not the dictate of sound policy. If life insurance is a good thing, and needs the eloquence of an agent, inspired by a commission, to impress this fact on a reluctant and procrastinating public,—both of which premises are unquestionably true,—he ought, by the same token, to have some inducement to perpetuate the benefits of the insurance by keeping the policy alive, as well as a very powerful one to get it at the first."

As bearing on these remarks we quote a statement which occurs early in the Report, that, "after making due allowance for the legitimate termination of policies by death, purchase and expiry," some 40,000 policies, insuring more than \$100,000,000, were allowed to drop in the year under review "from a mere want of persistency on the part of the assured." These policies were exactly a third of the whole number issued in the preceding year, in which the great bulk of them was doubtless issued. We will not now give an opinion as to the propriety of the Commissioner's decision in rigidly excluding all these "commuted commissions" from the list of assets; but content ourselves with remarking that, if this course be taken, it is clearly unjust, if not absurd, to require of the Offices a reserve in realized assets computed by the Experience 4 per cent net premiums, in respect of the policies on which the commuted commission has been paid. At the end of the first year of such an insurance, after the commuted commission and other expenses have been paid, and the current claims provided for, is it possible that a sum will be left at all approaching the Experience 4 per cent reserve? On the other hand, what actuary would venture to say that, under the circumstances, it was in any way necessary that the Office should possess that reserve? In these and some other respects it appears to us that the Commissioner shows a not unnatural tendency to magnify his office and exceed his authority.

Judging by results, it would appear that the wholesome restraint of publicity is not enough to exclude doubtful assets from figuring in the yearly balance sheets. Whilst in the case of a great and prudent Company like the Mutual of New York, there is hardly an item in its tale of securities, vast as they are, to which the most captious could take exception, there are other Companies whose list of assets affords hardly one on which the eye can rest with satisfaction. Railway, Bank, and Fire Insurance Companies' *shares,—shares* in a Linen Company, a Live-Stock Insurance Company, a Gas Company, Adams' Express Company, a Carpet Company, a Mail Steamship Company, a Telegraph Company, a Printing Company, "Wells, Fargo & Co."—are specimens of securities, often for considerable amounts, that meet one on almost every page of the returns, though most frequently and most largely in those of inferior Offices, which the public has been taught to avoid. These, let it be remembered, are assets which the Commissioner has had no alternative but to admit, their character having been stated with distinctness, and the legal obligation having been thus fulfilled. Amongst those which, lacking such definiteness, the Commissioner has been able to exclude, we find some which speak volumes for the ingenuity of their contrivers ;—"printing and stationery, estimated, \$4,000," "stationery in office and with agents, \$7,500," are examples.

Having investigated the condition of solvency of each Office, and exhibited in detail the data on which the investigation was made, it might on the first blush be thought that the labors of the Insurance Department were ended. But the State, or the Commissioner in interpreting the duty it imposes on him, has wisely thought otherwise. In what has been already done, there has been little to indicate in what degree there exists prudence of management, which, even in regard to solvency, is almost as important to an Office in the future, as is a sufficiency of assets in the present. Especially, however, is it important in relation to profits. Prudence lies at the root of gain ; "and among a people with whom thrift "is a cardinal virtue, and whose practical shrewdness is quick to "detect the fact that a thing costs more than it is worth, life "insurance cannot expect to recommend itself to the public "favour for any great length of time, unless its benefits can be "secured without too large an outlay on its machinery."

Returns have, therefore, to be made in such a form as to exhibit the whole internal economy of the Offices, without concealment or gloss. Again we have ventured to give a specimen, which includes the returns of two Companies at once amongst the largest and the most economically conducted, the Mutual Benefit Company holding, in fact, the very highest place in the latter respect.

## LIFE INSURANCE COMPANIES OF OTHER STATES.

Abstract of Annual Statements for the Year ending December 31, 1867,—with Detailed Statements of Assets and Liabilities.

	MANHATTAN. New York.	METROPOLITAN. New York.	MUTUAL. New York.	MUTUAL BENEFIT. Newark, N. J.
<b>GUARANTEE CAPITAL.</b>				
Whole amount of guarantee capital, . . . . .	\$100,000 00	\$200,000 00	.. ..	.. ..
of capital actually paid up in cash, . . . . .	100,000 00	200,000 00	.. ..	.. ..
consisting of stockholders' notes, . . . . .	.. ..	.. ..	.. ..	.. ..
par and cash market values of each share, . . . . .	\$50 ..	\$50 ..	.. ..	.. ..
<b>ASSETS.</b>				
Gross present Assets, . . . . .	\$4,305,112 72	\$313,193 94	\$23,995,057 97	\$14,391,258 54
<b>LIABILITIES.</b>				
Gross present Liabilities, . . . . .	\$3,619,842 95	\$90,865 02	\$22,500,781 19	\$12,187,277 15
<b>INCOME.</b>				
Whole amount received for premiums in cash, . . . . .	\$1,060,571 06*	27,264 87	\$9,073,890 77*	\$3,401,636 31
in promissory notes and securities, . . . . .	346,626 01	.. ..	.. ..	1,726,844 61
for interest on premium notes and securities, . . . . .	.. ..	.. ..	1,082,964 11	245,013 73
for interest on mortgages of real estate, . . . . .	277,379 12	.. ..	546,361 62	182,220 31
for interest and dividends from other sources, . . . . .	.. ..	644 54	.. ..	499,680 43
from other companies for claims on re-insurance policies, . . . . .	.. ..	.. ..	.. ..	.. ..
for rents, . . . . .	.. ..	.. ..	67,375 00	2,768 80
from all other sources, . . . . .	.. ..	.. ..	300 00	.. ..
<b>Gross Income during the year, . . . . .</b>	<b>\$1,684,476 19*</b>	<b>\$27,969 41</b>	<b>\$10,770,891 50*</b>	<b>\$6,058,164 19</b>
<b>EXPENDITURES.</b>				
Amount paid for losses and claims on policies in cash, . . . . .	\$471,747 79†	.. ..	\$1,258,293 56a	\$932,152 91e
in premium notes and securities, . . . . .	28,878 96	.. ..	.. ..	76,821 88
on lapsed, surrendered or purchased policies in cash, . . . . .	57,344 11	.. ..	249,483 02	67,705 74
in premium notes and securities, . . . . .	122,651 04	.. ..	.. ..	254,827 32

to the stockholders of the Company in cash, for dividends to the assured in cash or in reduction of cash premiums, . . . . .	\$40,000 00†	.. ..	.. ..	.. ..	\$595,791 33 193,809 23 372,404 91 29,899 33 47,034 32 76,610 23 .. .. .. .. 82,033 50
in reduction of premium notes and securities, for brokerage and commissions on premiums, for medical examinations, . . . . .	191,096 86 14,000 00   198,466 92§ 17,998 83 44,419 22 11,062 57 .. .. .. .. 48,228 22	.. .. .. .. \$3,789 64 791 00 .. .. .. .. 94 50 3,288 42 .. .. 205,531 46d	.. ..	.. ..	\$2,313,647 77b 1,000,860 48c 73,664 06 110,049 83 79,436 39 .. .. .. .. 82,033 50
for salaries and pay of officers and employes, for National, State and local taxes and fees, for premiums on re-insurance policies, for rents, . . . . .	.. .. .. .. .. .. .. .. .. .. .. .. .. .. .. .. 48,228 22	.. .. .. .. .. .. .. .. .. .. .. .. .. .. .. .. 48,228 22	.. ..	.. ..	.. .. .. .. .. .. .. .. .. .. .. .. .. .. .. .. 82,033 50
for office, agency and incidental expenses, . . . . .	.. ..	.. ..	.. ..	.. ..	.. .. 82,033 50
Gross Expenditures during the year, . . . . .	\$1,245,894 52*	\$7,963 56	\$5,290,916 57*	\$2,728,595 70	\$2,728,595 70
GENERAL ITEMS.					
Whole amount insured by existing policies, . . . . .	\$42,500,835 00	\$862,800 00	\$179,608,120 05	\$104,649,625 18	\$104,649,625 18
Number of policies terminated during the year by death, amount insured thereby, . . . . .	106* \$497,327 00	.. ..	363* \$1,068,900 00	297 \$997,550 00	297 \$997,550 00
of policies issued during the year and so terminated, amount insured thereby, . . . . .	8 \$31,000 00	.. ..	26 \$81,500 00	17 \$59,500 00	17 \$59,500 00
Amount of funds set apart for distribution during the year, to stockholders, . . . . .	292,699 24 41,836 62	.. ..	2,124,000 75	1,213,795 29	1,213,795 29
rate per cent of dividends declared to stockholders, Highest rate of interest received, . . . . .	250,862 62 83 per cent	.. ..	2,124,000 75	1,213,795 29	1,213,795 29
average rate received, . . . . .	7 per cent .. ..	7 per cent .. ..	7 per cent .. ..	8 per cent .. ..	8 per cent .. ..

\* The statement of this Company covers fourteen months from Nov. 1, 1866.  
† Including \$31,250, "percentage or share of premium or other receipts."  
‡ Estimated.

b Including \$23,399.62, *post mortem* dividends.

c Including \$641,305.50 in commutation of all future commissions, and in purchase of agents' annuities.

d An item of \$20,000 (not included in this amount), is charged to expense account for "rents," and credited to building sinking fund account.

e Including \$355.07 for annuities.

The facts, in regard to expenses, disclosed by these returns, are then thrown into five tables, with results which, had they reference to Offices in this country, might well give rise to the most lively apprehensions. They are serious, even from an American point of view, and after allowing for counterbalancing advantages existing there which are not possessed by us.

The tables are—A, showing "*Ratio of Commissions to Premium Receipts*"; B, "*Ratio of Gross Expenses (excluding Dividends on Guarantee Capital,) to Premium Receipts*"; C, "*Ratio of Expenses (including Net Cost of Guarantee Capital) to Entire Receipts*"; D, "*Ratio of Gross Expenses to Gross Receipts*"; and E, "*Synopsis and Average of Expense Ratios for 1867.*" The multiplicity of these tables is, we think, due either to the weakness or the pity of the Commissioner, who,—regarding "comparisons as odious, especially "on a point so sensitive as this," and believing that "the relative "cost of insurance in the several companies cannot fairly be judged "of from a single stand-point,"—exhibits the ratio of expenses in several aspects, "in the hope that both the strong and the weak points "of each company, on the score of economy, may crop out somewhere." The three intermediate tables have a special value, inasmuch as by contrasting the year 1867 with both the years 1866 and 1865 they show us that matters are but little on the mend. Taking the years in the order just stated, the ratios in table B, for all the Offices combined, are 19.05, 19.97 and 17.50 respectively; in table C they are 17.31, 17.98 and 15.03; and in table D they are 17.58, 18.32 and 14.59. But the most interesting table is the first. If there be a lack of economy anywhere, it is sure to show itself, in its most vicious form, in the matter of commission. The gross premium receipts of the 47 Offices for the year 1867, were \$55,419,164, and the gross expenses, excluding dividends on capital, \$10,558,731, or 19.05 per cent of the whole. Of these expenses, \$6,648,773—equal to 12.00 per cent on the premiums—were for commissions. The percentages varied, of course, in the different Offices, the lowest being 6.34 and the highest 24.99. In common fairness, it should be stated that the sums returned by the Companies as commuted commissions, and treated by them as investments, have here been charged in the expenses of the year. Giving the Companies the benefit of their view of the case, \$881,550 would have to be deducted, thus reducing the percentage of commission on the premiums to  $10\frac{1}{2}$  per cent, and that of the gross expenditure to  $17\frac{1}{2}$  per cent. To one Office, the Mutual of New York, such a correction would

be of the highest value. It is in these returns charged with \$1,000,860 as commission; it claims credit for no less than \$641,305 as commuted commissions, thereby reducing the 11.03 percentage of commission on premiums, as given by the Commissioner, to under 4 per cent. Such a correction in the case of any Office would, however, be far more unduly lenient than the plan of the Commissioner is unduly severe; for it is obvious that much of the commission commuted, is commission which must have been earned in connection with the business of 1867, and, as such, is chargeable on the receipts for that year. The importance of this consideration will be seen when we recall to our recollection that the new business of that year was, in amount, rather over a third of the whole sum remaining assured at the close of it.

Substantially, the fact remains that the ratio of the expenses is, in America, extremely high,—certainly not less than 18 per cent on the premiums. In the case of the Mutual Benefit Company, 12 per cent of the total premiums has sufficed for its expenses, and it might, therefore, suffice for others; and yet, in the face of this, we find such enormous ratios of expenses to premiums as 40 and 50 per cent, even in the case of established Companies. These, be it remembered, too, are all Companies that, with the aid of their capital, are solvent as regards their policy-holders. Well may the Commissioner point out that “the need of “reform in certain directions is too obvious to be overlooked,” though, with something approaching to unworthy timidity, he contents himself with showing the facts in regard to the expenses of the past year, without “intending . . . to give aid and “comfort to those alarmists who predict that this is the rock on “which the system is destined sooner or later to split.” We ourselves, without being alarmists, certainly think that it is. Following out the metaphor, there is, we are convinced, many a well found ship, heavily laden with the treasures of Assurance, now struggling with the fast whelming waters, whose broken keel and riven sides tell the tale, but too truly, that the once stately galleon has already fallen, in an evil hour, on this sunken and deadly rock. Publicity—the publicity which we contend for—may not, it is true, save these torn and shattered barks, or much of their costly freight, but it will at least serve to notify the world of their unseaworthiness, and, by such timely warning, keep others from venturing, in their unsafe bottoms, the sacrifice of their toil and, it may be, the sole inheritance of their children.

These lines had not been fully written when the introduction

into Parliament of Mr. Cave's Bill, showed that the task of legislating effectually for Assurance Companies had come to be regarded as within the possibilities of English statesmanship. We shall be glad if anything we have said may help forward that measure. Without approving of all its details, which this is neither the time nor place to discuss, we regard it as having taken hold of the right principle, and therefore as sufficient, if shaped with care, to lessen, if not to correct, the evils at which it is aimed. If it become law, we believe that it will be efficacious either to justify or to remove the suspicions which now attach, whether fairly or unfairly, to so many Offices.

The length to which our remarks have run, leaves us less space than we could desire for our notice of the discussion, to which we have already adverted, as to the comparative merits of the "percentage" and the "contribution" methods of dividing surplus. The percentage plan has its advocates amongst the seventeen correspondents of the Commissioner, as have one or two altogether novel methods of division; but the majority of the writers, and those certainly of most authority, give preference to the contribution system, and it is on that that the chief interest of the discussion concentrates itself. Mr. Homans has already in these pages said, so well and so fully, all that can be said on behalf of the system which he has the merit of having in whole or in part devised; and we have also ourselves, on a previous occasion, so frankly admitted its advantages, that lengthened repetition of its claims to consideration is needless now. But, still differing from its admirers as to its absolute perfection, and as to the need for its universal employment, we deprecate the tone adopted in the discussion by some of its adherents, which we think but little to their credit. It exhibits not merely the proverbial zeal, amounting to rashness, of recent converts, but that worst form of the tyranny of numbers—the desire it too often exhibits to compel that uniformity of thought, which results always in the unfruitfulness of monotony. We can pardon, readily enough, the enthusiasm of Mr. McAdam, who asserts that the contribution plan "affords strict mathematical justice to all," and that "it is easily understood, —more comprehensible than the percentage system, which is inexplicable,"—that being merely excess of zeal. We can forgive, whilst taking leave to doubt, the assertion of Mr. Homans, that, in addition to being simple and easily explained, "it is popular with agents and with policyholders, and, what is of more importance, is acknowledged, almost universally, to be just and equitable,"—that being but the natural partiality of a parent for his own offspring.

But when Mr. McCay links the word "honest" to "just," and appeals to one's "moral sense" to support his view that "the percentage plan is unjust, and monstrously so," being "systematic robbery on a gigantic scale involving millions upon millions every year," whilst "the contribution plan is just,—perfectly just";—and when, above and beyond all this, the Commissioner himself casts aside all judicial gravity, and, with the airs, adopts the language of stump oratory, permitting himself to say of the percentage plan, that, in dealing with the money of the insured, it "is like the hospitality" of the famous old robber of Attica, who, if the legs of his unwilling guests were too long for his bed, lopped them off, and "stretched them to the requisite length if they were too short";—we think it time to protest. Such language is not scientific, nor philosophical, nor true. Is it likely to be convincing?

Not in these columns will it be denied that in their efforts to perfect a more equitable plan of dividing surplus, the American Actuaries have done wisely and well, both for their own fame and the interests of assurance itself. But it must not be forgotten that, in this respect, circumstances have pressed closer on them than they have done, or are ever likely to do on most of us. The economic condition of assurance widely differs in the two countries. As we all know, a large portion of any surplus is made up of interest realized in excess of the expected rate, and is derived mainly from the older policies having the largest reserves,—a fact of which the percentage system of distribution takes no note. In America this is a far more pregnant circumstance than it is here. We assume 3 or sometimes  $3\frac{1}{2}$  per cent interest as the basis of our premiums, and value our liabilities most of us at 3, some at  $3\frac{1}{2}$ , a few at 4 per cent; whilst we realize, it may be, a net interest of something, though not much, over the last named rate, the cases being rare in which the interest approaches closely to 5 per cent. The Americans almost universally base their tables and valuations on 4 per cent interest, and as generally realize fully 7 per cent. The right disposal of the surplus interest, therefore, is with most of us of little moment in comparison with its importance to them. Hence, there has been a stimulus acting on their inventive faculties either wholly wanting, or existing in a much smaller measure, amongst us. They have, as we have said, wisely acted on its promptings; but, nature being frail, they have fallen into the error of thinking that it has led them to perfection.

In claiming for this new system that it "works strict mathematical justice"—"perfect justice," they display neither accu-

racy nor modesty. In the important particulars of expenses and cost of assurance, it makes no attempt to deal out justice *inter pares*, as we pointed out in our July number; and its expediency we regard as more than doubtful, if not subject to checks that almost uproot its principle. It may appear ungracious to join issue with Mr. Homans on what would appear at first sight to be a question of fact; but to ask us, as he does, to believe that the plan is simple, popular, and easily understood, is to draw unduly on our powers of belief. A principal, ever-present object in Assurance, is to get rid of violent fluctuations, by taking large averages. In his method of distribution, the very contrary course is pursued. Almost of necessity, the division is yearly. The period of observation is, therefore, of the briefest. The actual mortality at a particular age is made the ground work of the assessment on each policy for the cost of assurance for the year. Instead, therefore, of an aggregation, there is segregation of numbers,—small numbers observed over a brief space of time. Here we have, as everybody knows from daily experience, the amplest scope for the extremest variations. Men of consecutive ages, assured at the same time for like amounts, and even the same individual, assured under almost identical policies, but standing at different ages in the Office books, might receive the most widely differing bonuses, owing to the differing mortality at the respective ages. Indeed, it may even be, that for purposes of bonus the one policy shall be taken and the other left,—that whilst a large addition shall accrue to the one, nothing whatever shall accrue to the other. Nay, it is easy to conceive the case of a policyholder being “mathematically” called on to make good a deficit, the transactions of the year in respect of his class having issued in a loss. Nothing can be stronger, or more pertinent, than the following remarks, on this point, of Mr. McCay, one of the most enthusiastic upholders of the system: “If by ( $m'_{x+n}$ ) is meant the ‘actual mortality’ at each age, as “( $r'$ ), (P), and (e) represent the ‘actual facts’ instead of ‘theoretical assumptions,’ there would be great objections to this understanding; because, even in a large company, there are “great irregularities in the actual mortality at each year of life, “and in a small company, there are still greater. At many of the “ages ( $m'$ ) might be zero in the largest company; and at other “ages, two or three or ten times larger than ( $m$ ). In both companies, it might easily be three times as large for 47 as for 57, “or for 45 as for 46. The charges for one year’s losses would “change very irregularly from year to year for different members.

“ At one time they might be very large for one, and for another very  
“ small, while in the next year the reverse might happen. One of  
“ the insured might be charged heavily for every successive year  
“ for ten years, while another, one year older or younger, might  
“ be charged very lightly during the same period.” And on the  
subject of expenses, we add here the strong language of Mr. Elizur  
Wright, not so much to confirm our own view, as because it raises  
more broadly than we have yet seen the question of the real injustice,  
as we have always thought it, of charging, at any age, the same  
ratio for expenses on large policies as on small. He says,  
“ A much more important difficulty is that of properly assessing  
“ the expenses. The usual method, which I have pursued in these  
“ examples, is open to the serious objection of not only not having  
“ any satisfactory or logical argument in its favor, but of utterly  
“ confounding the important distinction between the self-insurance  
“ or savings bank deposit, and the insurance by the company. I  
“ need not say what would be thought of a savings bank charging  
“ fifteen per cent. on all the deposits for expenses. . . . .  
“ This plan of dividing according to contribution, is sure to call  
“ attention before long to the proper mode of assessing expenses ;  
“ and when it does, we may expect some change of practice in  
“ the direction of equity.”

Perhaps a sufficient provision would be to fix a limit of the sum assured—say £200—below which no profits will be allowed to a policy, and in calculating the profits on the policies, to deduct that limit in all cases from the sum assured, and compute the profits with reference to the balance only.

But a more serious objection, viewed in relation to its equity, has been made to the contribution system. We find it in the number of the “ Insurance Times ” of New York, for December, 1868, and, from internal evidence, we shall hardly be wrong in attributing its authorship to Mr. Meech, who is one of those who incline to the method of division which has generally and so long prevailed in England. His objection is that the contribution system is capable of very arbitrary and unequal application by the employment of different rates of interest in estimating the reserve. We subjoin his own explanation of it. The formula used in the calculations throws off, as will be seen, any gain from diminished mortality. As time goes on, and the Offices grow in bulk as well as in age, this gain will, he thinks, cease to be of much moment. We agree with his opinion, expressed elsewhere, that “ the mortality of insured lives evidently must approximate to that of the general

"community, as life insurance becomes universally practised," though we cannot follow him in his succeeding assertion that, "in the long average, therefore, the experience of American companies will conform most nearly to the natural standard." The following is the argument which we have ventured to attribute to Mr. Meech:—

"Since the assets of American Companies now correspond to a three per cent net valuation, let us compare the dividends of that and higher rates of interest.

"A party aged thirty-five years takes out an ordinary life policy of \$10,000, at an annual premium of \$273 (Carlisle, four per cent net, with 35 per cent added). Required, his annual dividends on the contribution plan, according as the premium reserve is reckoned at three, four, five, or six per cent interest, the company receiving six per cent on investments, above expenses.

#### THREE PER CENT CONTRIBUTION PLAN.

End of Year.	Premium Reserve.	Contribution Dividend.	Premium, less Dividend.
1 . .	\$128 70	\$63 25	\$209 75
2 . .	259 90	67 19	205 81
3 . .	393 20	71 19	201 81
4 . .	539 10	76 56	196 44
5 . .	663 80	79 30	190 70
10 . .	1,322 60	99 07	173 93
20 . .	3,100 50	152 40	120 60
30 . .	4,898 00	207 83	65 17

#### FOUR PER CENT CONTRIBUTION PLAN.

End of Year.	Premium Reserve.	Contribution Dividend.	Premium, less Dividend.
1 . .	\$108 50	\$81 26	\$191 74
2 . .	230 00	83 69	189 31
3 . .	334 60	85 78	187 22
4 . .	451 20	88 11	184 89
5 . .	567 40	90 44	182 56
10 . .	1,136 06	101 89	117 19
20 . .	2,782 10	134 73	138 27
30 . .	4,538 40	169 86	103 14

#### FIVE PER CENT CONTRIBUTION PLAN.

End of Year.	Premium Reserve.	Contribution Dividend.	Premium, less Dividend.
1 . .	\$92 20	\$96 16	\$176 84
2 . .	188 60	97 12	175 88
3 . .	285 90	98 12	174 88
4 . .	386 90	99 11	173 89
5 . .	486 90	100 11	172 88
10 . .	977 90	105 02	167 98
20 . .	2,498 90	120 23	152 77
30 . .	4,205 90	137 30	136 70

## SIX PER CENT CONTRIBUTION PLAN.

"End of each year, contribution dividend, \$108 33; premium less dividend, \$164 67; the same in each year after the first. With the Carlisle Table, if  $i$  denote the assumed rate of interest to which the net premium  $P$  and net reserve  $R$  must correspond.

$$(273 - P) 1.06 + (R + P) (.06 - i) = \text{Dividend.}$$

"It is evident that when  $i$  is made equal to .06, the term containing  $R$  vanishes, leaving the premium less dividend \$164 67, uniform in each year, as in the common percentage plan.

"Comparing, now, the above scales of dividend, the first, which is the more equitable, with the second, the differences are remarkable and apparently greater than any errors of the common percentage plan."

But may it not be said that the American Actuaries have set themselves an impossible task? Following them within certain bounds with respect and thanks, we part company with them when they attempt this mathematical hair-splitting. Their purpose, they say, is to establish a perfect equity amongst the assured, forgetting that that idea, if followed to its logical issue, would be fatal to assurance itself. We assure, at the same rate, the dwellers in cities and the dwellers in the country, measuring by a common standard, life in Liverpool, for example, where it prematurely fades, and life as it flourishes to the extreme of age on the hills of Surrey or the downs of Sussex. For the purposes of assurance we take little or no account of occupation. In this respect, we winnow with a sieve so coarse, that through its meshes can pass, with equal freedom, the county squire or parson, who leads his peaceful life amid ease and plenty, and the over-tasked brain-worker of towns, "who scorns delights and lives laborious days." Where in all this is the equity? And this is but a sample of the inequities running through the assurance principle,—which the Americans, were they consistent, would term iniquities. We do not, indeed, think them so; but neither do we think it "monstrously unjust," or "systematic robbery on a gigantic scale," that bonus should be allotted on the percentage, or on any other system well understood by those who adopt it.

We are ourselves very much disposed to think, that the Americans would do well to diminish the importance of this perplexing bonus question, by more nearly approximating, in their tables of rates and modes of valuation, to obvious facts. Bonuses, we are convinced, have worked a moral mischief, by withdrawing the atten-

tion of the assured from the safety of his Office, on which it should be chiefly fixed, and concentrating it on the accident of gain. It is a question, worthy of some thought, how much the large marginal addition to the premium known to be sufficient, has tended to degrade the conduct of assurance business. With loaded premiums, an unscrupulous actuary may play as a gambler plays with loaded dice. Much of the extravagance, the easy admission of lives, the imprudent investments, to which Companies have owed their decay or fall, we believe to have been due to the knowledge that the premium demanded was far more than that required for the risk. We are inclined to think that nothing would purify and benefit the cause of assurance so much as a well-considered reduction of the premiums. By circumscribing the area of chance, the arts of prudence and caution would succeed the contrivances of reckless ambition.

We stay but to notice the fact that, amongst other matters which we have not time even to glance at, there is embodied in the Report what is termed a National Life Table, as constructed by Mr. Meech from statistics of white males in the whole United States. Its close agreement with the Carlisle Table is, says Mr. Meech, "truly surprising, and evinces the sagacity and good fortune of his (Mr. Milne's) selection." Mr. Meech acknowledges the assistance afforded by Mr. Makeham's papers in his graduation of the Table, the facts of which he throws into D and N Columns after the fashion of Mr. Chisholm and Dr. Farr.

Our opinion of this Report will have been gathered as these pages have been perused; but, as a final word, we cannot refrain from adding that nothing on the subjects of which it treats more interesting or more able, has ever come under our observation.

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## HOME AND FOREIGN INTELLIGENCE.

*On the Condition and Progress of the German Life Assurance Offices in the Year 1867; being an Article which appeared in No. 881 of the Bremen "Handelsblatt." Translated and abridged by G. W. BERRIDGE, of the London and Provincial Law Assurance Society.*

IT appears that the amount of life assurance business transacted in Germany in the year 1867, considerably exceeded that of the previous year, and was also more profitable; for in 1866 it had to contend with both War and Cholera, which not only lessened the new insurances and increased the number of lapses, but also greatly increased the claims. These influences extended in some degree to the year 1867, but the Reports of the Companies have mostly expressed gratification at the results of the year, although the high price of the necessities of life, arising from a partial failure of the harvest of 1866, and a want of confidence in the permanence of the peace of Europe, had acted adversely.

The author of the article (Herr Finanzrath Hopf of Gotha) attributes this spread of assurance, in the first place, to the experience which the German public have acquired, of the practical benefits of assurance; of which the amounts paid for claims by death may be taken as some measure. In the year 1866, almost all the Companies paid claims above the expectation, and some considerably so; and the amount paid by 17 Companies alone in the year 1866 was more than £130,500 in excess of the expectation; the amount paid in the years 1860 to 1867 having been as follows:—

	£	Paid in respect of 2,062 deaths.
1860	350,179	
1861	390,714	" " 2,366 "
1862	411,491	" " 3,063 "
1863	513,462	" " 3,408 "
1864	545,626	" " 3,852 "
1865	633,373	" " 4,553 "
1866	904,695	" " 6,573 "
1867	786,776	" " 5,921 "

From these figures it will be seen how greatly the payments in the year 1866 exceed those in the previous year.

Another reason given for the increase of the business is the greater freedom in the transaction of business, resulting from the removal of many burdensome restraints. The change for the better in this respect, dates from the year 1848, when the local governments began to remove various restrictions, imposed under the idea of protecting the public from fraud. The relaxation has been gradually progressing since that time; and made a marked advance in the year 1867. It is true the founding of new life assurance societies still depends, in nearly all German states, on the permission of the local government; but once founded, it is far easier than formerly to obtain permission to transact business in the other states, Austria alone excepted: and permission once obtained, the transaction of business is less hampered: one especial point being, that the agents no longer need a particular concession running in their names, and any person can now act as an agent without a special government licence.

The activity of the agents has also been greatly stimulated, not only by their increased and still increasing numbers, but by the greatly increased rate of commission allowed: the effects of which are noticeable in previous accounts, and particularly so in those for 1867. It has been found that assurances obtained by the payment of a high rate of commuted commission soon lapse; and an increase of business of such a character is hardly a matter for congratulation; for the cost of management is in some instances so greatly increased, that the profit on the earlier premiums (assuming them to be of moderate amount) no longer suffices to meet it; and consequently a charge on the future is created. Some Offices treat the amount of commuted commissions as a charge to be spread over several years, which is at least an open course, as the balance sheet arranged on this principle shows the amount of such commission which still remains to be written off. Others, however, conceal its amount by deducting it from the reserve, the latter being reduced by various ingenious devices, such as tampering with the net premiums or valuing the policies at a high rate of interest, &c.—a most perilous course when carried to an extreme.

The above-mentioned facilitation of life assurance business has arisen, partly from a less rigorous practice on the part of the government officials, and partly from special decrees of the several governments. But no general statutory (or legal) regulation for

life assurance business has yet been made, either in North or South Germany, or in Austria. The governments of North Germany and Austria have had a general assurance law under consideration for some time; and though no result has been arrived at, the prospect of free action, in the transaction of assurance business, has improved in the course of the year. By the fourth article of the constitution of the North German Confederation, it is enacted that questions as to trade, inclusive of insurance, shall be subject to the control and legislation of the Bund. But unhappily this has not yet been carried into effect. When a bill for the regulation of trade was submitted at the last meeting of the federal assembly, provisions respecting assurance business (with the exception of agency) were not included; but a stipulation was inserted that assurances should remain subject to the local legislatures. It was evident beforehand that such a diminution of the efficiency of the Bund, could not receive the assent of the assembly, and the opposition manifested was so decided, that the bill was withdrawn and an amended bill may be confidently anticipated.

In the meantime it is reported that Prussia will shortly proceed to enact a law for the regulation of assurance; and that a bill, having that end in view, will be submitted to the next Prussian parliament. This would be to act in opposition, if not to the words, at least to the spirit, of that stipulation in the constitution of the Confederation; the object of which is, that assurance business should be withdrawn from diverse regulations in separate states, and placed under a uniform general law for the whole Confederation. As this cannot be misapprehended, either by the Prussian government or house of representatives, it is to be expected that the bill for a general insurance law, will be submitted, not to the local, but to the general Parliament, as part of the trade regulations for North Germany. Should Prussia proceed with a special assurance law of its own, there is great danger that something similar will take place in the other States of North Germany, especially in Saxony, where such strange principles prevail as to fire insurance business; and the difficulty of the introduction of a general law will be still more increased. It is to be hoped that this misleading course of decentralization will not succeed, as the reverse is urgently required for a prosperous and economical developement of assurance business in North Germany.

The annexed table shows the amount of life assurance business (assurance of a capital sum at death) transacted in Germany last year, and its increase on the previous year. Small burial assurances (those under £15) are excluded, when they could be separated. But as they form a considerable proportion of the business of some Companies, their amount is given in a foot note. The Companies named are those situated within the limits of the former Confederation, and in German Switzerland. They are 35 in number, of which—

20	are situated in	North Germany,
4	„	South Germany,
9	„	Austria, and
2	„	German Switzerland.

Of these Companies many publish full and well-arranged accounts, from which the particulars given in the various columns of the table, may be accurately obtained. In the accounts of some Companies however, the various kinds of assurance undertaken, are not separated, but life, endowment, annuity, and tontine business are thrown together confusedly, although in unimportant matters the reports are very detailed, as if with the object of preventing an exact insight into their position being obtained. Lastly, a third part, to which the Companies in Trieste belong, publish perfectly useless accounts; which do not permit even an approximately correct view of their position and management. The accounts of several Companies do not appear till twelve months after the close of their financial year, it has therefore been necessary to make general estimates for these Companies, which are based on their earlier accounts and general progress. It is believed that the figures given are pretty near the truth.

The following Table shows that 78,552 persons effected new assurances for £10,009,360 in the course of 1867, inclusive of further insurances on lives already assured. This is a greater amount than that of any previous year, as is shown by the following Table:—

*Development of Life Assurance in the German Companies,  
from 1852 to 1867.*

Year.	No. of Companies.	New Business in the course of the Year.		Number and Amount in force at the end of the Year.	
		Lives.	Amounts.	Lives.	Amounts.
			£		£
1852	12	5,236	883,936	46,980	8,655,337
1853	13	5,558	986,846	50,019	9,187,751
1854	14	5,224	883,532	52,816	9,608,429
1855	18	9,366	1,429,796	61,832	10,932,126
1856	18	12,778	1,714,935	71,169	12,061,861
1857	19	13,601	2,027,181	81,348	13,537,740
1858	20	14,645	2,457,314	90,128	15,102,165
1859	20	13,122	2,173,667	101,758	16,570,785
1860	24	24,730	3,738,750	129,589	20,631,342
1861	25	35,246	4,280,386	152,121	23,200,012
1862	26	42,209	5,343,198	183,812	26,491,142
1863	27	47,368	6,534,585	194,818	30,496,015
1864	27	55,357	7,597,445	230,394	35,240,962
1865	30	68,607	8,869,310	280,476	41,642,165
1866	32	55,981	7,611,455	305,433	45,083,948
1867	35	78,552	10,009,361	351,851	50,448,333

Although the practice of life assurance has not increased so rapidly in Germany as in North America, where this branch of assurance, formerly almost wholly neglected, has suddenly received an extravagant increase—in one Company, the Mutual of New York, the new assurances amounted in 1867 to £12,055,457 [in nominal amount, but in reality much less on account of the paper money being considerably below par]—yet it will be seen from the foregoing Table, that the practice of life assurance in Germany increases with a steady growth. Not only do the existing assurances increase, but also the amount of new assurances in each year, the increase of the latter being only interrupted in the years of war 1854, 1859 and 1866.

The number of policies discontinued in 1867 was very large, larger than in the war year 1866. They amounted to £3,909,250 assured on 81,601 lives, whilst in 1866 the amount of the sums assured under discontinued policies was £3,487,160 and the number of lives 25,251. Almost the half of the new assurances were lost again by discontinuance. The next Table shows in what various degrees the Companies suffered this loss. It gives the total amount in force at any time in 1867, (being the amount in force at the beginning of the year, plus the new business,) and the discontinuances from any cause. The Gotha Office had the smallest

proportion of discontinuances, these amounting only to '825 per cent; the highest proportion 12'662 per cent is attained in the Anchor in Vienna. The average amount discontinued in the Companies included in the Table is 5'844 per cent.

*Discontinuances in the Year 1867.*

Company.	Total Assurances during 1867.	DISCONTINUED.	
		Amount.	Per cent. on Sum Assured.
	£	£	
Gotha .....	8,710,065	71,865	'825
Lubeck .....	2,967,786	118,815	4'003
Leipzig .....	2,036,880	61,290	3'009
Hanover .....	398,205	9,270	2'328
Berlin .....	2,171,213	52,882	2'436
Brunswick .....	139,617	2,325	1'665
Janus (Hamburg) .....	1,906,859	114,132	5'985
Teutonia (Leipzig) .....	966,379	101,633	10'517
Concordia (Cologne) .....	3,651,124	148,724	4'073
Schwerin .....	206,625	5,325	2'577
Iduna (Halle) .....	1,055,615	87,792	8'317
Magdeburg .....	1,593,186	109,046	6'845
Erfurt .....	1,598,929	121,406	7'593
Germania (Stettin) .....	7,185,546	823,031	11'454
Providentia .....	977,699	121,196	12'396
Railway Assurance Co... ..	643,103	75,650	11'763
Munich .....	453,193	12,866	2'839
Stuttgart .....	1,801,238	34,097	1'893
Darmstadt .....	137,928	3,214	2'330
Janus (Vienna) .....	654,551	56,119	8'574
Anchor ( " ) .....	2,414,425	305,707	12'662
Zurich .....	1,353,826	44,564	3'292
Basle .....	1,090,285	96,949	8'892
Total .....	44,114,277	2,577,898	5'844 } Average {

From the above Table No. (2) it is found that the average amount assured on each life at the end of 1867 was £143, an increase of about £2 over the previous year.

The income received by the German Companies from the 351,851 persons assured, inclusive of interest on their former payments, was £2,024,450; the average receipt from each person was therefore £5'75, which is rather less than that of 1866, when the average receipt was £5'90. The income from premiums and interest was—

In the year 1867 . . . £2,024,450

„ „ 1866 . . . 1,803,250

Increase . . . £221,200 or 12'27 per cent.

The payments on claims by death amounted  
 in 1866 to . . . £904,695  
 and in 1867 to . . . 786,776

Shewing a decrease of £117,919 or 13·03 per cent.

And as the number of deaths in 1867 was 5,921, the average amount of each claim was £133, being about £10·5 less than in the previous year.

The year 1867 is the first in which the amount of death claims has been smaller instead of larger than that of the previous year; and this anomaly is explained by the considerable increase in the mortality of 1866, by war and cholera. Whilst in the year 1866 almost all the Companies had paid more for claims by death, than the tables on which their premiums are based led them to expect, the reverse has been the case in 1867, as the next table shews—

*Payments for Claims in the Year 1867.*

Company.	Expected Amount.	Amount Paid.	DIFFERENCE BETWEEN THE ACTUAL AND EXPECTED LOSSES.			
			More.		Less.	
			Amount.	Per Cent.	Amount.	Per Cent.
	£	£	£		£	
Gotha .....	191,920	169,665	..	..	22,255	11·596
Lubeck .....	46,010	46,216	206	·447	..	..
Leipzig .....	38,968	32,220	..	..	6,748	17·316
Hanover .....	10,757	10,680	..	..	77	·715
Janus (Hamburg) .....	26,572	30,646	4,074	15·329	..	..
Teutonia (Leipzig) .....	10,262	6,766	..	..	3,496	34·068
Concordia (Cologne) ..	43,934	40,249	..	..	3,685	8·388
Iduna (Halle) .....	11,834	13,654	1,820	15·383	..	..
Magdeburg .....	17,773	14,201	..	..	3,572	20·099
Erfurt .....	16,566	19,955	3,389	20·459	..	..
Germania (Stettin) ....	70,952	71,320	368	·518	..	..
Providentia (Frankfort)	10,289	9,457	..	..	832	8·080
Railway Assurance Co.	5,962	6,743	781	13·099	..	..
Prussian Life Assurance	2,023	750	..	..	1,273	62·930
Friedrich Wilhelm ....	2,742	1,222	..	..	1,520	55·418
Nordstern .....	1,557	1,335	..	..	222	14·258
Munich .....	8,622	9,240	618	7·173	..	..
Stuttgart .....	24,590	19,086	..	..	5,504	22·386
Darmstadt .....	2,110	1,483	..	..	627	29·712
Janus (Vienna) .....	17,973	13,414	..	..	4,559	25·366
Anchor (Vienna) .....	33,596	32,713	..	..	883	2·629
Zurich .....	18,856	17,422	..	..	1,434	7·606
Basle .....	8,112	6,214	..	..	1,898	23·398
	621,980	574,651	11,256		58,585	

According to this table £58,585—£11,256=£47,329 less than the expected amount, was paid in the year 1867 by these Companies; but in 1866 on the contrary, the actual payments exceeded the expectation by £115,625. This excess has therefore been partly compensated by the more favourable mortality of 1867, though not to the extent of one half.

Turning now to the general table given at the end of this article, we find a column showing the cost of management, and the proportions which it bears to the amount assured, and to the yearly income. Both of these proportions show great variations; the ratio of the cost to the income ranging between 4·8 per cent (Gotha) and 40·12 per cent (Basle); or, when compared with the amount assured at the end of the year, between 2·22 per mille and 14·01 per mille; which rates are shown by the same Companies.

It follows from the nature of the case, that in young Companies, of as yet small extent, the expenses should assume a higher ratio than in the older ones; but there are among the latter not a few, in which the expenses are so large, that they do not appear to be covered by the loading of the premiums. 'On account of imperfect information, the cost of management of many companies could not be stated at all, or not with the accuracy to be desired. As regards the Companies in Schwerin, Stuttgart, Darmstadt and the Anchor, it is also left out; because in addition to life assurance they carry on other business of considerable extent, to which the expenses, given in the published accounts, extend.

This time also, as in 1866, the Directors' fees and Managers' salaries have been added to the expenses of management, and also the sinking fund for the year, for the expenses of formation and earlier management.

The assets given in the next column consist of the actual amount of funds attained by the assurance business of each Company—including reserve, surplus if it exists, and provision for liabilities incurred but not yet discharged.

These assets amounted at the end of

1867 to £6,779,218

1866 to 6,154,073

Increase in the year . . . 625,145 or 10·16 per cent.

To state in a few figures the progress of life assurance in Germany in the last five years, it appears from this and the similar

statements issued in former years, that the increase per cent from year to year, (with the exception of claims, which have decreased in the last year) has been as follows:—

	1862.	1864.	1865.	1866.	1867.
In the Number Assured .	15·83	18·26	18·65	8·59	13·20
„ Sum Assured .	14·85	15·56	15·32	7·68	11·77
„ Yearly Income .	15·29	12·87	14·88	11·58	12·27
„ Claims .	24·78	6·26	16·08	42·84	13·03
„ Assets .	12·21	9·32	11·83	7·09	10·16

*[Remarks relating to the general table.]*

The figures marked \* rest upon approximate estimates. All the others are taken from official accounts. In the figures marked † are included the amounts arising from Annuity, Endowment, and Tontine business; as they cannot be separated from the life assurance business proper: they are however of no great importance.

Besides these life assurances proper, the following Companies have granted burial assurances:—

Lubeck . . . . .	£5,717
Teutonia . . . . .	181,701
Schwerin . . . . .	407
Iduna . . . . .	400,523
Magdeburg . . . . .	88,585
Thuringia . . . . .	74,586
Germania . . . . .	186,806
Prussian . . . . .	1,883
Friedrich Wilhelm . . . . .	13,162
Nordstern . . . . .	13,405
	<hr/>
	£966,775

## Business and Position of the German Life

	COMPANY.	Estab- lished.	Assurances existing at the beginning of the Year.		New Assurances during the Year.		Assurances existing at the end of the Year.		Income from Premiums, Interest, &c.
			Persons.	Amount. £	Persons.	Amount. £	Persons.	Amount. £	
1	Gotha.....	1827	29,563	7,952,160	2,377	757,905	31,029	8,464,770	391,214
2	Lubeck.....	1828	20,412	2,568,800	3,272	398,986	22,386	2,801,345	102,197
3	Leipzig.....	1830	9,734	1,641,960	2,033	394,920	11,195	1,941,870	83,454
4	Hanover.....	1830	3,475	365,250	326	32,955	3,620	378,255	15,682
5	Berlin.....	1836	10,110	1,961,318	1,116	209,895	10,702	2,072,640	100,358
6	Brunswick.....	1842	1,468	134,652	53	4,965	1,464	134,357	5,625
7	Frankfort-on-the-Main.....	1844	4,920	795,138	700*	120,000*	5,324	875,921	37,787
8	"Janus" (Hamburg).....	1847	13,810	1,696,859	1,500*	210,000*	14,535	1,790,406	68,177
9	"Teutonia" (Leipzig).....	1852	7,189	566,304	5,619	400,075	11,304	857,402	33,750*
10	"Concordia" (Cologne).....	1853	11,800*	3,169,077	2,142	482,046	12,800*	3,462,850	126,000*
11	Schwerin.....	1853	757	162,750	341	43,875	1,034	199,695	6,251
12	"Iduna" (Halle).....	1854	{ 8,369 Policies.	794,134	{ 3,271 Pols.	261,482	{ 10,356 Pols.	953,404	54,646*
13	Magdeburg.....	1856	9,711	1,278,870	{ 2,780 Pols.	314,316	11,409	1,473,545	51,177
14	"Thuringia" (Erfurt).....	1856	8,942	1,378,801	{ 1,445 Pols.	220,128	9,425	1,456,293	44,122
15	"Germania" (Stettin).....	1857	58,335	5,772,421	{ 19,300 Pols.	1,416,041	67,792	6,291,195	226,916*
16	"Providentia" (Frankfort) ..	1857	4,609	777,610	1,337	200,089	5,091	846,468	28,219
17	"Railway" Assurance (Berlin)	1861	{ 3,744 Policies.	464,583	{ 1,146 Pols.	178,519	{ 4,292 Pols.	560,260	18,399
18	"Prussian" (Berlin).....	1865	{ 764 Policies.	145,087	{ 1,450* Pols.	189,000*	{ 1,783 Pols.	280,896	10,680*
19	"Friedrich Wilhelm" (Berlin)	1866	?	?	2,500*	412,500*	2,291	376,732	12,000*
20	"Nordstern" (Berlin).....	1867	..	..	{ 2,627 Pols.	344,645	2,562	316,336	22,366*
Total I.....			207,712	31,625,774	55,335	6,592,342	240,394	35,534,640	1,439,020
21	Munich.....	1836	3,333	431,560	105	16,633	3,262	430,959	17,126
22	Stuttgart.....	1854	7,150	1,458,665	1,673	342,574	8,530	1,747,798	76,404
23	Darmstadt.....	1855	1,855	130,243	94	7,684	1,884	133,230	5,523
24	"General Annuity" (Stuttg.)	1861	{ 901 Policies.	114,482	{ 554 Pols.	56,313	{ 1,398 Pols.	160,640	5,700*
Total II.....			13,239	2,139,950	2,426	423,204	15,074	2,472,627	104,753
25	"Janus" (Vienna).....	1839	11,293	571,804	1,112	82,748	11,128	585,228	24,629
26	"Anchor" (Vienna).....	1858	10,579	2,105,633	{ 2,205 Pols.	308,792	10,590	2,075,206	79,500*
27	Generali, Azienda and Riunione in Trieste as well as the first Aus- trian Assurance Company (now the Donau) the Austrian Gresham, Phoenix and Patria in Vienna, together about.....		60,000*	7,050,000*	14,000*	1,800,000*	64,000*	7,500,000*	300,000*
Total III.....			81,872	9,727,437	17,317	2,191,540	85,718	10,160,434	404,129
28	Zurich.....	1857	5,998	1,151,552	{ 1,074 Pols.	202,274	6,665	1,291,580	42,000
29	Basle.....	1865	2,000*	490,285	2,400*	600,000*	4,000*	989,052	34,548
Total IV.....			7,998	1,641,837	3,474	802,274	10,665	2,280,632	76,548
Total I. II. III. and IV.....			310,821	45,134,998	78,552	10,009,360	351,851	50,448,333	2,024,450

## Assurance Companies in the Year 1867.

Claims paid.		Expenses of Management.			Assurance Fund.				Average Bonus for the last 10 Years. Per Cent. on Premiums.	Share Capital.	
Per-sons.	Amount.	Actual.	Per Cent on Annual Income.	Per Mille on Amount Assured.	Total.		Reserve.	Surplus.		Nominal.	Paid-up.
	£	£			Amount.	Per Cent. on Sum Assured at the end of the Year.	£	£		£	£
648	169,665	18,791	4.80	2.22	2,197,089	25.96	1,743,320	408,928	33.9	Mutual	
337	46,216	12,000*	11.74	4.28	366,226	13.07	349,748	16,479	0.397 } On sum ass.	76,500	7,650
188	32,220	9,016	10.80	4.64	381,915	19.67	320,005	51,888	27.6	Mutual	
105	10,680	1,758	11.21	4.65	64,998	17.18	56,492	4,691	None	Mutual	
262	45,525	9,706	9.67	4.68	568,235	27.42	476,980	77,508	16	150,000	30,000
38	2,935	368	6.54	2.74	32,922	24.50	?	?	?	Mutual	
98	12,060	3,837	10.15	4.38	160,901	18.37	155,270	5,630	10.6	257,143	25,714
241	30,646	9,172	13.45	5.12	200,969	11.22	195,301	5,667	10½	75,000	7,500
116	6,766	9,039	26.78	..	50,184†	..	Dr. 72,338†	None	None	1,500,000	300,000
181	40,249	?	?	?	586,534†	16.94	426,886†	151,349†	None	15,000	15,000
9	1,605	?	?	?	30,724	15.39	15,857	5,596	51½		
154	13,654	11,673	21.36	..	121,151†	..	113,122†	None	14	Mutual	
156	14,201	10,629	20.77	7.21	111,642	7.58	106,046	2,580	(1 year) } None	296,100	59,220
130	19,955	8,250*	18.70	5.67	81,682	5.61	78,365	None	None	336,450	67,290
802	71,320	43,796	19.30	6.96	382,122†	6.07	368,683†	13,439†	None	450,000	90,000
79	9,457	6,000*	21.26	7.09	55,802	6.59	55,802	None	None	685,714	68,803
63	6,743	3,750*	20.38	6.69	36,517	6.52	25,186	6,385	None	112,500	22,500
5	750	3,100	29.03	11.04	8,590†	3.06	6,774†	1,706†	None	150,000	30,000
9	1,222	?	?	?	8,059	2.14	13,479	None	None	150,000	37,950
6	1,335	6,917	30.93	..	11,378†	..	11,909†	None	None	187,500	37,500
3,627	537,204	..	..	..	5,457,640	..	..	..	..	..	..
73	9,240	?	?	?	88,120	20.45	73,562	13,169	None	Capital of the Loan Bank.	
92	19,086	?	?	?	251,927	14.41	177,746	64,882	39.7 } (9 years)	Mutual	
36	1,483	?	?	?	23,355	17.53	18,649	4,707	None	Capital of the Annuity Society.	
?	?	?	?	?	16,612	10.34	13,759	2,854	16 } (6 years)	Mutual	
201	29,809	..	..	..	380,014	..	..	..	..	..	..
280	13,414	?	?	?	103,643	17.71	84,437	19,206	14½	Mutual	
193	32,713	?	?	?	164,294	7.92	151,157	13,137	None	99,999	30,000
200*	150,000*	?	?	?	570,000*	..	?	?	..	..	..
1,973	196,127	..	..	..	837,937	..	..	..	..	..	..
96	17,422	3,418	8.14	2.65	72,000*	5.57	62,196	9,000*	None	Capital of the Swiss Credit Co.	
24	6,214	13,860	40.12	14.01	31,627	3.20	30,841	None	None	351,600	35,160
120	23,636	..	..	..	103,627	..	..	..	..	..	..
5,921	786,776	..	..	..	6,779,218	..	..	..	..	..	..

## LONDON AND PROVINCIAL LAW ASSURANCE SOCIETY.

## BONUS REPORT, 1865.

The period for the Third Division of Profits having arrived, the Directors have caused a careful investigation to be made into the Society's affairs, and they have the pleasure to report to the Proprietors and the Assured its results.

The Society has now completed twenty years of its existence. The Directors take the opportunity to make some general remarks on its operations in that period; and, adverting first to the new business, they desire to call attention to the progress of the Society in the intervals preceding the three periodical valuations, directing attention to the fact that, in accordance with the provisions of the Deed of Settlement, ten years elapsed before the first Division of Profits was made.

The New Premiums received in the three periods have been as follows:—

Term.	New Premiums received.			Average per Annum.		
	£	s.	d.	£	s.	d.
1846—1855....	29,191	7	10	2,919	2	10
1856—1860....	26,138	14	11	5,227	15	0
1861—1865....	41,674	9	9	8,334	18	0

The Renewal Premiums received during the last five years amounted to £208,046 16s. 2d. The Claims paid during the same period amounted to £95,307, less £8,686 13s. received under Re-Assurances with other Companies.

Since the foundation of the Society 2,572 Policies have been issued, assuring £2,805,073 10s. 10d., and £1,675 per annum Contingent Annuities, at annual Premiums amounting to £82,404 9s. 7d.; and of these 905, assuring £1,129,796 14s., and £200 per annum Contingent Annuities, were issued during the quinquennial period under review.

The Policies which have become void are 922, for £881,725 10s. 2d., and £700 per annum Contingent Annuities, classed as follows:—

Number of Policies.	Classes.	Sum Assured.			Contingent Annuities.
		£	s.	d.	
175	Claims by Death.....	179,473	5	2	£130 per annum
200	Surrendered Policies.....	252,849	17	0	—
428	Lapsed by Non-payment of Premium.....	348,923	9	0	£570 per annum
119	Void by Expiration of Term.....	100,478	19	0	—
Total 922	Total.....	881,725	10	2	£700 per annum

The Assurances remaining in force on 31st December last, were therefore 1,650 Policies, assuring £1,923,348 0s. 8d., and £975 per annum Contingent Annuities, the annual Premiums on which are £57,462 1s. 2d. The existing Bonuses declared at former divisions of Profits, amounting to £60,713 10s., must be added to the above sums assured, making a total of £1,984,061 10s. 8d.

The following is a classified Statement of the existing Assurances:—

CLASS.	Number of Policies.	Sum Assured.			Bonus.			Annual Premiums.		
		£	s.	d.	£	s.	d.	£	s.	d.
Life—With Profits .....	1,157	1,160,123	15	8	58,217	10	0	37,770	2	11
Life—Without Profits .....	292	363,513	0	0	..	..	..	12,712	1	6
Assurances for Terms of Years .....	26	29,185	0	0	..	..	..	620	9	5
On Death of last Survivor .....	20	51,280	0	0	740	0	0	1,006	9	0
Joint Lives .....	16	9,250	0	0	..	..	..	507	10	10
Descending Scale of Premiums .....	3	4,000	0	0	..	..	..	166	4	1
Ascending ditto .....	19	35,450	0	0	286	0	0	982	14	2
Endowments for Children .....	2	600	0	0	..	..	..	3	10	0
Assurances against Issue .....	51	177,516	5	0	..	..	..	..	..	..
Limited number of Premiums .....	9	14,000	0	0	1,042	0	0	947	13	6
Endowment Assurances .....	8	5,230	0	0	..	..	..	354	8	11
Contingent Assurances .....	35	62,560	0	0	..	..	..	1,250	7	5
Commuted Premiums .....	4	10,700	0	0	428	0	0	272	19	10
Contingent Annuities (£975 per ann.) .....	8	..	..	..	..	..	..	268	8	0
Foreign Residence .....	..	..	..	..	..	..	..	599	1	7
	1,650	1,923,348	0	8	60,713	10	0	57,462	1	2

From the foregoing statements, it will be seen that no less than two-thirds of the Assurances granted by the Society are still in force, and also that the Claims by Death have not reached 7 per cent. on the total amount assured.

The Assets and Liabilities of the Society have been carefully valued, and the principles adopted on former occasions have been followed.

The Directors have thought it prudent to write off £500 from the value of the Society's House, reducing this item in the Balance Sheet to £5,300. The realised Assets of the Society on the 31st of December, 1865, amounted to £390,582 11s. 5d., and at that date were invested to pay £4 13s. 7d. per cent. (excluding the Society's House.) The Renewal Premiums due and the current interest have been added, making, as will be seen by reference to the annexed Balance Sheet, an aggregate total of £396,458 5s. 8d.

The Assurance liabilities have been valued according to the tables founded on the experience of the Equitable Society, with interest at the rate of £3 per cent., and additional precautions have been taken to meet the contingency of a higher rate of mortality than has hitherto been experienced. After deducting Re-Assurances with other offices for £190,900, the net value of the Liabilities under Policies amounts to £226,425 12s.; and the value of the Annuities payable is £7,767 14s. The Balance Sheet includes also a reserve of £12,401 9s. for claims admitted, and after debiting every other liability the available Balance or amount of Profit now divisible is £73,358 11s. 11d. On the last quinquennial division the amount of the divisible Profit was £42,785 5s. 10d.

In accordance with the provisions of the Deed of Settlement, one-fifth of this profit, equal on this occasion to £14,671 14s. 5d., will be added to the Proprietors' Fund, by which the amount will be increased from £75,560 4s. 9d. to £90,231 19s. 2d. This addition is equivalent to 15s. 10d. per Share, and the amount paid up in respect of each Share will be £4 17s. 8d.

The Annual Dividend, which is paid from the interest of the Proprietors' Fund, will be 4s. 6d. per Share, as compared with 3s. 8d. during

the past five years, and is at the rate of  $11\frac{1}{4}$  per cent. on the original paid-up Capital.

The remaining four-fifths of the divisible surplus, being £58,686 17s. 6d., will be appropriated to those Policies of the Assured which are entitled to participate in the present Bonus. This sum has been, according to the constitution of the Society, converted into equivalent Reversionary Bonuses, which will be added to the sums assured and payable therewith, or, at the option of the Policy-holders, may be commuted either for an equivalent present payment in cash, or for a reduction of the future Annual Premiums. The Policies entitled to participate on this occasion are for £970,749, and the Bonuses amount to about £97,000; equivalent, on the average, to 60 per cent. on the Premiums paid, or to rather more than £2 per cent. per annum on the sums assured.

The corresponding amounts at the last Division of Profits were Reversionary Bonuses for £53,547 on Participating Policies assuring £590,768.

We also learn from the Chairman's address that the addition to the shares in the year 1855—£2 having been the original payment upon them—was, in the shape of interest, £1 6s. 2d., no dividends having been paid in the first ten years, and 6s. 4d. from the profits of the assurance fund. In 1860, at the end of the next five years there was an addition of 9s. 4d.; and in 1865 of 15s. 10d.; so that each share now represents £4 17s. 8d., of which £2 only was originally paid.

**Specimens of Bonuses on Society's Policies to 31st September, 1865.**

Date of Policy.	Age when Assured.	Sum Assured.	Former Bonus Additions.	Reversionary Bonus in respect of last 5 years.	Total Bonus Additions to 31st Dec., 1865.	Amount of Premiums paid in last 5 years.	Per Centage of present Bonus on Premiums paid.
		£	£	£	£	£	
1846	24	1,000	236	94	330	111	85
"	45	1,000	287	118	405	191	62
1851	26	1,000	162	94	256	116	81
"	57	1,000	248	158	406	300	53
1856	30	1,000	89	96	185	126	76
"	54	1,000	117	131	248	264	49
1861	25	1,000	..	88	88	110	80
"	65	1,000	..	177	177	446	40

N.B.—The above Reversionary Bonuses slightly exceed, on the average, 2 per cent. per annum on the Sum Assured.

**BALANCE-SHEET,**

**31st December, 1865.**

**LIABILITIES.**

	£	s.	d.
To Proprietors' Fund . . . . .	75,560	4	9
" Dividends unclaimed . . . . .	327	11	8
" Claims admitted (less Re-Assurances) . . . . .	12,401	9	0
" Annuities due . . . . .	117	2	4
" Value of Liabilities under Policies (less Re-Assurances) . . . . .	226,425	12	0
" Value of Life Annuities granted . . . . .	7,767	14	0
" Sundry outstanding Accounts, say . . . . .	500	0	0
" Balance, being the amount of divisible surplus . . . . .	73,358	11	11
	<b>£396,458</b>	<b>5</b>	<b>8</b>

## ASSETS.

	£	s.	d.
By Government Stock (£34,190. 16s. 3d.)	31,456	14	10
„ Mortgages . . . . .	258,405	4	6
„ Railway Debentures . . . . .	50,500	0	0
„ Life Interests . . . . .	18,166	0	8
„ Reversions purchased . . . . .	13,608	9	7
„ Society's House . . . . .	5,300	0	0
„ Balance at Union Bank—			
Deposit Account . . . . .	8,000	0	0
Drawing Account . . . . .	5,012	3	4
„ Do. at Bank of England . . . . .	100	0	0
„ Do. in hand . . . . .	33	18	6
„ Premiums due 31st December, 1865, and in course of payment	3,871	0	10
„ Interest due 31st December, 1865, and in course of payment	2,004	13	5
	<u>£396,458</u>	<u>5</u>	<u>8</u>

## ROCK LIFE ASSURANCE COMPANY.\*

*Established 1806.*

## REPORT OF THE DIRECTORS.

The value of the Assets of the Assurance Fund is . . . . .	£2,068,339	17	3
The total value of outstanding Liabilities . . . . .	<u>1,530,264</u>	<u>14</u>	<u>1</u>
Leaving a Surplus Profit for the last Seven Years of . . . . .	<u>£538,075</u>	<u>3</u>	<u>2</u>

Out of which they recommend the appropriation of £532,031 15s. 9d. as a Bonus; one-third, viz., £177,343 18s. 7d. to be added to the Subscription Capital Stock, and two-thirds, £354,687 17s. 2d. to be distributed amongst the Policies, which will yield 11s. 8d. per Cent. per Annum to each Policy (in addition to any previous Bonus thereon) for the number of years it has existed, up to the year 1861 inclusive, commencing from the 31st of December of the year following the date of the Policy, leaving a reserve of £6,043 7s. 5d.

The Directors also have to announce their intention (if the addition of £177,343 18s. 7d. be made to the Subscription Capital Stock), in pursuance of the power given to them by a Resolution passed by the Court of Proprietors on the 26th of August, 1840, of declaring a Bonus of Two Shillings and Sixpence per Share per Annum, to be paid with the Half-yearly Dividend in October, free of Income Tax, to commence in October next; and in conformity with a desire expressed by the Proprietors, the Dividends will in future be paid on the same days as those on the Government Funds are payable to the public in April and October.

	POLICIES.	SUMS ASSURED.	PREMIUMS.
The New Policies issued for the Seven Years ending the 19th ultimo were . . . . .	1410	£1,510,180 15 0	£57,839 18 5
For the preceding Seven Years ending 19th August, 1854 . . . . .	1429	1,666,597 15 0	66,905 15 0
And for the Seven Years ending 19th August, 1847 . . . . .	638	636,025 0 0	24,301 7 6

\* This report has been standing in type for a considerable time; but its insertion in this Journal has been delayed by the press of other matter. — Ed. J. I. A.

## THE CLAIMS PAID IN THE SEVEN YEARS.

## SUMS ASSURED.

1840 to 1846 inclusive were	£714,015 0 0	{ with the Bonus additions paid and redeemed thereon and on current Policies.		£380,280 12 11
1847 to 1853 inclusive were	808,955 0 0	ditto	ditto	389,439 18 6
1854 to 1860 inclusive were	923,459 18 0	ditto	ditto	451,892 11 1

These Bonus additions include £164,415 6s. 1d. redeemed on current Policies.

The above statement shows that the claims do not reach their maximum till an Office has been established at least fifty years, and also that the Bonus paid and redeemed on Claims was 43 per cent., whilst if the amount of Bonus redeemed on current Policies were added, it would show that the entire amount of Bonus paid and redeemed was equal to 50 per cent.

The total claims paid from the commencement of the Office to the 19th ultimo amounted to £5,442,241 10s. 10d., including therein Bonus additions of above a Million and a half.

The number of Policies now in existence is 3,333, the Amount Assured being £3,888,908 12s., and the Yearly Premiums for same £131,873 9s. 5d., the average duration of which at the present period is fifteen years and three quarters.

The Directors have much pleasure in announcing to the Proprietors the fact that no loss has ever been sustained by any investment since the commencement of the Company.

*State of the Accounts of the Assurance Fund of the Rock Life Assurance Company on the 20th August, 1861.*

*Dr.*

	£	s.	d.
To the present Value of £8,532 outstanding of the Appropriations made the 20th August, 1819, in respect of 125 Policies, the amount assured thereby being £132,840 . . . . .	7,147	11	0
To the present Value of £19,578 10s. outstanding of the Appropriations made the 20th August, 1826, in respect of 327 Policies, the amount assured thereby being £430,810 . . . . .	15,958	0	3
To the present Value of £73,322 19s. 9d. outstanding of the Appropriations made the 20th August, 1833, in respect of 604 Policies, the amount assured thereby being £818,310 . . . . .	58,274	11	11
To the present Value of £86,988 11s. 3d. outstanding of the Appropriations made the 20th August, 1840, in respect of 976 Policies, the amount assured thereby being £1,235,235 . . . . .	67,613	10	6
To the present Value of 140,105 11s. 2d. outstanding of the Appropriations made the 20th August, 1847, in respect of 1,332 Policies, the amount assured thereby being £1,658,935 . . . . .	106,174	11	7
To the present Value of £231,043 10s. 2d. outstanding of the Appropriations made the 20th August, 1854, in respect of 2,030 Policies, the amount assured thereby being £2,495,789 15s. . . . .	171,193	4	11
Carried forward . . . . .			

	£	s.	d.
Brought forward			
To the present Value of all the Assurances now existing; viz., 3,333 Policies for £3,888,908 12s. Assured . . . . . Value £1,057,330 14 7 7 Annuities; viz., 4 Immediate of £180, and 3 Contingent of £762 . . . . . " 3,543 13 2			
To the Amount of Outstanding Claims; viz., 30 Policies for £31,749 Assured, and £11,279 16s. 2d. Bonus . . . . .	1,060,874 43,028	7 16	9 2
To Balance forming the Amount of Surplus Profits, out of which a Bonus may be declared, after deducting £5,000 at least .	1,530,264 538,075	14 3	1 2
	<b>£2,068,339</b>	<b>17</b>	<b>3</b>

Cr.

	£	s.	d.
By Value of £3,725 2 0 per Annum Government Annuity, 1880 . . . . .	52,585	0	0
By " 5,801 19 6 " Government Annuities on Lives .	40,138	0	0
By " 200,000 0 0 Canada 4 per Cent. Debentures . £205,240 0 0 Less Temporary Loan thereon . . . . . 40,000 0 0	165,240	0	0
By " 30,000 0 0 Canadian Consolidated 5 per Cent. Loan . . . . .	30,000	0	0
By " 65,000 0 0 British Guiana 4 per Cent. Loan .	68,565	0	0
By " 8,824 8 0 per Annum Terminable Annuities	175,071	19	4
By " Railway and other Debentures and Guaranteed Railway Stock . . . . .	699,511	6	11
By Advances on Mortgage and Interest thereon . . . . .	642,459	9	0
By " on Policies and Interest thereon . . . . .	59,037	4	5
By 11,093 Shares of this Company, valued at . . . . .	94,290	0	0
By Lease of the Company's House, valued at . . . . .	8,883	0	0
By Amount of Premiums due prior to 20th August, not yet received . . . . .	2,937	2	11
By Amount due on Policies effected to Assure Bonus of 1861 .	2,168	0	5
By Balances due from Agents . . . . .	2,245	11	5
By Cash at Bankers . . . . .	25,103	13	5
By Petty Cash and Stamps on hand . . . . .	104	9	5
	<b>£2,068,339</b>	<b>17</b>	<b>3</b>

Amount of Bonus recommended by the Directors.

£532,031 : 15 : 9

One-third to be added to the Subscription Capital Stock.	Two-thirds to be distributed amongst the Policies: Amount	At what rate to each Policy.	Reserve.
£177,343 : 18 : 7	£354,687 : 17 : 2	11s. 8d. per Cent per Annum	£6,043 : 7 : 5

for the number of years the Policy has  
existed, up to the year 1861 inclusive, com-  
mencing from the 31st of December fol-  
lowing the year of the date of the Policy,  
being an addition to the Bonus previously  
appropriated.

## THE SCOTTISH PROVIDENT INSTITUTION.

*Instituted 1837.*

## THIRD SEPTENNIAL INVESTIGATION WITH DIVISION OF PROFITS.

Up to the close of the year 1866, there had been issued in all 16,339 Policies, assuring £7,525,372 : 13s. The Policies in force at the close of the year were 12,346 in number, while the Capital Sums remaining assured amounted to £5,582,322 : 11s., with £149,209 : 11 : 6 of yearly Premiums. The Premiums of all kinds received in the year (after deducting those paid for reassurances), amounted to £152,320 : 2 : 8. The total Receipts of the year, including interest, were £205,357 : 10 : 11.

The Realised Fund, arising entirely from the accumulated Premiums of the members, was, at 31st December, £1,245,372, 16s. 8d. The increase in the Fund in the course of the year was thus £111,901 : 13 : 8.

THE PROGRESS of the INSTITUTION in each of the four Septennial Periods is shown in the following TABLE:—

	No. of New Policies issued.	Assuring.	Accumulated Fund at end of Period.
In <i>First</i> Period—			
To 31st Decem. 1845—(8 years)	2136	£942,899	£69,009
In <i>Second</i> Period—			
To 31st December 1852 . . .	3762	1,628,429	254,675
In <i>Third</i> Period—			
To 31st December 1859 . . .	4357	2,018,972	633,514
In <i>Fourth</i> Period—			
To 31st December 1866 . . .	6034	2,935,073	1,245,372

In tracing out the sources of this satisfactory progress of the Fund, the most important inquiries are those which relate to the Mortality among the Members, and the rate of Interest which has been secured on the Investments.

The average rate of Interest has been somewhat less than  $4\frac{1}{2}$  per cent, having been under that rate in the earlier, and above it in the later, years of the septennial period. The present rate is considerably higher, and, looking to the numerous openings which are now available for the employment of money, the past rate is likely to be at least maintained. The Directors have long been of opinion that it would be a desirable arrangement, both for borrower and lender, that a fair rate of Interest should be fixed for a term of years, in place of the system now followed in Scotland, by which the rate is liable to fluctuate with each half-yearly term; and they are prepared to entertain proposals on this footing.

In regard to Mortality, the Claims which have emerged since the commencement of the Society in 1837 amount, including Bonus Additions, to £806,952 : 5s. These have resulted from the deaths of 1437 members. A minute analysis has been made by Mr. Meikle, the Actuary of the Institution, from which it appears that this number is just 75 per cent of what might have been expected according to the estimate in the office tables, and 85 per cent of what would have occurred according to the Carlisle Table of Mortality. In the last seven years the number of deaths has been 720—the ratio for that period being 73 per cent of the office expectation, or rather less than it has been over the whole period.

THE Directors have now to report the arrangements in connection with the third Septennial Investigation.

Every item in the Balance-Sheet has been carefully considered, and such of the investments as were not of a fixed character have been valued.

#### ABSTRACT STATEMENT OF THE REALISED FUND.

Loans on Heritable Securities and Mortgages.	£801,826	6	3
Loans on security of Trust-Funds . . . . .	26,470	0	0
Loans on assignment of Rates, of Preference Stocks, etc. . . . .	76,053	6	4
Loans on personal securities with Policies of Assurance . . . . .	29,771	10	9
Loans to Members on their Policies—within the Surrender Value . . . . .	97,569	0	10
Value of Advances on Reversions . . . . .	16,678	2	3
First-class Debentures, and Guaranteed and Preference Stock . . . . .	53,787	10	0
Value of business premises in Edinburgh, London, Glasgow, and Dublin, and property held in connection therewith . . . . .	56,381	3	10
Office-furniture, stamps, and cash in hand . . . . .	1,873	8	8
Current Premiums (mostly due in the month) and Interest on loans, etc., to 31st Dec. . . . .	34,824	0	7
Balances in Bank—			
National Bank of Scotland—on deposit and current account	£28,211	13	6
Union Bank of London	7,500	0	0
		35,711	13 6
Sum . . . . .	£1,280,946	3	0

It will be observed that the great proportion of the investments consists of loans on heritable mortgages (all of them within the United Kingdom), or other securities not liable to the disturbing fluctuations which attend investments in the public funds or other stocks. In stating the value to be set on their property in Edinburgh and at the chief Branches, the Committee have made a considerable abatement from the cost price, at which it stood in the books; although, besides providing ample accommodation for the business, it produces a rental which yielded a return of above 3 per cent on that original price; and they have excluded from the fund a sum which had accrued as profit on a reversionary transaction—a question having arisen in regard to the succession on which the security depends. It may be noted, however, that there cannot in any event be a loss on the transaction—the Office having been already fully re-imbursed for its advances.

BEFORE proceeding to give the details of the Investigation, a few words may be said in regard to the Valuations.

The principle which was adopted in fixing the rates of the Office Premiums, and which has been followed in the subsequent computations of this Society, was—that the estimates both of Mortality and Interest should as nearly as possible agree with what was really to be expected in the

future—leaning in both cases to the side of safety. When the Institution was commenced, the Northampton Table of Mortality, with Interest at 4 per cent, was that mostly in use among Actuaries for the calculations of Assurance Offices. That Table, it has long been known, shows too high a rate of mortality, and in its place is now very generally substituted a Table, founded on the Carlisle Observations of Mortality, which is understood to measure pretty accurately the expectation of a select class of lives, with Interest at the rate of 3 or  $3\frac{1}{2}$  per cent. The adoption of the former rate must be due to the necessity of providing for the future continuance of the Bonus Additions which Members paying a high rate of Premium have a right to expect. The Surplus to yield these, formerly derived from the excessive estimate of Mortality, is now to be drawn from an opposite excess in the estimate of Interest. The arrangement may be a proper one in some circumstances, but it is not suitable to the case of an office which, like this, is opposed to the charge of excessive rates for the purpose of creating Surplus for after-division. And, in accordance with this view, when valuing for their non-participating class of Assurances, offices are found using the same Carlisle Table in combination with a higher rate of Interest. The Table used in valuing the Policies of this Society is one deduced from the Office Premiums. These, as was fully explained in the Report of the first Investigation, were based chiefly upon the Government Table of observations for males, with a general addition of ten per cent for expenses and casualties, but with a further graduated addition, increasing from middle age upwards, to correct what appeared to be defects in its estimate of the mortality at the more advanced ages. The Table thus constructed has been found to agree very closely with the Table since framed by the Registrar-General from his observations of the mortality of the general population. It shows a considerably higher rate of Mortality than the Carlisle Table, and it is therefore so far a safer guide—producing a lower estimate of Surplus. The Directors have, on this as on former occasions, tested the results brought out according to this Table, by calculations based upon a Carlisle  $3\frac{1}{2}$  per cent Table, with the effect of confirming their confidence in the Office Valuations.

“STATE OF THE AFFAIRS OF THE INSTITUTION,  
AS AT 31ST DECEMBER 1866.

ASSETS.

1. Realised Fund—Amount as in separate State . . . . .	£1,230,946	3	0
2. Value of future net Premiums receivable . . . . .	1,526,947	1	9
Sum . . . . .	£2,757,893	4	9

LIABILITIES.

1. Claims under Policies emerged, but not yet paid, and other Sums outstanding . . . . .	£49,174	17	7
2. Present Value of Annuities payable . . . . .	43,023	18	4
3. Present Value of Sums Assured under the Society's Policies, including Bonus Additions . . . . .	2,484,149	16	5
Sum . . . . .	£2,576,348	12	4

## ABSTRACT.

The Assets being, as above . . . .	£2,757,893	4	9
And the Liabilities . . . .	2,576,348	12	4
<hr/>			
There remains a SURPLUS of	£181,544	12	5
<hr/>			

"By the existing Laws it is provided that the 'Surplus, under deduction of such proportion as the Directors shall consider necessary and proper in the circumstances to be retained as a guarantee, shall be made available to the members entitled thereto.' Up to the last General Meeting the Laws provided that the reserve to be so retained should be 'not less than one-fourth nor more than one-third.' Though a larger discretion is now entrusted to the Directors, the Committee think it right at this Septennial Period to adhere to the rule followed on previous occasions; and they therefore recommend that one-third of the above Surplus, or £60,514 : 17 : 5, be retained to accumulate for division at a future Investigation; and that the remaining two-thirds, or £121,029 : 15s., be now apportioned in terms of articles 29 and 30 of the Laws.

"The Committee find from the states, that the number of Policies now entitled to participate, either immediately or prospectively in the course of the next seven years, is 2492, among which accordingly the above sum of £121,029 : 15s. falls to be divided. In the distribution of this Surplus they have, in accordance with the Laws, allotted shares to members who participate for the first time, in proportion to the absolute value of their Policies; and to those who had previously participated, in proportion to the increase in the value of their Policies (including the previous additions) since the date of the last Investigation.

"The Surplus is divisible not by the usual system of an equal percentage on each £100 assured, without reference to the age of the members or the duration of their Policies, but in proportion to the values of the Policies respectively; hence the rate of addition necessarily varies in each case. As the circumstances which affect the values of the Policies—such as the member's age, the scale of premium, and the number of years the Policy has endured—are so various, it is not possible to quote individual cases which might be assumed as 'normal examples. The Committee will only state generally that Policies now participating for the first time receive additions varying from 16 or 18 to about 30 per cent, while the additions to Policies which have already participated, range from 10 or 12 to about 20 per cent.\* It will be observed that the number of Policies participating is much larger at this than at the previous Investigation, yet the rate of addition to each is not materially affected.

\* In a few exceptional cases, owing to the circumstances which determined the value, the additions were smaller.

It may be stated, as interesting to the Members, that Policies which share at both the present and last Investigations have received additions varying from 30 to 45 per cent on the sums assured; and those which have shared at all the three Investigations have received additions in all of 40, 60, and even so much as above 80 per cent.

## CORRESPONDENCE.

## "EVILLY DISPOSED."

*To the Editor of the Assurance Magazine.*

SIR,—In your recent review of my book on the Law of Fire Insurance your reviewer protests against the use of an expression which he seems to consider clearly indefensible. "Evilly disposed" is the offender; and as I gather from the letters italicised, the "ly" is the offence. As I cannot deny the charge, perhaps you will allow me in legal language "to confess and avoid" it by submitting that two authors of undoubted authority—Shakespeare and Jeremy Taylor—use the obnoxious termination. Thus we find in "King John," Act iii., s. 4—

"This act, so evilly born, shall cool the hearts  
Of all his people, and freeze up their zeal."

And in Taylor's "Holy dying," Book iv., s. 1—

"It will be an unhandsome injustice evilly to requite their care by thy too anxious and impatient spirit."

The review is so friendly in its tone, and so far overestimates the merits of my little Essay, that in this instance I would willingly kiss the rod, if I felt that it was deserved—but it appears to me, that a verbal criticism thus made, should not be allowed to pass unchallenged unless plainly right. The expression "evilly disposed" may be slightly archaic; but I contend that it is as structurally correct as its convertible term "evil disposed," if indeed it is not more so, as well as more euphonious; since it is to an unwillingness to use the word "evil" adverbially, that we may perhaps trace its almost invariable modern contraction to "ill" when in composition.

I remain, Sir,  
Yours,

19, *Serjeant's Inn, Fleet Street,*  
6th March, 1869.

C. J. BUNYON.

CORRECTION OF AN ERROR IN THE ENGLISH LIFE TABLES  
(No. 1 MALES).

*To the Editor of the Assurance Magazine.*

SIR,—As some Offices adopt the English Life (No. 1 Males), 4 per cent annuities, when converting annual into single payments, it may be advisable to point out that, at age 35, the value of an annuity should be 15·7006, and not, as it is printed in the Sixth Registrar General's Report, 15·6645.

I remain,  
Yours obediently,

Glasgow, 6th Jan., 1869.

T. M.

JOURNAL  
OF THE  
INSTITUTE OF ACTUARIES  
AND  
ASSURANCE MAGAZINE.

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*On the rates of extra premium for foreign travelling and residence.*  
By ARTHUR H. BAILEY, ESQ., *Actuary of the London Assurance Corporation.*

[Read before the Institute, 22nd February, 1869.]

FEW who are engaged in the practice of Life Assurance will deny that the prevailing restrictions imposed on, and charges made for foreign travelling and residence are unsatisfactory to all parties concerned. A considerable and increasing portion of the public whose pursuits or inclinations, especially in early life, take them to different parts of the world, are deterred from effecting life assurances on this account. The value of policies as securities is diminished owing to the uncertainty of the extra premiums which may at any time be imposed, and the risk of the assurances being forfeited altogether by causes over which mortgagees have no control. And actuaries feel that this subject is the opprobrium of their profession; for as in general no sufficient data exist whereby the risks incurred may be measured, the extra charges are therefore arbitrary and unscientific, for the most part little better than random guesses made on no intelligible principles. For these reasons it is hoped that any suggestions on the subject, however imperfect, may not be unacceptable, and may be the means of eliciting some useful discussion.

The subject may be conveniently divided into three heads. (1) What has been considered to constitute extra risk. (2) The general principles on which extra risks should be treated. (3) The rates of premium which should be charged. It is proposed in the present paper to limit the discussion to times of peace, and to persons following civil occupations.

Under the conditions of the early policies the persons whose lives were assured were restricted to Great Britain. The language of one in my own possession, which bears date the 25th November 1721, and is believed to be the oldest life policy extant, is as follows:—"Provided always, that this Policy shall be utterly void and of no effect in case the said ——— shall voluntarily go to sea." From the early records of some old Offices to which I have had access, the following specimens of the extra premiums charged about the year 1741, are extracted.

Flanders and back . . . . .	£1 6 per cent.
Ireland and back . . . . .	0 10 "
Scarborough by sea . . . . .	0 10 "
Gibraltar . . . . .	2 2 "
(And the same as late as 1821.)	
Holland, Germany, and Russia . . . . .	0 10 "

Even during the present century some charges have been made which now seem almost ludicrous—*e.g.* in 1821—for leave to proceed once to Paris by way of Dover and Calais and return once from thence by the same way—10s. per cent. In 1817, for sea risk and residence in Brussels—10s. per cent. From time to time the restrictions have been relaxed; residence allowed in any part of Europe; and voyages permitted, first, from one port of the United Kingdom to another, then to and from certain foreign ports, generally those between Hamburgh and Bordeaux, and afterwards to and from any European port. Even now the conditions of several Offices restrict the lives assured to Europe; others however allow of residence in Australia, British North America, the northern of the United States, and some other temperate climates, a charge being generally made for the voyage to and fro. Latterly a practice has grown up, convenient but not scientific, of bounding the prohibited regions by two lines drawn round the globe parallel to the equator, and corresponding generally with the 33rd degrees of north and south latitude respectively.

When however the prohibited regions are once touched, all system is at an end. I have examined several lists of extra premiums,—*rudis indigestaque moles*,—imposed without either

knowledge or skill, and of which it would be useless to attempt any analysis. Extra rates varying from 5s. to £10 per cent (£2 per cent appears to be the favourite,) are demanded in the most arbitrary way, a reasonable ground of complaint to the policyholder, who in this matter is almost entirely in the power of the Offices, not having the benefit of competition, which, if it could have full play, would bring about at all events a greater uniformity of practice.

Rather more than twenty years ago an improvement was introduced by the managers of the Scottish Offices, by classifying most of the climate risks into four main divisions. Class A, including Europe and other temperate regions, B, the parts of North America not included in class A: for these the extra rate proposed was £1 per cent. C, the East Indies and China, for which £2. 10s. extra was proposed; and D, the West Indies, for which no rate was recommended. About the same time a Company was formed for the express purpose of giving increased facilities for foreign travelling and residence. This Company adopted the same classification of risks as the Scottish managers, but instead of levying extra premiums by a uniform annual addition to the home rate, tables of premium varying according to age were published in the prospectus for each of the four classes. But the classification was incomplete; several places were excepted, and a general power was reserved of charging increased rates in particular circumstances. Since then little progress seems to have been made either in reducing the number of prohibited places or in better defining the risks. The laudable attempt made several years ago by the Council of the Institute to collect data whereby these risks could be measured, an account of which is given in the *Journal* for October 1857, was barren of result.

In endeavouring to arrive at some general principle of treating extra risks, I venture to lay down the maxim that all problems in Life Assurance must be considered in a broad spirit, remembering that they deal with numbers not individuals. Minute classification is impracticable. If any additional premium is imposed it should be considerable in amount, as small extra charges annoy the assured without corresponding benefit to the Office. It is already the practice to charge the same premium for men of almost every occupation and station in life, and to allow of residence in a great variety of climate. Experience has justified this practice, and it seems to me that it may be further extended with safety and advantage.

Extra risk is divisible into sea and climate risk. As regards

the former, is there really much greater danger to the lives of passengers in sea voyages than in other modes of travelling? or than in hunting, shooting, mountain climbing, or even in walking about the crowded streets and alleys of great cities. The last is not an imaginary risk. We learn from the Registrar General's report that during the year 1867, 164 persons, including 49 children under the age of 10, were killed by horses or carriages in the streets of this metropolis. And on the other hand from the valuable statistics compiled from the records of the Scottish Equitable Life Assurance Society, it appears that out of 126,672 lives exposed to risk, the deaths from drowning were 19 only, and of these 8 are reported as "found drowned," so that the mortality from sea risk was not more than 11 in 126,672, '009 per cent, and in these mariners by profession are of course included. From the equally valuable statistics published by the Scottish Amicable Society it appears that 10 deaths from drowning were recorded out of 56,300 lives at risk. How many of these deaths occurred at sea is not stated; but even if all be taken into account, the mortality from this cause would have been only '018 per cent. Other Assurance Societies which have published statistics of the causes of death, have not at the same time furnished the corresponding number of lives at risk, so that the materials they supply are of little avail for the purpose of this inquiry. But it may be confidently asserted that deaths from accidents at sea are of rare occurrence in the experience of Life Offices, so that I have come to the conclusion that all extra charge for sea risk may safely be dispensed with, except for seafaring men.

We have next to consider climate risks. Here we find one rate of premium almost invariably adopted for such an immense tract of country as the East Indies and China, the former alone extending over 26° of latitude. Is it then a very startling proposal to adopt one uniform rate for tropical climates generally? We are content to measure life risks in the whole of British North America and a great part of the United States by observations made in England. Why then should we object to deduce premiums for Central America and the West Indies from records of mortality in the East Indies, especially when the materials available for the purpose are ample and trustworthy in the latter case, few or none in the former?

The Offices seem to have put some parts of the world under a sort of ban, often without any good reason. Take for instance the West Indies. In 1824 I find an extra charge of £8. 8s. per cent made for Jamaica, and even in my own time 6 per cent was a

common rate. But sufficient experience has now been acquired to show how unwarranted are the notions which have been prevalent as to the mortality of Europeans in the West Indies. Thus, in the report of the Colonial Assurance Society in 1864, it is stated that the amount of assurances effected with that Society in the West Indies was £794,189; and in the East Indies £499,546; that the number of deaths was 108 in the former, and 97 in the latter; which seems to indicate a more favourable mortality for the West than the East Indies. From the late Mr. Spens's report on the mortality of the Scottish Amicable Society, it appears that the deaths in the West Indies among the members of that Society were 31, while the number of computed deaths was 32·5 by the Northampton, and 17·4 by the 17 Offices' Experience table. And again from Mr. Burnett's carefully compiled statistics of the Barbados Mutual Life Assurance Society it will be observed that he records 106 deaths during the 25 years ending in 1865, while the number expected by the Northampton table was as many as 149.

The settlements on the West coast of Africa are no doubt extremely unhealthy, but then the European settlers there are extremely few. And as regards particular places, such as New Orleans at one season of the year, may we not confidently trust to the instinct of self preservation implanted in mankind that such places will be carefully shunned whenever possible? And it must be remembered that these exceptional cases bear so small a proportion to the whole number, as scarcely to affect the general result.

As almost any authentic information on this subject is worth having, the result of 45 years' experience of the London Assurance may not be without interest. In that period assurances were granted on 12,200 lives, of whom 1186 (nearly 10 per cent,) had at one time or other incurred extra risk—climate, military, or maritime. The annual mortality per cent was, for the ordinary risks 1·99, for the others 2·91, and for the two combined 2·07. So that the effect of what is believed to be an unusually large proportion of extra risks was to raise the ordinary mortality ·08 per cent only.

Considering then the importance both of simplifying and systematising the charges for climate risk, I have come to the conclusion that for this purpose it will be sufficient to divide the world into two portions only; one, containing what may be considered healthy climates, to which the common premiums will apply; the other, unhealthy districts, for all of which a uniform scale of premiums may I think safely be charged.

To distinguish between the healthy and unhealthy districts is

no doubt a difficult task; the line wherever drawn must be arbitrary, and to some extent governed by considerations of expediency. In the Northern hemisphere, Madeira, Egypt and the Holy Land, all now places much resorted to by visitors, may I think be included in the healthy districts with the region north of the 33rd parallel of latitude. Otherwise this boundary, which has been adopted for several years, seems judicious; it excludes Shanghai in the east, and Charleston in the west. In the southern hemisphere, it seems to be now generally admitted that the whole of Australia, the Cape colony, and Natal should be included in the healthy districts; and I would suggest that the general boundary should be extended to the 31st parallel of latitude, so as to allow of residence in such parts of South America as Valparaiso, on the west coast, and Uruguay and the River Plate region, on the east.

The remaining consideration is what scale of premiums should be adopted for the unhealthy districts. The materials available as a guide for this purpose are by no means abundant. Reference may be made to the official reports which have been published on the sickness and mortality of the troops at different stations, the principal results of which have been embodied in a series of papers by the late Sir Alexander Tulloch which will be found in the early volumes of the *Journal of the Statistical Society*. Valuable and interesting as these papers undoubtedly are, the information they contain is unsuitable for life assurance purposes. For several reasons the common soldier in tropical climates is placed in very unfavourable circumstances in regard to health when compared with the upper and middle classes. To such an extent is this the case that the Royal Commission which reported in 1863 on the sanitary condition of the army in India found that the rate of mortality in that country was nearly twice as great among the private soldiers as among the officers, being 6·9 per cent for the former, and 3·8 per cent for the latter.

Fortunately however for our purposes several Funds were established under the auspices of the East India Company for the members of the different branches of their service. The financial affairs of these Funds having become extremely complicated, the assistance of several actuaries has at different times been required for their investigation, among whom Griffith Davies, Mr. Neison, and Mr. Brown have been conspicuous. In the elaborate reports of these gentlemen will be found most authentic information, derived from the recorded experience of the Funds, on the mortality

of the civil, military, and medical officers of the Indian services. Most of these reports I have examined with care and interest. My attention was first directed to the mortality among the civil servants as most suitable for the present purpose. But these being a select body, the numbers are small. Mr. Neison enumerates in the Madras Civil Service 335 deaths only in a period of 91 years; in the Bengal Civil Service 398 deaths only in 52 years; and Mr. Brown, in his more recent report dated in 1865 on the Bengal Civil Fund, records only 478 deaths, including 99 annuitants, in the period 1801-58. These seemed to me insufficient data on which to found a table of mortality for practical use.

I therefore at last decided to make use of Messrs. Brown, Hardy and Smith's report on the Madras Military Fund. The period embraced in that investigation comprises 50 years from 1808 to 1857 inclusive, and the number of deaths recorded is 2251. But with great deference to the knowledge and experience of the authors of the report, it seems to me that the materials at their disposal have not been judiciously used. The facts they record show conclusively that there has been a marked improvement in the mortality of Europeans in India, not merely since the first establishment there of British rule, but ever since the beginning of this century. Instead therefore of deducing a table of mortality from the entire observations, it seemed to me better to use the recent facts only, wherever the numbers were sufficient. The summaries in quinquennial periods of age given in the report, pp. 162, 163 have accordingly been taken, going back no further in point of time than was sufficient to obtain a minimum of 50 deaths for each quinquennial period of life. The facts thus obtained with the resulting rates of mortality deduced, contrasted with Mr. Brown's rates and those of the 17 Offices' Experience table, are subjoined.

*Madras Military Fund.*

Entered the Fund between the years	Age.	Exposed to risk.	Deaths.	Mortality per cent.	Do. by Mr. Brown's table.	Do. by 17 Offices' Experience.
1838-57	20-24	6573	173	2·631	3·26	·747
Do.	25-29	4797	119	2·480	3·16	·802
1828-57	30-34	4257·5	86	2·019	3·20	·875
1818-47	35-39	5173·5	98	1·894	2·94	·970
Do.	40-44	3600	72	2·000	2·80	1·095
1808-37	45-49	4368	117	2·679	2·63	1·356
Do.	50-54	2867	79	2·755	2·75	1·800
Do.	55-59	1635	50	3·058	3·06	2·475

As with most other observations on Anglo-Indian mortality, the rates thus obtained seem to set at defiance the prevailing ideas as to the relative mortality of different periods of life. Any elaborate graduation appeared therefore to be out of the question; and in order to obtain money results I have been content to adopt the method suggested in the report. The rate of mortality for each quinquennial period has been taken to represent the probability of dying in the year at the mean age of the period. The logarithms of these numbers being taken, the logarithms of the probabilities for the intervening ages were interpolated by first differences. To complete the table it was assumed that the annual mortality is constant from 18 to 22; and that above the age of 60, where the facts are very few, it corresponds with the English Life Table No. 2. On these data tables have been computed of the values of annuities and annual premiums at 3 per cent interest. Specimens of the latter are subjoined, and the corresponding premiums by the 17 Offices' Experience table placed in juxtaposition.

*Net Annual Premiums.*

Age.	Proposed whole world rates.	17 Offices' Experience.
25	2.594	1.665
35	2.890	2.210
45	3.748	3.133
55	5.026	4.767

These rates may in my judgment be safely adopted as the basis of a table of whole world premiums for civilians in time of peace. Of course it must be clearly understood that these are net premiums only; the amount of loading must be a matter of individual judgment. For myself I think that this loading should be heavy, not on account of the mortality risk, but on account of the expense of foreign business, which, according to my experience, has increased, is increasing, and is not likely to diminish.

No one can be more conscious than myself how much this scheme is open to criticism both in principle and detail. To give one instance only;—parallels of latitude and territorial divisions are unsuitable boundaries of climate; isothermal lines would certainly be an improved substitute. But how are boundaries defined by isothermal lines to be brought within the four corners of a legal contract? The whole question is a choice of difficulties; as has been before observed, all our arrangements, and not ours

only, must be a compromise between what is correct in theory and what is expedient in practice. If these suggestions tend to introduce some approximation between theory and practice, some attempt at order out of the present chaos, they will not have been altogether offered in vain.

*Net Annual Premiums—Whole World Risk.*

Age.	Premium for £100.	Age.	Premium for £100.
20	2·580	40	3·257
21	2·581	41	3·347
22	2·583	42	3·442
23	2·585	43	3·543
24	2·588	44	3·646
25	2·594	45	3·748
26	2·601	46	3·852
27	2·611	47	3·953
28	2·623	48	4·052
29	2·641	49	4·160
30	2·666	50	4·275
31	2·698	51	4·401
32	2·737	52	4·540
33	2·783	53	4·690
34	2·834	54	4·852
35	2·890	55	5·026
36	2·953	56	5·214
37	3·021	57	5·418
38	3·095	58	5·639
39	3·174	59	5·877

The following discussion is abridged from the *Insurance Record*.

Mr. PORTER stated that he had been engaged for some time in collecting information for his own Office with regard to extra premiums, and that his Directors had paid the Institute the compliment of deferring consideration of the question until after Mr. Bailey's paper had been read. He had procured a list of the extra premiums charged by most of the principal Offices, and was surprised to find the extraordinary variation between the rates of different Offices of equal standing and respectability. In one first class Office, presided over by one of the ablest Actuaries, he was told that the experience with regard to extra risks had been highly satisfactory—so much so, that they thought charging extra premiums to be wholly unnecessary, although in deference to the views of other Offices and because they found they could get them, they would charge say 10s. per cent. In another Office, the experience has been so extremely bad, that they would rather not take extra risks at all, and therefore they charged as much as they could get. He found that, for the same place (the Mauritius), the rates varied from 10s. to £4. per cent with all intermediate rates. What would the public think of Assurances Offices when they found this to be the case? They would conclude that they did not know what they were about. He did not consider this to be a satisfactory state of things for either the Offices or the Institute in the year 1869.

He remembered that twenty years ago he was much perplexed at the different values put by Actuaries upon reversionary property—so much so that he was almost afraid to name a definite sum when consulted. Just about that time the Institute was established, and in consequence of the papers read and the discussions which took place upon them, and of the greater freedom of intercourse which was brought about between Actuaries, this scandal has been got rid of. A greater uniformity of practice now prevails, and there is scarcely a single point of difficulty which can occur in the ordinary professional practice of an Actuary which is not elucidated in the pages of the *Journal*. He should like to see the question of extra premiums set at rest in the same way. The Institute should first agree upon a scale of rates. They should next consult with the Actuaries' Club—and he gathered there would be every disposition on the part of that body to meet the Institute. Lastly, they should induce the Offices to adopt the common tariff. Mr. Bailey in his paper divided the world into two parts—healthy and unhealthy. His own and several other Offices had adopted as the limits of the healthy districts the lines of 33 degrees north and south of the equator. He regretted that Mr. Bailey had selected the 33rd parallel north and the 31st south, as the uniformity and simplicity of the scheme was thereby marred, and the public would confound the two lines and never be sure whether they might go within 31 degrees north or south. No doubt, the intention was to include Buenos Ayres, Valparaiso and the River Plate. For measuring the risk of the unhealthy districts, Mr. Bailey proposed to take as a standard the mortality experience of the Madras Military Fund. There was, of course, great difficulty in finding anything to answer the purpose; but he did not think that the Directors of Assurance Companies would like to fix their premiums according to the mortality of any particular place. The mortality even of troops differed widely in different places. In 1866 the annual mortality per 1000 of the mean strength of the white troops was 9.62 for the United Kingdom. Taking this as the standard, the rate in Gibraltar and Malta was 8.89; in America, 9.58; in Bermuda, 24.01; the West Indies, 26.94; the Cape and St. Helena, 10.46; the Mauritius 14.01; Ceylon, 21.44; Australia, 12.53; China and Japan, 32.46; and India, 21.7. Therefore while the rate for the United Kingdom was under 10 per thousand, that in China and Japan was 32, or three times as much. The average for all stations was 14.3 per thousand. There were reasons for these different rates of mortality, in the case of the army. The position of the barracks was sometimes very bad, and the removal of the forests by permitting the spread of miasma sometimes rendered places very unhealthy which had previously been perfectly salubrious. For it was well established that forests resisted the spread of miasma, but how this was brought about was unknown. Again, enthetic disease was a principal cause of mortality to the troops—while in the course of his experience amongst Assurance Offices he had not known a single case of a death recorded from that class of disease. The pioneer on the question of extra risks was the late Mr. Bidder, who drew up a report on the subject in the year 1840. This was circulated amongst Actuaries, some of whom used it as the basis of their own extra premiums. Mr. Bidder explained that, except for the East and West Indies, he possessed no data, and that his results must be taken as rough approximations only. He however deemed that the “moral risk,” or liability to be imposed upon

by fraud, was an important element. He (Mr. Porter) considered that this latter risk could not be measured, and that therefore there was a reason for avoiding any affectation of precision. For such a risk as Timbuctoo one Actuary might quote 2 per cent; while Mr. Bailey would require £2. 7s. 4d. per cent if the life were 45—but if the birth-day be only just passed an additional 3s. 2d. per cent beyond. This was not in keeping with Mr. Bailey's statement in the outset that he wished to treat the subject on a broad basis.

It would have been better, in his opinion, had Mr. Bailey deferred the publication of his scheme until it had been more fully discussed. It was so desirable that all should agree upon a uniform scale; and this was quite possible now, since each independent worker could give the benefit of his lucubrations to the general body. He was surprised to see that the mortality of the West Indies was so much less than that of the East Indies, but for some improvement of this nature he had been prepared, for he remembered that 8 guineas used to be charged for the former risk and only 3 to 3½ for the latter. The West Indian rate was now greatly reduced, and having felt some doubt as to the safety of the reduction he was now glad to find it justified by the statistics. He thought that the proportion of lives exposed to foreign risk in the "London Assurance"—10 per cent—was excessive: in his own office, it was probably 2 per cent. With a proportion at once so large and returning only an additional mortality of .8 per cent, he considered that we should soon entirely get rid of extra premiums. Mr. Bailey's paper was thoroughly well done, and the Institute owed him one more debt. To the Assurance Offices it was no less valuable, as it might be the means of inducing a more rational and less empirical method of dealing with an important practical question.

In answer to an enquiry, Mr. Bailey said that his Office would not grant an insurance on the published terms to a man who proposed his life and stated he was going to the West Coast of Africa.

Mr. EMMENS had been led by the paper into the belief that, in Mr. Bailey's partition into healthy and unhealthy districts, there would be no exceptions whatever, even in the case of new assurers. He now gathered that special contracts would be made in some cases, just as is now done with regard to the question of health. Mr. Bailey's objection to the existing rates was they were empirical; but, he had himself shown in his paper that no data existed upon which a scale might be founded. In the selection which he had made for the basis of his whole world rates, he was at issue with Mr. Brown. Finding therefore that two such eminent authorities disagreed, he (Mr. Emmens) scarcely thought that either the data, or the subsequent results would command the confidence of the profession. The practice in his own Office was to take as a standard Indian premiums deduced from a combination of original observations made by Neison and Naylor. For other places, the risk would be compared with the Indian risk as far as practicable and the premium fixed accordingly. These rates were about the average of those usually charged, and gave general satisfaction. With such limited data, the question of acclimatization did not weigh with them. Upon this point, he would remark that, the superior vitality observed amongst the older lives was due to exceptional circumstances, and did not militate against the normal law of mortality as propounded by Gompertz and Makeham.

Mr. BADEN thought that there were other influences to take account of besides those of climate. In one Office that he was acquainted with, there was a considerable agency at Truro, and the risks consisted chiefly of miners. Many of these emigrated to Chili to follow their avocation there, and an additional premium of 2 per cent was charged without any special consideration. The mortality among these men turned out to be exceedingly high, and the Office in reviewing its extra rate for Chili determined on doubling it. This would supply an instance of the necessity of regarding other things besides the mere climate. With a universal scale, such as Mr. Bailey proposed, no consideration would be taken of the circumstances under which a life went abroad, nor of the peculiar influences to which he was likely to be subjected. He thought that, as the expense of maintaining foreign agencies was considerable, there was no reason why an extra rate should be confined to the unhealthy districts only. Why should not an extra premium be charged for Valparaiso without assigning any particular reason for it—we being satisfied that it is necessary to meet the extra expenditure there.

Mr. HARBEN thought that the subject was one of practice rather than of theory; and could not therefore agree with Mr. Porter that it was brought forward prematurely. It was a means of getting it well ventilated. He preferred Mr. Bailey's method to that of Mr. Emmens. It had the charm of simplicity and would free the policies from some of the conditions usually attaching to them, and assignees would know exactly what rate would be added for any particular place. At present, the extra rates charged were almost prohibitory. Analogous to the reduction of the fire duty, a considerable augmentation of business might be expected from a removal of restrictions. Mr. Bailey's principle might be safely adopted, after the period of acclimatization had been got over. He would charge a small extra premium for the first five years, and then allow the premium to revert to the ordinary rate. The difficulties of an Indian climate would be overcome in five years, and the life would then be as good as a European one—except in a few localities. Extra premiums should cease at the expiration of a specified time, and then people would know what they had to pay. He should adopt a rough principle that within certain limits there should be certain fixed rates, and beyond those limits certain other rates.

Mr. SPRAGUE said that two views had been taken of the subject. One by Mr. Porter, who urged them to wait till they had agreed upon a common course. The other by Mr. Bailey, who, having judged for himself what were the proper rates for tropical parts, had adopted them. This latter course was to be preferred as the more practical and business-like, but Mr. Bailey had, he thought, gone too far in some respects. The present scheme had already been noticed in the *Journal*, under the heading of "Conditions of Assurance." It was new to him to learn that many Offices allowed their assured to reside in any part of Australia, free of extra premium. He thought that even when residence was permitted up to 33 degrees from the Equator, Australia was sometimes excepted, and that Queensland was usually charged for at the rate of 10s. per cent. In the southern hemisphere, the limit might, he thought, with propriety be extended to 31 degrees, which in point of temperature would nearly correspond with 33 degrees in the north. The emigration of Cornish miners to Chili, referred to by Mr. Baden, was to be accounted for by the failing

state of the mining industry in Cornwall. The expense of working had been increased by the great depth to which it was now necessary to go, the mines having been so long worked; and it was found that mines in other parts of the world could be worked cheaper. Numbers of miners were consequently thrown out of employ and might naturally emigrate to Chili, which now sent a very large quantity of copper to England.

He was wholly opposed to the suggestions which had been thrown out as to the total abolition of extra premiums. That would have the effect of making those who stay at home pay for all those who go abroad and are subject to extra risk. Such would be a fair condition if at the outset all had the same expectation of going abroad: but in practice it would not be fair, because some of the assured were more likely to go abroad than the others. The inducement to go to a place like India might be the larger salary which can be obtained there: there was no reasonable objection to charging a man himself with the extra premium instead of leaving those who stay at home on lesser salaries to pay it for him. A good deal, however, might be done in the direction which Mr. Bailey had taken. A gentleman of position and wealth travelling for mere pleasure might have *carte blanche* to go where he pleased, without the Insurance Offices losing thereby. But this would be a different thing from relieving from extra premiums all who were compelled by business to visit places that inclination and health would have made them shun. He could not adopt Mr. Harben's suggestion with regard to acclimatization, that after a certain period lives are not exposed to heavier mortality than those who stay at home. Looking at the table before us, we find that lives from 20 to 24 have a mortality in India of 2·6 per cent against 77 per cent in England; from 40 to 44, the mortality was 2 per cent in India and 1 per cent in England, and from 55 to 59, 3 per cent against  $2\frac{1}{2}$  per cent, and so on throughout all the Table. So that although many of the persons have been the greater part of their lives in India, they exhibit a greater mortality than is seen amongst persons of corresponding ages in England.

Mr. BUNYON said that he had paid a great deal of attention to the subject of acclimatization, and was satisfied that there was a considerable fallacy in the idea that a person becomes as well able to withstand an adverse climate as if he had originally been a native. The effect of acclimatization was of this sort. There are certain diseases which attack all new comers to a climate. But after these have been got over, the climate begins to take an insidious effect upon the constitution, till it can no longer be borne. It is then necessary to come home for complete restoration of the health. But on return, the first acclimatization was of no avail, for the same process has to be gone through; and the older a man was, the less able was he to withstand the climate. In the case of Indian lives, it might be noticed that each stay in the country became shorter. Were it not for the experience which old Indian residents acquire in measuring their power of resistance, the mortality at the higher ages would be just as great as that at the younger. This shows that the effect of foreign unhealthy climate continues during the whole of life. The question of rates seemed to him to be one of contract and convenience. The late Mr. Downes told him that, after careful enquiry, 2s. 6d. per cent would have met the extra risks of the "Economic." No doubt this was correct, and as only comparatively a few go abroad, a very small addition to the ordinary premiums would allow

people to go all over the world. But how could you get parties to pay such higher rates as you might propose? The Fire Offices had pretty well agreed to the rates for various risks: but life premiums were charged upon different scales and different principles, and the question would be how to deal with those persons who have broken the conditions of the contract upon which they had entered. Mr. Bailey had hit upon the right principle, that there should be some identification of the various interest and some modification of the general contract. There should, however, he thought, be more than one division. If you allow persons to travel in unhealthy districts at a uniform rate, you charge many such persons more highly than you ought, in order to let others off more easily: that is, should a risk which would be met by 1 per cent be rated at 2 per cent, in order to let another man go to the West Coast of Africa for the same premium? If we are to modify our present ideas, it must be by an accommodation of various interests—and a closer enquiry will enable us to do so without taking such a very broad division as Mr. Bailey has done.

Mr. ADLER thought that one set of rates for extra risks would not work satisfactorily. An officer insured in his own Office was charged £10 per cent for the Gold Coast—and he was the only officer of his regiment who returned, and he acknowledged he had not been charged too much. He objected to Mr. Bailey's deviations from the simplicity and uniformity of the general line laid down; and thought that if exceptions were made in one direction they should also be made in the other. Mr. Brown's results did not present the great fluctuations from the ages 20 to 39, which Mr. Bailey attached so much importance to in his own. Such roughly deduced figures should not be made the ground of a sweeping assertion against the Gompertzian theory as propounded by Gray and Makeham. Woolhouse's rate of mortality unadjusted for age 20 was 2.66; for 25, 2.78; for 30, 2.91; for 35, 3.15; for 40, 3.44; for 45, 3.81; for 50, 4.26. Neison gave also, without adjustment, for age 20, 2.32; 25, 2.50; 30, 2.78; 35, 2.86; 40, 2.97; 45, 3.79; 50, 2.98. There are here fluctuations at different ages, but not particularly between the ages 20 and 39. We make no restriction upon the occupations of lives after they are once assured: so that the rest of the assured pay for the risk incurred by those who subsequently embrace more dangerous pursuits. He believed that eventually we should have to do away with extra premiums, and that considering the increased facilities of communication such was merely a question of time.

Mr. NEWBATT stated that, in the experience of his Office those lives which returned to India appeared to be subject, even in a more marked degree, to the effects of the climate than those who had gone out for the first time. This fact was adverted to and brought out very prominently in the Report of the Commissioners of military mortality in India. The lines of division proposed by Mr. Bailey had been adopted, in the main, by his Office since the year 1851, though not quite to the same extent. The free limits included the Holy Land, Madeira and Egypt, and generally all places south of 31 degrees south latitude. Free residence in Australia was also conceded. The charge for voyages to America was remitted, but that to Australia retained till about seven years ago, when it was abolished. The great object to be gained in any alteration of the existing system is that people should find themselves as much as possible

unfettered. Giving leave to reside in North America and Australia, and then imposing a penalty for unauthorized voyaging between the two was likely to mislead people to their cost. As a matter of practice, his Office would allow the Panama route to be taken.

The rate of charges for the zone reserved by Mr. Bailey would not be easily settled in this generation. *Prima facie*, there would be no equity in charging the same premium for all parts of the world embraced in it,—for it contains different and varying risks. The risk on the West Coast of Africa could not be compared with that of Ceylon. But these varying and unequal risks exist in our every day business. We assure the life of a butcher upon the same terms as we do that of a clergyman—and therefore Mr. Bailey has considerable justification for putting forward a uniform scale. His own leaning was to an abolition of extra premiums upon the Scotch plan, that is after the policy had subsisted a certain number of years without having been the subject of an additional premium. To this, no doubt, we should ultimately come. This would still involve payment by the stayers at home for those who went abroad; but, as Mr. Bailey pointed out, the home residents incur risks of their own, not of the same kind but in degree more perilous than those of foreign residence. There was no question of equity, and there was no establishing a perfect equity. That which in itself would work the least harm—that which was the simplest—that which, upon the whole, would give the greatest satisfaction to the great body of life insurants, was the plan which we should endeavour to carry out. Such a plan would be one which said to a policy holder, who had been upon the books a certain number of years, that he was free from any trammel in respect of residence, provided his pursuits were not of a kind necessarily to lead him into circumstances and places that might be considered highly dangerous.

Mr. AUGS. HENDRIKS wished to express an opinion as to what he considered a want of appreciation of the extra risk incurred in travelling by sea. It was pretty generally admitted that it was advisable that the additional premium for such risks should be abolished, and most Offices had now ceased to charge for them. But this had been done on the ground of expediency, and from the idea that if we could disregard the risk of railway and other accidents, we could also afford to allow sea-travelling. Recently he had examined the experience of his own Office in the matter. They had about £1,000,000 assured in Australia, at an annual premium of about £30,000. During a year about £50,000 of these risks passed the sea either way, and the average of the losses (excluding all cases of falling overboard) in the past ten years had been £300 per annum, or nearly 1 per cent upon the premiums. Reducing this sum by the reserve upon the policies, the loss was £250 about—so that an extra rate of 10s. per cent on the sum assured, which was assumed at hap-hazard, turned out to be very correct. For foreign residence, it was desirable to reduce the rates so as to bring assurance within the means of the largest portion of the community: but he objected to its reduction below cost price. The risks could not be divided into two classes only—the difference between them was too wide. Probably some five or six classes would have to be established, with a separate tariff for each, to be revised as circumstances might occasion. Many of the good American Offices restricted their operations to the north of a point just below Washington. California used to be

excluded, but recently it has been proved to be a very healthy district and admitted. It will be found impossible to fix foreign rates upon a durable basis: we have not got the same numbers to deal with, and therefore have to treat these isolated cases more as underwriters than as actuaries. The Panama route, he thought, was one of considerable risk, and should be charged at the rate of one guinea per cent.

Mr. HODGE thought that the general body of the assured were under as great obligations to Mr. Bailey for bringing the matter forward as the Offices were. It was of pressing importance that, in these days, the question of extra premiums should be settled upon some more satisfactory basis than at present. He did not complain of the existing system as compared with that formerly adopted. He remembered a case which occurred 30 years ago, of a gentleman assured for £30,000 who went to pay a visit to his father, the then Governor of the West Indies. The premiums paid amounted to £1,800, and the gentleman stayed only two or three weeks—but the only return allowed by the Office was about £300. He remembered when the West Indian rate was ten guineas, and 5 per cent was considered to be ruinously low. He could not agree with the deduction made by Mr. Hendriks from his facts. An extra premium of 10s. per cent for the voyage, with £50,000 constantly at risk afloat, would have been charged upon a larger amount—because the voyage lasts only half-a-year. He thought the proposition with respect to the southern limit supported by the facts with which we are acquainted. The regulations of many Offices allow the assured to go to any place, not within 33 degrees of the Equator. What if any one should propose to go to the North Pole? It was impossible to follow strictly the isothermal lines; but he did not see why the limitations as to latitude should not be varied with those of longitude, so as to exclude peculiarly unhealthy places. Notwithstanding what had been said by Mr. Bunyon, there was a good deal in a person's being accustomed to a climate, because he gets a knowledge of what is beneficial, which a new comer does not possess. He approved of the line of 31 degrees south, but thought that justice would not be done to all parties by adhering to the principle of a single rate. It was sound to enlarge the limits as much as possible, and to fix them in the way which Mr. Bailey had done. He thought that there was a good deal of information yet to be obtained, and that considerable light would be thrown upon the question if such be properly classified.

The PRESIDENT was pleased that so interesting a paper on so important a subject had elicited such an animated discussion. The subject was one of great moment, and he trusted that we were soon likely to arrive at a period when there would be no extra premiums whatever. There were difficulties in one Office alone commencing so utopian a scheme; but if others also agree, the undue excess arising from an accumulation of risks would vanish. The losses arising from the risks of foreign residence would be easily met by the most trifling addition to the ordinary premiums paid by residents in this country. On looking into the experience of a large Office, he found that the extra premiums amounted to only sixpence per cent on the total sums assured. In that case would it be thought necessary to put on an extra charge, if all were to participate in the risk? He had adopted the limit of 33 degrees on each side of the Equator, with some large increase of licence in certain parts as Buenos Ayres, Natal and Australia. The tropical risks, in his own Office, amounted to £1,400. a year, and many of these

were introduced through Indian connections: he did not think that the time had arrived for accepting these at the ordinary rates. He agreed with Mr. Bailey that it was advisable to reduce the whole of the foreign risks as much as possible to a single rate. The rate proposed was, he thought, rather too high, for he was under the impression that there was a great improvement going on in the rate of mortality during residence in tropical climates. This was observable in his Reports upon the Indian Funds. In the case of the Madras Military Fund, where the experience extended over all kinds of military service during the last 50 years, the rate taking both climate and war was scarcely more than 3 per cent through the entire period of service, and was nearly constant at all ages under 65. It started at a little over 3 per cent and diminished slightly to about the middle period of life; and if the individuals be traced on their return home, it will be found that they merged into the ordinary mortality of the country. Examining the data, and subdividing the classes into married and unmarried, a great difference will be found in the mortality of the latter. This was easily explainable. In the early part of the period over which the observations extended, the habits of society were different, and the unmarried were most exposed to those influences which are injurious to health after their first arrival. The mortality of the married men, under age 65, in the whole 50 years' experience, was little more than from 60 to 73 per cent of that of the unmarried. He believed that a great change was going on in civilian life also. The real rate of mortality was from  $1\frac{1}{2}$  to 2 per cent above that of this country in the case of the military; and scarcely over 1 per cent for civilians. These rates include the average of all the climates in India, some of the localities of which are quite as healthy as any in this country. No permanent extra premiums could be charged for the Himalayas, nor for some other places. With regard to the effects of acclimatization, he thought that those who survived were better lives from the beginning and not that because they became accustomed to the climate. Such as survived and returned to England were quite on a par with other lives—the climate does not appear to deteriorate them to any extent. This point has been controverted; but the experience of nearly all the Funds showed that the retired officers received their annuities as long as residents here. These facts confirmed the conclusions he had arrived at, that the nearer we could come to the pleasing result of getting rid of all our extra premiums the better it will be for all parties. In these days of rapid and easy travelling it was impossible to say where we may be tempted to go, and it was a great hardship to put a heavy charge upon persons, which would prevent their taking pleasure excursions. The climate risks are now generally less than they were, and business representatives were frequently sent abroad on the slightest occasions. Heavy charges for extra premiums were likely to affect commercial enterprise, or check the practice of assurance amongst a class in which it was very desirable to extend it.

Mr. BAILEY, in reply, expressed his gratification that his paper had elicited so interesting a discussion. Mr. Porter objected to taking the experience of military risks as the basis of the premiums for the unhealthy districts. But there was a distinction between the mortality of privates and officers. In those districts which he had termed unhealthy he did not think that the difference in mortality was very marked—probably not 10 per cent. But even if it were more, it would hardly affect his argument, because the mortality of the troops is not to be taken as a

standard. He did not use the mortality in the Indian Civil Service, as the members were too few and the results appeared to him to be too favourable. No exception ought to be taken to the military statistics, on the ground of insufficiency: they are the most recent, and the best that can be got. The resulting rates are moderate, more moderate than those generally adopted. He had endeavoured to contend that rates derived from observations upon the mortality of officers in the Madras Presidency would suffice for the whole world. The essence of his plan was there should be no exception—once introduce exceptions and the whole scheme is destroyed. By adopting two rates only, one for healthy and one for unhealthy districts, a man knows exactly what will happen to him wherever he goes. As a matter of fact, how many out of the total number of the assured went to the West Coast of Africa? A man was allowed at present to go to the North Pole, as pointed out by Mr. Hodge. But why? Because he did not go there. The Cornish miners referred to by Mr. Baden were probably assured for £50. upon the same terms as charged to a country gentleman for £5,000: such small policies ought, no doubt, to pay a much higher rate, since the expense to the office would be the same as for the larger cases. But these inequalities do exist, and to a certain extent are unavoidable. He had had his scheme under consideration for some time past, and had consulted skilled underwriters and others conversant with sea risks, all of whom would be content to allow passengers to go by sea without any extra charge whatever. He did not understand Mr. Hendriks's statistics, unless it was meant to be asserted that one out of every 200 persons going to Australia is drowned. From the evidence he had obtained, the risk of drowning was very small and not worth considering. He had not referred to Mr. Bidder's report, for he had not heard of it: but the notion of there being a "moral risk" involved in foreign assurances reminded him of Lord Burleigh's shake of the head. He did not at all understand Mr. Emmens' process of fixing extra premiums: it was certainly not logical. There was considerable force in Mr. Baden's suggestion that premiums should be loaded, not simply on account of the additional mortality, but on the score of the increased expense of conducting a foreign agency. But the difficulty is that there are many places for which it would be impossible to charge any extra premium—Canada for instance—although the expense of an agency there would be as considerable as at Valparaiso or Buenos Ayres. His figures were deduced from Mr. Brown's report and could be easily tested. There were many objections to Messrs. Woolhouse and Neison's statistics. Mr. Woolhouse commenced from 1760, soon after the establishment of the British rule in India, and extracted his data from Dodwell and Miles' lists, and thus included a period which he thought it desirable to exclude, because the class of men first sent out to India were a totally different class to those now going out there. The interesting question of acclimatization was still unsettled. Sir Ranald Martin, the medical examiner at the India Office, has no belief in acclimatization, and thinks that a new comer is better able to resist the ravages of the climate than an old resident of the same age. The mortality amongst men aged 35 landing in India for the first time would be less than that amongst men of the same age who had been there for ten years. He did not think that a common agreement would ever be arrived at by the Offices. Let each actuary act upon his own judgment, and do what he deemed equitable between the public and his own Office.

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*On an Improved Theory of Annuities and Assurances.**By W. S. B. WOOLHOUSE, F.R.A.S.*

[Read before the Institute, 29th March, 1869.]

**THE** existing theory and practice of Annuities and Assurances, and the tables ordinarily constructed for the purpose of expediting the various computations relating to the same, are radically founded on annual payments, or upon a series of mathematical values which appertain to certain annual periods that are specifically defined. Indeed, the methods employed are not strictly applicable unless the prescribed condition of annual periodicity is realized or assumed; and this, in fact, is the reason why assurances, as well as annuities, are necessarily supposed to be payable on the exact completion of one of the recurring periods, viz., at the end of the year of age in which the death takes place.

Again with respect to annuities, it cannot but appear to be remarkable that the investigation of so elementary a value as that of an annuity, when diverted from its usual course and made payable half-yearly or quarterly, should not have been determined with greater accuracy in works on Annuities and Assurances. This may in some measure perhaps be ascribed to a sort of vague impression that a rough approximation might be considered sufficient for the cases which may be expected to arise in the course of actual business. But, looking at the formidable character of some of the investigations, there can be but little doubt that a more potent reason is to be traced in the mathematical difficulty encountered in previous efforts to arrive at anything like an accurate formula, which would at the same time be sufficiently simple for the ordinary routine of Office calculation.

The new theory might appropriately be called the **CONTINUOUS METHOD**. According to the principles laid down, all moneys invested, in place of receiving yearly increments of interest, are considered to be continuously growing. This is undoubtedly the only true way to assimilate our computations with actual facts, since moneys as they come to hand are invested in various securities and at all seasons, and should therefore not be assumed to bear interest at established and immoveable periods, but should be regarded as realizing the accumulations from the exact dates at which they are received. Also, lives, instead of being subjected to successive yearly decrements are, in like manner, properly considered to be diminishing continuously. In fact all our investigations are founded on the general and self-evident principle, that any

function whatever, of which a series of values, geometrically speaking, constitute the ordinates of a continuous curve line, must, as an indispensable consequence, have all its intermediate values accurately represented by the corresponding interposed ordinates of the functional curve.

It is no longer necessary to be dependent upon such gratuitous suppositions as that the deaths which take place during any year shall be equally distributed throughout the year; or that when two lives  $x$  and  $y$  both fail in a stated year, it is equally probable whether  $x$  shall die before  $y$  or  $y$  shall die before  $x$ . These specious suppositions, though expedient as regards calculation, are, in fact, not strictly true. The inaccuracy introduced by their adoption would be made very conspicuous were we to deal with quinquennial in lieu of annual intervals, since no experienced calculator would for a moment tolerate the hypothesis of deaths being equally distributed throughout a quinquennial period, and yet the principle is the same for one year as for five. The error consists in the rude substitution of a polygon in place of the mortality curve.

Another important advantage of the new as compared with the existing methods, is that it is not only more accurate in principle, and in all respects philosophically consistent, but that the various formulæ are generally more simple and commodious for calculation. As a remarkable evidence of comprehensive simplicity it may be instanced that the formula hereafter established for determining the value of an assurance of one life against another, a calculation usually somewhat complicated, is precisely the same as that of the most ordinary case, viz., an assurance of one life. The values obtained are also those of assurances payable at the instant of death, and admit of special adaptation to the actual stipulations of the contract. If the assurance be payable at three months or any other prescribed time after death, the adjustment of the result is only a question of interest of money during the interval. To these preliminary observations I have only further to add that one of the most prominent recommendations of the method is the extreme facility with which it can be adapted to all existing Life and Annuity Tables.

Before, however, proceeding with the general subject of enquiry, I shall briefly advert to a former application of an equivalent method to the case of an annuity payable by instalments. If  $a$  denote the present value of an annuity of £1 payable annually, the like annuity when payable in half-yearly instalments is known

to be approximately  $a + \frac{1}{4}$ . That the true value is something less than  $a + \frac{1}{4}$  may easily be shown as follows. Let the successive periods of payment, at the end of the first, second, &c. years, be distributed along a horizontal line, representing the order of time, and let each yearly payment be analysed into  $\frac{1}{4} + \frac{1}{2} + \frac{1}{4}$ . Then,  $a^{(2)}$  denoting the annuity when payable in half-yearly instalments, the series of prospective payments in the two respective cases will be

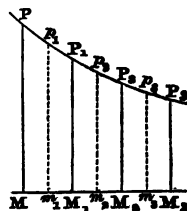
	Comm't.	1st year.	2nd year.	
$\frac{1}{4} + a =$ present value of	$\left(\frac{1}{4}\right)$	$\frac{1}{4} \left(\frac{1}{2}\right) \left(\frac{1}{4}\right)$	$\frac{1}{4} \left(\frac{1}{2}\right) \left(\frac{1}{4}\right)$	&c.
$a^{(2)} =$	„ „ $\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ &c.

Here it may first be observed that the present values of any three consecutive payments of an annuity or instalment are very nearly in geometrical progression, and secondly that in all such progressions twice the middle term is less in magnitude than the sum of the two extremes. Now, inspecting the above representations, we observe that the second series, which exhibits  $a^{(2)}$ , is deducible from that above it, which exhibits  $\frac{1}{4} + a$ , by substituting in place of each pair of terms  $\frac{1}{4}$ , enclosed within a parenthesis, a double middle term  $\frac{1}{2}$ . Hence, as each of these substitutions diminishes the present value of the series, it manifestly follows that the result  $a^{(2)}$  is less than  $a + \frac{1}{4}$ .

The true relation between the value of an annuity payable yearly and that of a corresponding annuity payable half-yearly may be conceived to be represented geometrically as follows:—Let the line of abscissas  $MM_1M_2M_3 \dots$  represent the time in successive yearly intervals; then, the annuity being £1 per annum, let  $PM = I$  be the value of £1 receivable immediately,  $P_1M_1$  the present value of £1 receivable at the end of one year,  $P_2M_2$  the value of £1 receivable at the end of two years, &c., &c. Then, through the several extremities  $PP_1P_2P_3 \dots$  of these ordinates a continuous curve line being supposed to be traced, which curve may be designated the Annuity Curve, let the intermediate or half-yearly ordinates  $p_1m_1, p_2m_2, p_3m_3, \dots$  be drawn; and the values of an annuity of £1 payable yearly and half-yearly will be

$$a^{(1)} = P_1M_1 + P_2M_2 + P_3M_3 + \dots$$

$$a^{(2)} = \frac{1}{2}(p_1m_1 + P_1M_1 + p_2m_2 + P_2M_2 + \dots).$$



In my paper on "Summation" (*Journal*, vol. xi., page 327) the value of the last-mentioned annuity is found to be

$$a^{(3)} = a + \frac{1}{4} - \frac{\mu + \delta}{16};$$

and generally, for an annuity of £1 when payable in  $m$  instalments,

$$a^{(m)} = a + \frac{m-1}{2m} - \frac{m^2-1}{12m^2} (\mu + \delta),*$$

where  $\mu$  and  $\delta$  denote what I call the forces of mortality and discount.

In the investigation of the subject before us I shall endeavour to develop the requisite formulæ in the simplest manner possible, and the better to accomplish this object, in a systematic manner, I propose to establish the principles of the method from the commencement, without any further reference to what has been previously done in the paper on Summation. I am the more induced to adopt this course as the mathematical processes are so effectively simplified that the whole of the matter will come within a reasonable compass.

#### *Principles of Summation.*

Let  $V$  denote a function of a variable quantity  $x$ , in which the symbol  $x$  may be conceived to be the abscissa of a curve and may be employed to represent an interval of time; and suppose the development of  $V$  in powers of  $x$  to be

$$V_x = V_0 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \dots$$

Then, by successive differentiation,

$$\left(\frac{dV}{dx}\right)_x = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \dots, \quad \left(\frac{dV}{dx}\right)_0 = A;$$

$$\left(\frac{d^2V}{dx^2}\right)_x = 2B + 6Cx + 12Dx^2 + 20Ex^3 + \dots;$$

$$\left(\frac{d^3V}{dx^3}\right)_x = 6C + 24Dx + 60Ex^2 + \dots, \quad \left(\frac{d^3V}{dx^3}\right)_0 = 6C;$$

&c.

&c.

\* Mr. Sprague has since gone at great length into the subject of annuities, and in confirming the accuracy of these results has been most liberal in his allusions to the decided improvement effected by my formulæ. It is well that Mr. Sprague has been induced to follow me in this particular investigation, as his methods are so essentially different from mine and afford such a scope for his acute mathematical talent, that his paper must be esteemed as a most valuable contribution to the *Journal*.

Therefore,

$$V_0 + V_x = 2V_0 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \dots$$

$$\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_x = -2Bx - 3Cx^2 - 4Dx^3 - 5Ex^4 - \dots$$

$$\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_x = -24Dx - 60Ex^2 - \dots$$

&c.

&c.

Also, by integration,

$$\int V dx = V_0 x + A \frac{x^2}{2} + B \frac{x^3}{3} + C \frac{x^4}{4} + D \frac{x^5}{5} + E \frac{x^6}{6} + \dots$$

This integral may be expressed in terms of the differential coefficients by replacing the preceding values of these coefficients: thus we obtain

$$\begin{aligned} \int V dx = & \frac{x}{2} (V_0 + V_x) + \frac{x^2}{12} \left\{ \left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_x \right\} \\ & - \frac{x^4}{720} \left\{ \left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_x \right\} + \dots \end{aligned}$$

Now suppose the quantity  $x$  to pass over successive intervals, each equal to  $\frac{1}{m}$ , viz., from 0 to  $\frac{1}{m}$ ,  $\frac{1}{m}$  to  $\frac{2}{m}$ ,  $\frac{2}{m}$  to  $\frac{3}{m}$ , &c. By applying the formula to each of these intervals we get

$$\begin{aligned} \text{From } 0 \text{ to } \frac{1}{m}, \int V dx = & \frac{1}{2m} (V_0 + V_{\frac{1}{m}}) + \frac{1}{12m^2} \left\{ \left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_{\frac{1}{m}} \right\} \\ & - \frac{1}{720m^4} \left\{ \left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_{\frac{1}{m}} \right\} + \dots \end{aligned}$$

$$\begin{aligned} \text{From } \frac{1}{m} \text{ to } \frac{2}{m}, \int V dx = & \frac{1}{2m} (V_{\frac{1}{m}} + V_{\frac{2}{m}}) + \frac{1}{12m^2} \left\{ \left(\frac{dV}{dx}\right)_{\frac{1}{m}} - \left(\frac{dV}{dx}\right)_{\frac{2}{m}} \right\} \\ & - \frac{1}{720m^4} \left\{ \left(\frac{d^3V}{dx^3}\right)_{\frac{1}{m}} - \left(\frac{d^3V}{dx^3}\right)_{\frac{2}{m}} \right\} + \dots \end{aligned}$$

$$\begin{aligned} \text{From } \frac{2}{m} \text{ to } \frac{3}{m}, \int V dx = & \frac{1}{2m} (V_{\frac{2}{m}} + V_{\frac{3}{m}}) + \frac{1}{12m^2} \left\{ \left(\frac{dV}{dx}\right)_{\frac{2}{m}} - \left(\frac{dV}{dx}\right)_{\frac{3}{m}} \right\} \\ & - \frac{1}{720m^4} \left\{ \left(\frac{d^3V}{dx^3}\right)_{\frac{2}{m}} - \left(\frac{d^3V}{dx^3}\right)_{\frac{3}{m}} \right\} + \dots \end{aligned}$$

&c.

&c.

&c.

By adding together these successive portions of the integral we get for the complete limits, in which  $\omega$  denotes the terminal value of  $x$ ,

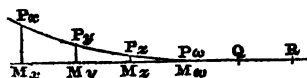
$$\int V dx = \frac{1}{m} \left( \frac{1}{2} V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} \dots + V_{\frac{m-1}{m}} + \frac{1}{2} V_m \right) \\ + \frac{1}{12m^3} \left\{ \left( \frac{dV}{dx} \right)_0 - \left( \frac{dV}{dx} \right)_m \right\} - \frac{1}{720m^5} \left\{ \left( \frac{d^3V}{dx^3} \right)_0 - \left( \frac{d^3V}{dx^3} \right)_m \right\} + \dots$$

Let  $\Sigma^{(m)}V$  denote  $\frac{1}{m} (V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} \dots + V_m)$ ; then

$$\Sigma^{(m)}V = \int_0^m V dx + \frac{1}{2m} (V_0 + V_m) - \frac{1}{12m^3} \left\{ \left( \frac{dV}{dx} \right)_0 - \left( \frac{dV}{dx} \right)_m \right\} \\ + \frac{1}{720m^5} \left\{ \left( \frac{d^3V}{dx^3} \right)_0 - \left( \frac{d^3V}{dx^3} \right)_m \right\} - \dots \quad (1)^*$$

which is a general development for evaluating the summation of any proposed consecutive series of functions.

In applying the formula to assurances and annuities, comprising the whole duration of life, it becomes considerably abridged and simplified, since in all such calculations it may be shown that the terminal values of the function and its differential coefficients severally vanish. Suppose the quantities  $V$  to be geometrically



represented by the ordinates of a curve line, and let  $P_x, P_y, P_z, P_\omega$  be the terminal portion of this curve,  $P_\omega$  being the point at which we first have

$V=0$ . Then, if we proceed onward to higher ages  $Q, R$ , &c., we shall still have  $V=0$ ; and since the curve must not only pass through  $P_x, P_y, P_z, P_\omega$  but must necessarily be made to pass also through  $Q, R$ , &c., it is obvious that the curve must be tangential to the line of abscissas at the point  $P_\omega$ , and there become finally deprived of its curvature, otherwise its continuity would be broken at that point, and it would cease to be a true representation of the values of the function  $V$ . And if the enquiry be treated as a question of pure analysis, it will be equally evident, from a consideration of Taylor's Theorem, that a continuous function cannot become permanently zero beyond a certain limit, unless the differential coefficients also vanish simultaneously with the function. It is obviously quite immaterial whether the value of the terminal age  $\omega$  be finite or indefinitely great, that is, whether the line of abscissas be a tangent or an asymptote to the curve. In either case the absolute extinction of all terminal values adapts the method in a special manner to annuities and assurances, and the formula (1), for summation, becomes simply

\* It has been pointed out to me by Mr. Merrifield that Legendre employs this formula in evaluating certain elliptic integrals in his valuable work on that subject. The formula is ascribed to Euler, and is also used by Laplace and others in the Theory of Probabilities.

$$\Sigma^{(m)}V = \int V dx + \frac{1}{2m}V - \frac{1}{12m^3} \frac{dV}{dx} + \frac{1}{720m^5} \frac{d^3V}{dx^3} - \dots \quad (2)$$

in which the terms following the integral are exclusively initial values.

Again, if the values of the function be taken at consecutive integer or annual intervals, the formula (2) becomes

$$\Sigma^{(1)}V = \int V dx + \frac{1}{2}V - \frac{1}{12} \frac{dV}{dx} + \frac{1}{720} \frac{d^3V}{dx^3} - \dots \quad (3)$$

This compared with (2) we obtain

$$\Sigma^{(m)}V = \Sigma^{(1)}V - \frac{m-1}{2m}V + \frac{m^2-1}{12m^3} \frac{dV}{dx} - \frac{m^4-1}{720m^5} \frac{d^3V}{dx^3} + \dots \quad (4)$$

which is the general formula for subdivision of intervals, or payments by instalments.

The formula (1) or (2) enables us to evaluate the sum of a series of values of a function, viz. :

$$\Sigma^{(m)}V = \frac{1}{m} \left( V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} + V_{\frac{3}{m}} + \dots \right)$$

Suppose this series of values to be deferred by an interval  $t$ , and let it be required to determine the sum of the series of deferred functions

$$\Sigma^{(m)}V_t = \frac{1}{m} \left( V_t + V_{\frac{1}{m}+t} + V_{\frac{2}{m}+t} + V_{\frac{3}{m}+t} + \dots \right)$$

This sum may be readily found by first getting that of the subtractive correction  $\Sigma^{(m)}(V - V_t)$ .

$$\text{Now, } V - V_t = - \left( t \frac{dV}{dx} + \frac{t^2}{2} \frac{d^2V}{dx^2} + \frac{t^3}{2.3} \frac{d^3V}{dx^3} + \dots \right)$$

By substituting this in place of  $V$  in (2) we at once obtain, for the sought correction,

$$\begin{aligned} \Sigma^{(m)}(V - V_t) &= tV + \frac{t^2}{2} \frac{dV}{dx} + \frac{t^3}{2.3} \frac{d^2V}{dx^2} + \frac{t^4}{2.3.4} \frac{d^3V}{dx^3} + \dots \\ &\quad - \frac{1}{2m} \left( t \frac{dV}{dx} + \frac{t^2}{2} \frac{d^2V}{dx^2} + \frac{t^3}{2.3} \frac{d^3V}{dx^3} + \dots \right) \\ &\quad + \frac{1}{12m^3} \left( t \frac{d^2V}{dx^2} + \frac{t^2}{2} \frac{d^3V}{dx^3} + \dots \right) \\ &= tV - \frac{t}{2} \left( \frac{1}{m} - t \right) \frac{dV}{dx} + \frac{t}{12} \left( \frac{1}{m} - t \right) \left( \frac{1}{m} - 2t \right) \frac{d^2V}{dx^2} \\ &\quad + \frac{t^2}{24} \left( \frac{1}{m} - t \right)^2 \frac{d^3V}{dx^3} \text{ \&c.} \dots \quad (5) \end{aligned}$$

Hence, subtracting (5) from (2) we get, for the summation of the successive values of a deferred function, the formula

$$\Sigma^{(m)} V_t = \int V dx + \left( \frac{1}{2m} - t \right) V - \frac{1 - 6mt + 6m^2 t^2}{12m^3} \cdot \frac{dV}{dx} \\ - \frac{t \left( \frac{1}{m} - t \right) \left( \frac{1}{m} - 2t \right)}{12} \frac{d^2 V}{dx^2} + \left\{ \frac{1}{720m^4} - \frac{t^2}{24} \left( \frac{1}{m} - t \right)^2 \right\} \frac{d^3 V}{dx^3} \dots (6).$$

These formulæ for summation may be usefully applied to computations of various kinds, but that of Annuities and Assurances, to which we now proceed, forms the special subject of the present enquiry, and to this particular discussion they are fortunately well adapted.

#### Notation.

For greater distinctness and facility of reference it may be desirable first to specify and define the notation employed, viz.

$i$  the rate of interest on £1 for one year;

$v$  the present value of £1 certain, payable after one year;

$d = 1 - v = \frac{i}{1+i}$  the discount on £1 certain for one year;

$\delta = \text{hyp. log}(1+i)$ , the force of discount; it is the rate of interest which if accumulated continuously or momentarily, would be equivalent to that which actually subsists; the developed relations between  $i$  and  $\delta$  are

$$\delta = \text{hyp. log}(1+i) = i - \frac{i^2}{2} + \frac{i^3}{8} - \frac{i^4}{4} + \dots$$

$$i = e^\delta - 1 = \delta + \frac{\delta^2}{2} + \frac{\delta^3}{2 \cdot 3} + \frac{\delta^4}{2 \cdot 3 \cdot 4} + \dots$$

also,  $v = (1+i)^{-1} = e^{-\delta}$ ;

$x, y, z$  the ages of lives similarly designated;

$l_x$  the "number living," at age  $x$ , in the table of mortality;

$\mu_x = -\frac{1}{l} \frac{dl}{dx}$  (approximately  $= \frac{l_{x-1} - l_{x+1}}{2l_x}$ ) the force of mortality at age  $x$ ;

$p_x = \frac{l_{x+1}}{l_x}$  the probability of the life  $x$  surviving a year, and attaining the age  $x+1$ ;

$p_{x,t} = \frac{l_{x+t}}{l_x}$  the probability of the life  $x$  surviving the time  $t$ , and attaining the age  $x+t$ ;

$\omega$  the limiting age in the table of mortality, or the age at which the lives first become totally extinct;

$D_x, N_x, M_x$  the numbers, for age  $x$ , contained in an ordinary "N and D table" at interest  $i$ , viz.

$$D_x = l_x v^x$$

$$N_x = D_{x+1} + D_{x+2} + D_{x+3} + \dots$$

$$M_x = (l_x - l_{x+1})v^{x+1} + (l_{x+1} - l_{x+2})v^{x+2} + \dots$$

$$= vN_{x-1} - N_x = D_x - (1-v)N_x;$$

$\bar{N}_x, \bar{M}_x$  continuous values of these numbers, viz.

$$\bar{N} = \int dt D_{x+t} = Da$$

$$\bar{M} = -\int dl_{x+t} v^{x+t} = D(1 - \delta a);$$

$a_x$  the present value of an annuity of £1 payable yearly during the existence of the life  $x$ ;

$a_x^{(m)}$  ditto when payable by  $m$  instalments in each year;

$a_x$  the present value of an annuity of £1 payable yearly and in advance;

$a_x^{(m)}$  ditto when payable by  $m$  instalments in each year, or in advance  $m$  times a year;

$\bar{a}_x$  the present value of an annuity of £1 per annum, when supposed to be payable continuously or momentarily;

$a_{xy}, a_{xyz}$  the present values of joint annuities of £1 payable yearly during the joint existence of the lives  $xy, xyz$ ;

$\bar{a}_{xy}, \bar{a}_{xyz}$  ditto when payable continuously or momentarily;

$a_{(xy)}, a_{(xyz)}$  the present values of annuities of £1 payable yearly during the existence of the longest of the lives  $xy, xyz$ ;

$\bar{a}_{(xy)}, \bar{a}_{(xyz)}$  the present values of the same annuities when supposed to be payable continuously or momentarily;

$c$  a characteristic prefixed to an annuity to indicate that it is complete, having a provision for a proportionate payment of the same up to the instant of decease;

$A_x$  the present value of an assurance of £1 payable at the end of the year of decease of  $x$ ;

$\bar{A}_x$  the present value of a continuous assurance of £1 payable at the instant of decease;

$A_{xy}, A_{xyz}$  the present values of assurances of £1 on the joint lives, payable at the end of the year of the first decease of the lives  $xy, xyz$ ;

$\bar{A}_{xy}, \bar{A}_{xyz}$  the present values of ditto when payable at the instant of decease;

$A_{xy}^1$  the present value of a contingent survivorship assurance of

£1 payable at the end of the year of decease of  $x$ , provided that  $y$  be then living;

$\bar{A}_{xy}^1$  the same when payable at the instant of decease;

$A_{(xy)}$ ,  $A_{(xyz)}$  the present values of assurances of £1 payable at the end of the year of decease of the last survivor of the lives  $xy$ ,  $xyz$ ;

$\bar{A}_{(xy)}$ ,  $\bar{A}_{(xyz)}$  the present values of ditto, when payable at the instant of decease;

${}_n\bar{a}_x = \frac{D_{x+n}}{D_x} a_{x+n}$ ,  ${}_n\bar{A}_x = \frac{D_{x+n}}{D_x} A_{x+n}$  the respective values of an annuity and assurance when the same is deferred  $n$  years;

$a_{x:n} = a_x - {}_n\bar{a}_x$ ,  $A_{x:n} = A_x - {}_n\bar{A}_x$  ditto when temporary for  $n$  years;

$e_x$  the curtate expectation, or average number of years a life  $x$  survives, the same being estimated up to the beginning of the year of decease;

$\bar{e}_x$  the complete expectation, estimated up to the instant of decease;

$E_{xy}^1$  the expectancy of survivorship, or probability that  $x$  shall die in the lifetime of  $y$ .

It may here be noted that, in investigations and formulæ for one life only, the subscript letter  $x$  may, for the most part, be omitted with advantage.

### *Forces of Decrement.*

In the following investigations the functional value  $V_x$  usually signifies the present value of a quantity that varies from year to year, and which generally decreases when the age  $x$  increases.

The negative differential coefficient  $-\frac{dV}{dx}$ , which is arithmetically positive, expresses the rate per annum at which  $V$  decreases at the precise age  $x$ . The intensity of this decrement is obviously the ratio which it bears to the corresponding value  $V$ , and is therefore found by dividing the same by  $V$ . Thus, the force of decrement of the function  $V$  is

$$-\frac{1}{V} \frac{dV}{dx} = -\frac{d \log V}{dx}$$

the logarithm being understood to be hyperbolic.

When  $V = l_x$ , we have

$$\text{Force of mortality } \mu = -\frac{d \log l}{dx} = -\frac{1}{l} \frac{dl}{dx}.$$

When  $V=v^x$ , the present value of £1 certain after  $x$  years,

$$\begin{aligned}\text{Force of discount } \delta &= -\frac{d \log(v^x)}{dx} = -\frac{d(x \log v)}{dx} \\ &= -\log v = \log(1+i)\end{aligned}$$

and is constant, and independent of  $x$ .

Suppose  $V=PQR \dots$  to consist of factors  $P, Q, R, \dots$ ; then since  $\log V = \log P + \log Q + \log R \dots$ , we shall have

$$\frac{d \log V}{dx} = \frac{d \log P}{dx} + \frac{d \log Q}{dx} + \frac{d \log R}{dx} \dots$$

Hence the following general theorem:—

**GENERAL THEOREM.**—*The force of decrement of a combined function is equal to the sum of the forces of decrement of the constituent factors.*

A practical case of this theorem, before given in the paper on "Summation," is the following.

**THEOREM.**—*The force of mortality of a joint existence is equal to the sum of the forces of mortality of the constituent lives.*

It also follows that

$$\begin{array}{llll}\text{The force of decrement of } D_x = l_x v^x & \text{is } \mu_x + \delta \\ \text{,,} & \text{,,} & D_{xy} = l_x l_y v^x & \text{,, } \mu_x + \mu_y + \delta \\ \text{,,} & \text{,,} & D_{xyz} = l_x l_y l_z v^x & \text{,, } \mu_x + \mu_y + \mu_z + \delta \\ & & \text{\&c.} & \text{\&c.}\end{array}$$

### *Annuities on One Life.*

For a Continuous Annuity\* we have by (3)

$$\bar{N} = \int D dx = \Sigma D - \frac{1}{2} D + \frac{1}{12} \frac{dD}{dx}$$

\* Otherwise, if  $p$  denote the probability of surviving the interval  $x$ , then by (3),

$$\begin{aligned}\bar{a} &= \int dx p v^x = \Sigma p v^x - \frac{1}{2} + \frac{1}{12} \frac{d(p v^x)}{dx} \\ &= a - \frac{1}{2} - \frac{\mu + \delta}{12} \\ &= a + \frac{1}{2} - \frac{\mu + \delta}{12}.\end{aligned}$$

And, since  $-dp$  is the proportion of deaths or probability of decease during the instant  $dx$ ,

$$\begin{aligned}\bar{A} &= - \int dp v^x = -p v^x + \log v \int dx p v^x \\ &\quad (\text{between limits}) = 1 - \delta \bar{a}.\end{aligned}$$

Also, by (2),

$$\begin{aligned}a^{(m)} &= \Sigma^{(m)} p v^x = \int p v^x dx + \frac{1}{2m} - \frac{1}{12m^2} \frac{d(p v^x)}{dx} \\ &= \bar{a} + \frac{1}{2m} + \frac{\mu + \delta}{12m^2}.\end{aligned}$$

And, after deducting the advance payment  $\frac{1}{m}$ ,

$$a^{(m)} = \bar{a} - \frac{1}{2m} + \frac{\mu + \delta}{12m^2}.$$

and, dividing by  $D$ , observing that

$$\frac{dD}{Ddx} = \frac{d(\log D)}{dx} = \frac{d(\log l + x \log v)}{dx} = -(\mu + \delta)$$

we get

$$a = a + \frac{1}{2} - \frac{\mu + \delta}{12} \dots \dots \dots (7)$$

Also for a Continuous Assurance, payable on the instant of decease, we have

$$\begin{aligned} \bar{M} &= -\int dl.v^x = -lv^x + \log v \int dxlv^x \\ &= lv^x - \delta \int dx D = D - \delta \int D dx = D - \delta \bar{N}. \end{aligned}$$

Therefore, dividing by  $D$ ,

$$\bar{A} = 1 - \delta \bar{a} \dots \dots \dots (8)$$

Again by (4)

$$\Sigma^{(m)} D = \Sigma^{(1)} D - \frac{m-1}{2m} D + \frac{m^2-1}{12m^3} \cdot \frac{dD}{dx} \&c. \dots \dots$$

And, dividing by  $D$ , we get the value of an annuity payable in advance  $m$  times a-year, viz. :

$$\begin{aligned} a^{(m)} &= a - \frac{m-1}{2m} - \frac{m^2-1}{12m^3} (\mu + \delta) \\ &= a + \frac{m+1}{2m} - \frac{m^2-1}{12m^3} (\mu + \delta). \end{aligned}$$

Hence deducting  $\frac{1}{m}$ , the annuity, with first payment after the interval  $\frac{1}{m}$ , is

$$a^{(m)} = a + \frac{m-1}{2m} - \frac{m^2-1}{12m^3} (\mu + \delta)$$

Otherwise by (2)

$$\Sigma^{(m)} D = \int D dx + \frac{1}{2m} D - \frac{1}{12m^3} \frac{dD}{dx} \dots \dots$$

and, dividing by  $D$  and deducting  $\frac{1}{m}$ ,

$$a^{(m)} = a - \frac{1}{2m} + \frac{\mu + \delta}{12m^3} \dots \dots \dots (9)$$

To find the correction, or increase in the value, when the

annuity is to be paid with a proportionate part to the day of death, we have, during an interval  $x$ ,

$$-\int d\bar{M}.x = -\bar{M}x + \int \bar{M}dx$$

which must be estimated for each successive interval  $\frac{1}{m}$ .

$$\text{Through } x=0 \dots \frac{1}{m} \text{ it is } -\frac{1}{m} \bar{M}_1 + \int_0^{\frac{1}{m}} \bar{M}dx$$

$$,, \quad x=\frac{1}{m} \dots \frac{2}{m} \quad ,, \quad -\frac{1}{m} \bar{M}_2 + \int_{\frac{1}{m}}^{\frac{2}{m}} \bar{M}dx$$

&c.

&c.

Hence by summing these together, we get

$$-\int d\bar{M}.x = \frac{1}{m} \bar{M} - \Sigma^{(m)} \bar{M} + \int \bar{M}dx.$$

That is, by (2),

$$-\int d\bar{M}.x = \frac{1}{2m} \bar{M} + \frac{1}{12m^2} \frac{d\bar{M}}{dx} - \frac{1}{720m^4} \frac{d^3\bar{M}}{dx^3},$$

the last term of which may however be rejected as insignificant.

But  $\bar{M} = D - \delta \int Ddx$

and  $\frac{d\bar{M}}{dx} = \frac{dD}{dx} + \delta D = -(\mu + \delta)D + \delta D = -\mu D;$

$$\therefore -\int d\bar{M}.x = \frac{1}{2m} \bar{M} - \frac{\mu D}{12m^2}.$$

And, dividing by  $D$ , we have

$$\text{Augmentation of annuity} = \frac{1}{2m} (1 - \delta a) - \frac{\mu}{12m^2} \dots \dots (10)$$

Therefore, adding this to the foregoing value of the annuity (9), we get the complete annuity payable with a proportionate part to the day of death, viz.,

$$ca^{(m)} = \left(1 - \frac{\delta}{2m}\right) a + \frac{\delta}{12m^2} \dots \dots \dots (11).$$

#### Current Annuities.

If the first payment of the annuity be due after the time  $t$ , the function for summation will be  $D_{x+t}$  or  $D_t$ ; and by (6)

$$\begin{aligned} \Sigma^{(m)} D_t = \int Ddx - \left(t - \frac{1}{2m}\right) D - \frac{1 - 6mt + 6m^2 t^2}{12m^2} \cdot \frac{dD}{dx} \\ - \frac{t}{12} \left(\frac{1}{m} - t\right) \left(\frac{1}{m} - 2t\right) \frac{d^3 D}{dx^3}. \end{aligned}$$

Hence, dividing by  $D$ , we have an annuity payable  $m$  times a year, first payment due after the time  $t$ , viz. :

$$a_{\overline{t}|i}^{(m)} = \bar{a} - t + \frac{1}{2m} + \frac{\mu + \delta}{12m^2} (1 - 6mt + 6m^2t^2) - \frac{t}{12} \left( \frac{1}{m} - t \right) \left( \frac{1}{m} - 2t \right) \frac{D''}{D} \dots \dots (12)$$

If with this annuity a proportionate part is payable up to the day of decease, these proportionate parts, during an interval  $\frac{1}{m}$ , will be represented by

$$\begin{aligned} & - \int_0^t d\bar{M} \left( \frac{1}{m} - t + x \right) - \int_t^{\frac{1}{m}} d\bar{M} (x - t) \\ & = - \int_0^{\frac{1}{m}} d\bar{M} . x - t (\bar{M} - \bar{M}_{\frac{1}{m}}) + \frac{1}{m} (\bar{M} - \bar{M}_t). \end{aligned}$$

By summing the values of this expression for all intervals we get

$$- \int d\bar{M} . x - t\bar{M} + \Sigma^{(m)} (\bar{M} - \bar{M}_t)$$

the evaluation of which, according to (10) and (5) gives

$$\begin{aligned} & \frac{1}{2m} \bar{M} - \frac{\mu D}{12m^2} - \frac{t}{2} \left( \frac{1}{m} - t \right) \frac{d\bar{M}}{dx} + \frac{t}{12} \left( \frac{1}{m} - t \right) \left( \frac{1}{m} - 2t \right) \frac{d^2\bar{M}}{dx^2} = \\ & \frac{1}{2m} \bar{M} - \frac{\mu D}{12m^2} + \frac{t}{2} \left( \frac{1}{m} - t \right) \mu D + \frac{t}{12} \left( \frac{1}{m} - t \right) \left( \frac{1}{m} - 2t \right) \left( \delta \frac{dD}{dx} + \frac{d^2D}{dx^2} \right); \end{aligned}$$

and, dividing by  $D$ , the required correction of the annuity (12) is

$$\frac{1 - \delta \bar{a}}{2m} - \frac{\mu}{12m^2} + \frac{t}{2} \left( \frac{1}{m} - t \right) \mu - \frac{t}{12} \left( \frac{1}{m} - t \right) \left( \frac{1}{m} - 2t \right) \left\{ \delta (\mu + \delta) - \frac{D''}{D} \right\} \dots \dots (13)$$

Therefore, adding this correction to (12), the value of a current annuity, payable  $m$  times a year, with the first payment falling due at the time  $t$ , and the annuity to be completed up to the day of death, is

$$ca_{\overline{t}|i}^{(m)} = \left( 1 - \frac{\delta}{2m} \right) \bar{a} + \frac{\delta}{12m^2} + \left( \frac{1}{m} - t \right) \left\{ 1 - \frac{\delta t}{2} - \delta t \left( \frac{1}{m} - 2t \right) \frac{\mu + \delta}{12} \right\} \dots \dots (14)^*$$

When  $t = \frac{1}{2m}$ , the annuity (12) becomes equal to  $\bar{a} - \frac{\mu + \delta}{24m^2}$ , the

\* The value of a life annuity, when it is to be completed by the payment of a proportionate part up to the day of death, was first investigated in Mr. Sprague's paper, before referred to. It may be added that the formulæ (11) and (14) I have deduced for Complete Annuities are identical with his results.

correction  $\frac{1}{2m}(1-\delta a) + \frac{\mu}{24m^3}$ , and the complete annuity (14) is then

$$ca_{\frac{1}{2m}}^{(m)} = \left(1 - \frac{\delta}{2m}\right)a + \frac{1}{2m} - \frac{\delta}{24m^3} \dots \dots \dots (15)$$

This formula expresses the value of the annuity when the interval  $t$  assumes its average value. It should however be understood that this result differs from the average value of the annuity for promiscuous intervals. When  $t$  takes indiscriminately all values, not exceeding the interval  $\frac{1}{m}$ , the true average value of the Curtate Current Annuity (12) is then

$$\begin{aligned} \frac{\int a_{\frac{1}{m}}^{(m)} dt}{t} &= \bar{a} - \frac{t}{2} + \frac{1}{2m} + \frac{\mu + \delta}{12} t \left(\frac{1}{m} - t\right) \left(\frac{1}{m} - 2t\right) \\ &\quad - \frac{t^2}{24} \left(\frac{1}{m} - t\right)^2 \frac{D''}{D} \\ \left(\text{when } t = \frac{1}{m}\right) &= \bar{a} \dots \dots \dots (16) \end{aligned}$$

In like manner the average value of the correction (13) is

$$\begin{aligned} \frac{\int (13) dt}{t} &= \frac{1-\delta \bar{a}}{2m} - \frac{\mu}{12m^3} + \frac{t^2}{2} \left(\frac{1}{2m} - \frac{t}{3}\right) \\ &\quad - \frac{t^2}{24} \left(\frac{1}{m} - t\right)^2 \left\{ \delta(\mu + \delta) - \frac{D''}{D} \right\} \\ \left(\text{when } t = \frac{1}{m}\right) &= \frac{1-\delta \bar{a}}{2m} = \frac{\bar{A}}{2m} \dots \dots \dots (17) \end{aligned}$$

Therefore the true average value of the Complete Current Annuity (14) is

$$ca_{\frac{1}{2m}}^{(m)} = \bar{a} + \frac{\bar{A}}{2m} = \left(1 - \frac{\delta}{2m}\right)a + \frac{1}{2m} \dots \dots \dots (18).$$

And these last formulæ (16), (17), (18) are those which in strictness should be employed for calculating the liability of an Insurance Company in respect of Current Annuities due and payable at promiscuous intervals from the date of valuation.

We have thus found that the average value of a curtate current annuity, payable  $m$  times a year, the first payment being due and payable at an unknown time, not exceeding the period  $\frac{1}{m}$ , is independent of  $m$ , and is precisely equal to the value of a continuous annuity commencing immediately.

It also appears that the correction required to make the annuity complete, up to the day of death, is equal to the present value of half an instalment payable at the instant of decease.

These last results, which are of great practical utility and are remarkable for their rudimentary simplicity, have been arrived at after a somewhat elaborate investigation. A little consideration will, however, show that they admit of an easy proof without the aid of mathematics. In the adjoining diagram the successive years

	Portion of Year when due						
	1	2	3	4	5	6	7 &c.
1st year	a	a	a	a	a	a	&c.
2nd year	b	b	b	b	b	b	&c.
3rd year	c	c	c	c	c	c	&c.
4th year	d	d	d	d	d	d	&c.
5th year	e	e	e	e	e	e	&c.
&c.				&c.			

are prospectively laid out in horizontal lines, and supposed to be divided into an indefinite number of equal portions by the vertical columns numbered 1, 2, 3, 4, 5, 6, 7, &c. Payments made in the first year are distinguished by the letter *a*; those in the second year by the letter *b*, &c.; and the beginning of the first year is of course the date of valuation. Suppose the annuities to be payable annually; then if the first payment were due in any stated portion of the year, say for instance the 5th, the prospective series of pay-

ments of the annuity would be represented by the letters *a*, *b*, *c*, *d*, *e*, &c., down the 5th vertical column. Now by hypothesis the particular portion of the year in which the annuity falls due is undefined, and it is equally probable that it may be any one of those represented, since it depends simply upon the chronological date at which the annuity originally commenced. Therefore the average value of the annuity is the same as if there existed one of every possible kind, and in such case they would collectively occupy all the vertical columns, and thus include the whole of the letters shown in the tabular diagram. But if the complete table of letters, instead of being partitioned in vertical columns, be read off along the successive horizontal lines they will clearly exhibit an annuity payable in every subdivision, the series of subdivisions commencing at the beginning of the first year, or immediately from the date of valuation; and this equivalent annuity will become ultimately con-

tinuous when the number of subdivisions is conceived to be increased indefinitely.

Again, as regards the payment of a proportionate part up to the time of death, it is evident that, at whatever period of the tabular year the death takes place, since the annuity may with equal probability be any one of the set of annuities severally due in all portions of the year, the average amount of the proportionate part will be just one-half of an annuity payment. Thus it appears that the average correction of the value of the Current Annuity is accurately equal to the present value of one-half an annuity payment made at the instant of decease.

And if the annuity be payable  $m$  times a year, by conceiving the successive intervals  $\frac{1}{m}$  to be in like manner severally divided into an indefinite number of portions, it will be perceived that in both cases the reasoning is analogous and the conclusion exactly the same.

In this section we have only to further enunciate the principle here advanced that the average present value of the current premiums of an Assurance Company, at a given present age, the same being due at promiscuous periods after the date of valuation, is the same as that of a Current Annuity under similar circumstances, and, in years' purchase, is equal to the present value ( $\bar{a}$ ) of a Continuous Annuity commencing immediately at the given present age, and that this is accurately and rigidly true whether the premiums be payable yearly or by instalments.

#### *Annuities on Two Lives.*

For the determination of a continuous single annuity and a continuous joint annuity on two lives, we shall evidently have

$$\left. \begin{aligned} l_x \bar{a}_x &= (l\bar{a})_x = \int dt l_{x+t} v^t \\ l_x l_y \bar{a}_{xy} &= (l\bar{a})_{xy} = \int dt l_{x+t} l_{y+t} v^t \end{aligned} \right\} \dots \dots \dots (19)$$

the evaluation of which by the formula (3) gives

$$\begin{aligned} (l\bar{a})_x &= \Sigma^{(1)}(l_{x+t} v^t) - \frac{1}{2} l_x + \frac{1}{12} \frac{d(l_{x+t} v^t)}{dt} \\ (l\bar{a})_{xy} &= \Sigma^{(1)}(l_{x+t} l_{y+t} v^t) - \frac{1}{2} l_x l_y + \frac{1}{12} \frac{d(l_{x+t} l_{y+t} v^t)}{dt}. \end{aligned}$$

Therefore, dividing by  $l_x$  and  $l_y$  respectively, we obtain

$$\left. \begin{aligned} \bar{a}_x &= a_x + \frac{1}{2} - \frac{\mu_x + \delta}{12} \\ \bar{a}_{xy} &= a_{xy} + \frac{1}{2} - \frac{\mu_x + \mu_y + \delta}{12} \end{aligned} \right\} \dots \dots \dots (20)$$

The former of these is the formula (7), and the latter is perfectly analagous to it, since  $\mu_x + \mu_y$  may be written  $\mu_{xy}$ .

By applying either of these formulæ to the particular case of an annuity certain, in which  $\mu=0$ , it becomes

$$\frac{1}{\delta} = \frac{1}{i} + \frac{1}{2} - \frac{\delta}{12} \dots \dots \dots (21)$$

$$\left. \begin{aligned} \therefore \frac{1}{\delta} - \bar{a}_x &= \frac{1}{i} - a_x + \frac{\mu_x}{12} \\ \frac{1}{\delta} - \bar{a}_{xy} &= \frac{1}{i} - a_{xy} + \frac{\mu_x + \mu_y}{12} \end{aligned} \right\} \dots \dots \dots (22)$$

This singular relation may be thus enunciated:—

*The present value of a continuous reversionary perpetuity of £1 per annum, commencing at the instant of decease, exceeds that of an annual reversionary perpetuity, first payment at the end of the year of decease, by one-twelfth of the force of mortality at the given age or ages.*

Again, by taking the formula (20) in the extreme case in which money is supposed not to yield any interest, the value of all annuity payments are alike and independent of time; the annuities  $\bar{a}$ ,  $a$ , hence become respectively equal to  $\bar{e}$ ,  $e$ , the complete and curtate expectations of life, and we deduce

$$\left. \begin{aligned} \bar{e}_x &= e_x + \frac{1}{2} - \frac{\mu_x}{12} \\ \bar{e}_{xy} &= e_{xy} + \frac{1}{2} - \frac{\mu_x + \mu_y}{12} \end{aligned} \right\} \dots \dots \dots (23)$$

Therefore the tabular expectations  $(e + \frac{1}{2})$ , usually given, are in excess of the truth by one-twelfth of the force of mortality at the respective ages.

The present value of a Contingent Survivorship Annuity of £1 payable annually on a life  $x$  after the decease of  $y$  is  $a_x - a_{xy}$ ; and, when the annuity is payable  $m$  times a year, the value is  $a_x^{(m)} - a_{xy}^{(m)}$ . It is obvious that the annuity so found is thus defined. If we assume the annuity  $a_x^{(m)}$  which commences immediately to be put in operation, the survivorship annuity will consist of all the payments of such annuity which fall successively due after the

death of  $y$ . Similarly the present value of a continuous survivorship annuity commencing at the instant of the decease of  $y$  is  $\bar{a}_x - \bar{a}_{xy}$ .

Mr. Sprague has recently investigated an approximate value of a Survivorship Apportionable Life Annuity conformable to the conditions usually observed in legal practice, viz., on the supposition that the annuity commences from the instant of the decease of  $y$ , is payable half-yearly, or more generally  $m$  times a year, and is ultimately to be completed by a proportionate payment up to the day of the decease of  $x$ . By ingenious approximate reasoning Mr. Sprague deduces the formula

$$\frac{a_x - a_{xy}}{\sqrt{1+i}} \left\{ 1 + \frac{i}{2} \left( 1 - \frac{1}{m} \right) - \frac{i^2}{12} \left( 1 - \frac{1}{m^2} \right) + \frac{i^3}{24} \left( 1 - \frac{1}{m^3} \right) \cdot \dots \right\}.$$

The true value of such annuity, otherwise apparently unattainable, is readily determined by the continuous method, as follows:—

At the instant of the death of  $y$ , provided that  $x$  be then living, let  $\theta$  denote the exact age attained by  $x$ , and at that instant the value of the apportionable annuity, according to the formula (11) will then be

$$Ca_{\theta}^{(m)} = \left( 1 - \frac{\delta}{2m} \right) \bar{a}_{\theta} + \frac{\delta}{12m^2}.$$

Now, this value in contingent reversion is to be entered upon only in the event of  $x$  surviving  $y$ , and the present value thereof is required so as to include a due consideration of all possible values of the variable age  $\theta$  of survivorship. It is also obvious, as already stated, that the present value of the continuous reversionary annuity  $\bar{a}_{\theta}$ , taking into account all values of  $\theta$ , is properly  $\bar{a}_x - \bar{a}_{xy}$ .

Hence if  $\bar{A}_{xy}$  denote the present value of a contingent assurance of £1 payable at the instant of the death of  $x$ , provided that  $y$  be then living, the present value of the Contingent Survivorship Apportionable Annuity of practice is found to be

$$\left( 1 - \frac{\delta}{2m} \right) (\bar{a}_x - \bar{a}_{xy}) + \frac{\delta}{12m^2} \bar{A}_{xy} \cdot \dots \dots \dots (24)$$

By (20) this formula may be written

$$\left( 1 - \frac{\delta}{2m} \right) \left( a_x - a_{xy} + \frac{\mu_y}{12} \right) + \frac{\delta}{12m^2} \bar{A}_{xy} \cdot \dots \dots \dots (25)$$

and so calculated with ordinary annuities.

Also, by (11), it may be stated by complete annuities, thus:—

$$Ca_x^{(m)} - Ca_{xy}^{(m)} + \frac{\delta}{12m^2} \bar{A}_{xy} \cdot \dots \dots \dots (26)$$

The value of the last term of each of these formulæ, in which the contingent assurance appears as a factor, is evidently so small that it must be quite immaterial whether the contingent assurance be continuous or otherwise.

It need hardly be added here that continuous life annuities observe the same laws of combination as ordinary annuities. For example, the continuous annuity on the longest of two lives,  $x, y$  is,

$$\begin{aligned}\bar{a}_{(xy)} &= \bar{a}_x + \bar{a}_y - \bar{a}_{xy} \\ &= a_x + a_y - a_{xy} + \frac{1}{2} - \frac{\delta}{12}, \text{ by (20)} \\ &= a_x + a_y - a_{xy} + \frac{1}{\delta} - \frac{1}{i}, \text{ by (21),}\end{aligned}$$

that is,  $\bar{a}_{(xy)} = a_{(xy)} + \frac{1}{2} - \frac{\delta}{12} = a_{(xy)} + \frac{1}{\delta} - \frac{1}{i};$

and so of all other like combinations.

#### *Assurances on One Life.*

Let  $x$  denote the age of the life, and  $l_x$  the "number living" in the table of mortality. Then for determining the value of a continuous annuity we shall have

$$l\bar{a} = \int dt l_{x+t} v^t \dots \dots (a)$$

And, for determining the value of a continuous assurance, payable at the instant of death,

$$\begin{aligned}l\bar{A} &= - \int dl_{x+t} v^t \\ &= - \frac{d(l\bar{a})}{dx}, \text{ by (a),} \\ &\left( \text{since } \frac{dl_{x+t}}{dx} = \frac{dl_{x+t}}{dt} \right);\end{aligned}$$

that is,

$$\begin{aligned}l\bar{A} &= - \frac{dl}{dx} \bar{a} - l \frac{d\bar{a}}{dx} = l \left( \mu \bar{a} - \frac{d\bar{a}}{dx} \right); \\ \therefore \bar{A} &= - \frac{1}{l} \frac{d(l\bar{a})}{dx} = \mu \bar{a} - \frac{d\bar{a}}{dx} \dots \dots \dots (27)\end{aligned}$$

A second formula is found thus,

$$\begin{aligned}l\bar{A} &= - \int dl_{x+t} v^t = - l_{x+t} v^t + \log v \int dt l_{x+t} v^t \\ &\text{(taken between the limits } t=0 \dots \infty)\end{aligned}$$

$$=l-\delta la, \text{ by (a);}$$

$$\therefore \bar{A}=1-\delta a \dots\dots\dots (28)$$

agreeing with (8).

This last formula is also evident from the following considerations. According as an annuity of £1 is payable yearly or continuously we shall obviously have

$$\begin{array}{l|l} \frac{1}{i} = \text{Perpetuity, annual;} & \frac{1}{\delta} = \text{Perpetuity, continuous;} \\ \frac{1}{i} - a = \text{Reversionary do.} & \frac{1}{\delta} - a = \text{Reversionary do.} \end{array}$$

Therefore, £1 = continuous perpetuity  $\times \delta$   
and  $\bar{A} = \text{£1 in continuous reversion}$

$$= \left( \frac{1}{\delta} - a \right) \times \delta = 1 - \delta a.$$

By the use of the relation (22) the formula (28) resolves into another neat expression involving the ordinary tabular annuity, viz.

$$\bar{A} = \delta \left( \frac{1}{i} - a + \frac{\mu}{12} \right) \dots\dots\dots (29).$$

We have thus arrived at three distinct formulæ (27), (28), (29) for determining the present value of an assurance of £1 payable at the instant of death.

To find the equivalent annual premium, the value of the assurance, or single premium, must be divided by  $a+1$ ; or, generally, if the premium be payable in  $m$  instalments, the divisor will be

$$\begin{aligned} a^{(m)} &= a + \frac{1}{2m} + \frac{\mu + \delta}{12m^2} \\ &= a + \frac{m+1}{2m} - \frac{m^2-1}{12m^2} (\mu + \delta). \end{aligned}$$

It is only further requisite to remark that as the annuity  $a$  diminishes when the age  $x$  increases, the term  $-\frac{da}{dx}$  is in reality positive, and that in using the formula (27) sufficient accuracy will be attained by making

$$\left. \begin{aligned} \mu &= \frac{l_{x-1} - l_{x+1}}{2l_x} \\ -\frac{da}{dx} &= \frac{1}{2} (\bar{a}_{x-1} - \bar{a}_{x+1}) \end{aligned} \right\} \dots\dots\dots (30)$$

in which it is tacitly assumed that for three consecutive points in a curve, the tangent at the intermediate point is parallel to the chord that connects the two extreme points.

*Example.*—Required the present value of an assurance, payable at the instant of death, on a life aged 36, according to Davies' Equitable Experience Table at 3 per cent.

Calculation by formula (27).	By formula (28).	By formula (29).
$\bar{a} \begin{cases} (35) & 18.7218 \\ (37) & 18.2397 \end{cases}$	$\mu \begin{cases} (36) & .01150 \\ (3) & .02956 \end{cases}$	$\frac{1}{t} \begin{cases} & 33.3333 \\ & 17.9879 \end{cases}$
$2.4821$	$12.04106$	$15.3454$
$-\frac{d\bar{a}}{dx}(36) \quad 24105$	$.0034$	$\frac{1}{15} \mu \quad 10$
$\bar{a} \begin{cases} (36) & 18.4845 \\ \mu & .01150 \end{cases}$	$a + \frac{1}{t} \quad 18.4879$	$\left\{ \begin{array}{l} \log \quad 1.18601 \\ \log \delta \quad 8.47069 \end{array} \right.$
$184845$	$\left\{ \begin{array}{l} \bar{a} \quad 18.4845 \\ \log \quad 1.26681 \\ \log \delta \quad 8.47069 \end{array} \right.$	$\left\{ \begin{array}{l} \log \quad 9.65670 \\ \bar{A} \quad .45363 \end{array} \right.$
$18484$	$\left\{ \begin{array}{l} \log \quad 9.73750 \\ \delta \bar{a} \quad 0.54639 \\ 1.00000 \end{array} \right.$	
$9242$	$\bar{A} \quad .45361$	
$\mu \bar{a} \quad 212571$		
$-\frac{d\bar{a}}{dx} \quad 24105$		
$\bar{A} \quad .45362$		

### Assurances on Two Lives.

Let  $x, y$  denote the respective ages of two lives, and  $\bar{a}_{xy}$  the present value of the continuous joint annuity; then we shall evidently have, as in (19),

$$l_x l_y \bar{a}_{xy} = (l\bar{a})_{xy} = \int dt l_{x+t} l_{y+t} v^t \dots \dots \dots (\beta)$$

And  $\bar{A}_{xy}^1$  denoting the present value of a continuous contingent assurance of £1 on the life of  $x$  against that of  $y$ , payable on the instant of the decease of  $x$  provided that  $y$  be then living,

$$\begin{aligned} l_x l_y \bar{A}_{xy}^1 &= - \int dl_{x+t} l_{y+t} v^t \\ &= - \frac{d}{dx} (l\bar{a})_{xy}, \text{ by } (\beta), \\ &\quad \left( \text{since } \frac{dl_{x+t}}{dx} = \frac{dl_{x+t}}{dt} \right); \end{aligned}$$

that is,

$$\begin{aligned} l_x l_y \bar{A}_{xy}^1 &= - \frac{dl_x}{dx} l_y \bar{a}_{xy} - l_x l_y \frac{d\bar{a}_{xy}}{dx} \\ &= l_x l_y \left( \mu_x \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dx} \right); \end{aligned}$$

$$\therefore \bar{A}_{xy}^1 = -\frac{1}{l_x} \frac{d(l_x \bar{a}_{xy})}{dx} = \mu_x \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dx} \left. \begin{array}{l} \text{Similarly,} \\ \bar{A}_{xy}^1 = -\frac{1}{l_y} \frac{d(l_y \bar{a}_{xy})}{dy} = \mu_y \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dy} \end{array} \right\} \dots \dots \dots (31)$$

Adding together these two values, we obtain the present value of an assurance on the Joint Lives, payable at the instant of the first decease, viz.

$$\bar{A}_{xy} = \mu_{xy} \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dt} \dots \dots \dots (32)$$

where

$$\mu_{xy} = \mu_x + \mu_y$$

Another formula for the joint lives is found thus,

$$\begin{aligned} l_x l_y \bar{A}_{xy} &= -\int d(l_{x+t} l_{y+t}) v^t \\ &= -l_{x+t} l_{y+t} v^t + \log v \int dt l_{x+t} l_{y+t} v^t \\ &\quad \text{(taken between the limits } t=0 \dots \infty) \\ &= l_x l_y - \delta l_x l_y \bar{a}_{xy} \text{ by } (\beta); \\ \therefore \bar{A}_{xy} &= 1 - \delta \bar{a}_{xy} \dots \dots \dots (33) \end{aligned}$$

By applying the second formula of (22) we hence also obtain another expression, involving the ordinary tabular annuity, viz.

$$\bar{A}_{xy} = \delta \left( \frac{1}{i} - a_{xy} + \frac{\mu_{xy}}{12} \right) \dots \dots \dots (34)$$

where, as before,  $\mu_{xy} = \mu_x + \mu_y$ .

To determine the equivalent annual premiums for the foregoing assurances, the present values, or single premiums, must in each case be divided by  $a_{xy} + 1$ ; and, if the premium be payable in  $m$  instalments, the divisor will be

$$\begin{aligned} a_{xy}^{(m)} &= \bar{a}_{xy} + \frac{1}{2m} + \frac{\mu_x + \mu_y + \delta}{12m^2} \\ &= a_{xy} + \frac{m+1}{2m} - \frac{m^2-1}{12m^2} (\mu_x + \mu_y + \delta). \end{aligned}$$

Sufficient accuracy is obtained in the use of the formulæ (31), (32) by making

$$\left. \begin{aligned} \frac{d\bar{a}_{xy}}{dx} &= \frac{1}{2} (\bar{a}_{x-1,y} - \bar{a}_{x+1,y}) \\ \frac{d\bar{a}_{xy}}{dy} &= \frac{1}{2} (\bar{a}_{x,y-1} - \bar{a}_{x,y+1}) \\ \frac{d\bar{a}_{xy}}{dt} &= \frac{1}{2} (\bar{a}_{x-1,y-1} - \bar{a}_{x+1,y+1}) \end{aligned} \right\} \dots \dots \dots (35)$$

In the first of these it will be perceived that a change to a year preceding and following the given age is assumed by the life  $x$  only; in the second by  $y$  only; and that in the third the change takes place simultaneously with both lives.

*Example.*—Required the present values of the contingent survivorship and joint assurances on two lives aged 65 and 25, payable the instant the claim arises, on the basis of the Northampton table with interest at 3 per cent.

To avoid reference to tables the requisite data are here stated.

Age	$\mu$
24	·01551
25	·01576
26	·01601
64	·04702
65	·04902
66	·05155

Ages	Ordinary Joint Annuity $a_{xy}$
64 24	7·62653
25	7·61057
65 24	7·38509
25	7·37023
26	7·35479
66 25	7·12477
26	7·11044

The values of  $a_{xy}$ , contained in (35), being prepared by the formula (20), the calculations according to (31) and (32) are as hereunder:—

65 against 25.	25 against 65.	Joint Lives.
$\bar{a} \begin{cases} (64, 25) & 8·10288 \\ (66, 25) & 7·61670 \end{cases}$	$\bar{a} \begin{cases} (65, 24) & 7·87725 \\ (65, 26) & 7·84691 \end{cases}$	$\bar{a} \begin{cases} (64, 24) & 8·11886 \\ (66, 26) & 7·60235 \end{cases}$
2)·48618	2)·03034	2)·51651
$-\frac{d\bar{a}}{ds}(65, 25) \quad \underline{\underline{24309}}$	$-\frac{d\bar{a}}{dy}(65, 25) \quad \underline{\underline{01517}}$	$-\frac{d\bar{a}}{dt}(65, 25) \quad \underline{\underline{258255}}$
$\bar{a}(65, 25) \quad 7·86237$	$\bar{a}(65, 25) \quad 7·86237$	$\bar{a}(65, 25) \quad 7·86237$
$\mu(65) \quad \underline{\underline{04902}}$	$\mu(25) \quad \underline{\underline{01576}}$	$\mu(65, 25) \quad \underline{\underline{06478}}$
3144948	786237	4717422
707613	393118	314495
1572	55037	55037
$\mu\bar{a} \quad \underline{\underline{3854138}}$	4717	6290
$-\frac{d\mu\bar{a}}{ds} \quad \underline{\underline{24309}}$	$\mu\bar{a} \quad \underline{\underline{1239109}}$	$\mu\bar{a} \quad \underline{\underline{5093244}}$
$\bar{A}_1 \quad \underline{\underline{62850}}$	$-\frac{d\mu\bar{a}}{dy} \quad \underline{\underline{01517}}$	$-\frac{d\mu\bar{a}}{dt} \quad \underline{\underline{258255}}$
$\bar{A}_{65,25}$	$\bar{A}_1 \quad \underline{\underline{13908}}$	$\bar{A}_{64,25} \quad \underline{\underline{76758}}$
	$\bar{A}_{25}$	

Calculation of joint lives otherwise by formulæ (33) and (34):—

$\bar{a}(65, 25)$	7.86237	$\frac{1}{i}$	33.33333
$\delta$	.02956	$a(65, 25)$	7.37023
	1572474		
	707613	$\frac{1}{ix}\mu(65, 25)$	25.96810
	39312		.00540
	4717		
$\delta\bar{a}$	2324116		25.96850
	1.00000	$\left\{ \begin{array}{l} \log \\ \log \delta \end{array} \right.$	$\left\{ \begin{array}{l} 1.41444 \\ 8.47069 \end{array} \right.$
$\bar{A}_{65:25}$	.76759	$\left\{ \begin{array}{l} \log \\ \bar{A}_{65:25} \end{array} \right.$	$\left\{ \begin{array}{l} 9.88513 \\ .76759 \end{array} \right.$

The calculation of the value of the Continuous Contingent Survivorship Assurance from the ordinary tabular annuities will be materially expedited, and the value obtained will be sufficiently correct for official purposes, if  $a_{xy} + \frac{1}{2}$  be substituted in place of  $\bar{a}_{xy}$ . This substitution, being a trifle in excess, and the coefficient  $\mu_x$  being small, the result will be but little affected by the change and will incline to the safe side. Thus we shall have the convenient practical formula

$$\bar{A}_{\frac{1}{xy}} = \mu_x \left( a_{xy} + \frac{1}{2} \right) + \frac{1}{2} (a_{x-1, y} - a_{x+1, y})$$

And, dividing by  $a_{xy} + 1$ , the corresponding annual premium, for the same assurance, will then be

$$= \mu_x + \frac{1}{2} \frac{(a_{x-1, y} - a_{x+1, y}) - \mu_x}{a_{xy} + 1} \dots \dots \dots (36)$$

These formulæ suggest the following simple rules, and the use of each is elucidated by an example worked out.

**PROBLEM.**—To find the Single Premium or present value of a Contingent Survivorship Assurance of £1 payable the instant the claim is determined.

**RULE.**—Increase the ordinary Joint Annuity, for the given ages, by  $\frac{1}{2}$  or 0.5; take out the corresponding logarithm, to which add the logarithm of the force of mortality at the age of the assured life; and then take out the corresponding natural number. Next put down the ordinary joint annuities with the assured life taken respectively one year younger and one year older. Then one-half the difference between these annuities added to the number before found will give the required net Single Premium.

Calculation of the foregoing example, stated on page 118.

65 against 25.				25 against 65.			
65, 25	7·37023	log $\mu$ (65)	8·69037	65, 25	7·37023	log $\mu$ (25)	8·19745
	0·5				0·5		
	7·87023	log	0·89599		7·87023	log	0·89599
64, 25	7·61057	{ log	9·58636	65, 24	7·38509	{ log	9·09344
66, 25	7·12477			{ numb.	·38580		
	·48580 $\times \frac{1}{4}$ =		·24290		·03030 $\times \frac{1}{4}$ =		·1515
Net Single Premium			·62870	Net Single Premium			·13915

PROBLEM.—To find the Annual Premium for a Contingent Survivorship Assurance of £1 payable the instant the claim is determined.

RULE.—Put down the ordinary joint annuities with the assured life taken first a year younger and then a year older; from the difference of these annuities subtract the force of mortality at the age of the assured life; and divide one-half this last difference by the ordinary joint annuity increased by unity. Then, the quotient added to the force of mortality, before made use of, will give the net annual premium required.

Calculation of the same example.

65 against 25.				25 against 65.			
64, 25	7·61057			65, 24	7·38509		
66, 25	7·12477			65, 26	7·35479		
	·48580	$\mu$ (65)	·04902		·03030	$\mu$ (25)	·01576
	·04902				·01576		
	2)·43678				2)·01454		
	·21839				·00727		
	·02609				·00087		
$a_{65, 25} + 1$	8·3702			$a_{65, 25} + 1$	8·3702		
Net Annual Premium			·07511	Net Annual Premium			·01663

The annual premiums, accurately calculated, are respectively ·07509 and ·01661.

The present values of Contingent Assurances payable on the instant of decease provided another given life shall have died previously may be immediately found from what precedes. For we shall evidently have, in such case,

$$\left. \begin{aligned} \bar{A}_{\frac{1}{xy}} &= \bar{A}_x - \bar{A}_{\frac{1}{xy}} \\ \bar{A}_{\frac{2}{xy}} &= \bar{A}_y - \bar{A}_{\frac{1}{xy}} \end{aligned} \right\} \dots \dots \dots (37)$$

And the sum of these gives also

$$\begin{aligned} \bar{A}_{(xy)} &= \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \\ &= 1 - \delta \bar{a}_{(xy)} \end{aligned}$$

$$= \delta \left( \frac{1}{i} - a_{(xy)} \right) = \frac{\delta}{d} A_{(xy)} \dots \dots \dots (38)$$

As one more example of two lives, a solution to the following problem is here annexed.

PROBLEM.—Determine the present value of a Contingent Assurance of £1 payable at the instant of the decease of a life  $x$ , provided he die before another given life  $y$ , or within  $n$  years after the death of  $y$ .

During the first  $n$  years the risk is evidently that of an absolute assurance on the life  $x$  and, for that period, the present value is  $\bar{A}_{x:n}$ . For the remainder of the life of  $x$ , after  $n$  years, the supplementary value is

$$\begin{aligned} - \frac{\int d l_{x+n+t} l_{y+t} v^{n+t}}{l_x l_y} &= - \frac{v^n}{l_x l_y} \frac{d}{dx} (l\bar{a})_{x+n, y} \\ &= \frac{v^n}{l_{xy}} (l\bar{A})_{\frac{1}{x+n, y}} = \frac{D_{x+n}}{D_x} \bar{A}_{\frac{1}{x+n, y}}. \end{aligned}$$

Therefore the required present value of the contingent assurance, under the proposed conditions, is

$$\left. \begin{aligned} &= \bar{A}_{x:n} + \frac{D_{x+n}}{D_x} \bar{A}_{\frac{1}{x+n, y}} \\ &= \bar{A}_x - \frac{D_{x+n}}{D_x} \left( \bar{A}_{x+n} - \bar{A}_{\frac{1}{x+n, y}} \right) \end{aligned} \right\} \dots \dots \dots (39)$$

which respectively express the equivalents of

$$\bar{A}_{x:n} + n | \bar{A}_{\frac{1}{x, y-n}} \text{ and } \bar{A}_x - n | \bar{A}_{\frac{2}{x, y-n}}.$$

### Assurances on Three Lives.

The formula for assurances on three lives are similar to those for one and two lives. Thus, for the Continuous Joint Annuity, we have

$$(l\bar{a})_{xyz} = \int dt l_{x+t} l_{y+t} l_{z+t} v^t \dots \dots \dots (\gamma)$$

And, thence, as before,

$$\begin{aligned} (\bar{A})_{\frac{1}{xyz}} &= - \int d l_{x+t} l_{y+t} l_{z+t} v^t \\ &= - \frac{d}{dx} (l\bar{a})_{xyz} \\ &= - \frac{dl_x}{dx} l_{yz} \bar{a}_{xyz} - l_{xyz} \frac{d\bar{a}_{xyz}}{dx}. \end{aligned}$$

Therefore, after dividing by  $l_{xyz}$ ,

$$\left. \begin{aligned} \bar{A}_{\overline{xyz}} &= -\frac{1}{l_x} \frac{d(l_x \bar{a}_{xyz})}{dx} = \mu_x \bar{a}_{xyz} - \frac{d\bar{a}_{xyz}}{dx} \\ \text{Similarly, } \bar{A}_{\overline{xyx}} &= -\frac{1}{l_y} \frac{d(l_y \bar{a}_{xyz})}{dy} = \mu_y \bar{a}_{xyz} - \frac{d\bar{a}_{xyz}}{dy} \\ \bar{A}_{\overline{yxs}} &= -\frac{1}{l_z} \frac{d(l_z \bar{a}_{xyz})}{dz} = \mu_z \bar{a}_{xyz} - \frac{d\bar{a}_{xyz}}{dz} \end{aligned} \right\} \dots \dots (40)$$

The sum of these gives

$$\bar{A}_{xyz} = \mu_{xyz} \bar{a}_{xyz} - \frac{d\bar{a}_{xyz}}{dt} \dots \dots \dots (41)$$

which again also admits of being expressed in the other two forms analogous to (33) and (34), viz.,

$$\bar{A}_{xyz} = 1 - \delta \bar{a}_{xyz} \dots \dots \dots (42)$$

$$\bar{A}_{xyz} = \delta \left( \frac{1}{i} - a_{xyz} + \frac{\mu_{xyz}}{12} \right) \dots \dots \dots (43)$$

where

$$\mu_{xyz} = \mu_x + \mu_y + \mu_z.$$

The other principal combinations may readily be deduced, from these and the preceding formulæ, in the following manner.

In the first place we have evidently

$$\left. \begin{aligned} \bar{A}_{\overline{1}_{xz}} &= \bar{A}_{\overline{1}_{xyx}} + \bar{A}_{\overline{1}_{xyx}} \\ \therefore \bar{A}_{\overline{21}_{xyz}} &= \bar{A}_{\overline{1}_{xz}} - \bar{A}_{\overline{1}_{xyx}} \\ \bar{A}_{\overline{21}_{xyz}} &= \bar{A}_{\overline{1}_{xy}} - \bar{A}_{\overline{1}_{xyz}} \end{aligned} \right\} \dots \dots \dots (44)$$

And, by addition,

$$\bar{A}_{\overline{2}_{xyz}} = \bar{A}_{\overline{1}_{xy}} + \bar{A}_{\overline{1}_{xz}} - 2\bar{A}_{\overline{1}_{xyz}} \dots \dots \dots (45)$$

Therefore also

$$\begin{aligned} \bar{A}_{\overline{2}_{xyz}} &= \bar{A}_x - \left( \bar{A}_{\overline{1}_{xyx}} + \bar{A}_{\overline{2}_{xyz}} \right) \\ &= \bar{A}_x - \bar{A}_{\overline{1}_{xy}} - \bar{A}_{\overline{1}_{xz}} + \bar{A}_{\overline{1}_{xyz}} \dots \dots \dots (46) \end{aligned}$$

Again, by adding together the values of  $\bar{A}_{\overline{2}_{xyz}}$ ,  $\bar{A}_{\overline{2}_{xyx}}$ ,  $\bar{A}_{\overline{2}_{xyz}}$ , according to the formula (45), the present value of the absolute assurance of £1 payable instantly on the second decease is

$$\bar{A}_{\overline{222}_{xyz}} = \bar{A}_{xy} + \bar{A}_{xz} + \bar{A}_{yz} - 2\bar{A}_{xyz} \dots \dots \dots (47)$$

And by collecting the values of  $\bar{A}_{\frac{3}{xyz}}$ ,  $\bar{A}_{\frac{2}{xyz}}$ ,  $\bar{A}_{\frac{1}{xyz}}$ , according to the formula (46) we in like manner obtain the present value of an absolute assurance of £1 payable immediately on the extinction of the last survivor of the three lives, viz.,

$$\bar{A}_{(xyz)} = \bar{A}_x + \bar{A}_y + \bar{A}_z - \bar{A}_{xy} - \bar{A}_{xz} - \bar{A}_{yz} + \bar{A}_{xyz} \dots \dots (48).$$

### Expectations of Survivorship.

By proceeding to the extreme hypothetical case in which money is supposed not to bear any interest, the values of all payments will become totally independent of time; and it is evident that, as before stated on page 112, annuities become expectations of life. In like manner it will appear that the values of contingent assurances become transformed into the corresponding abstract probabilities of survivorship. Hence the probabilities of survivorship are at once derived from the formulæ for contingent assurances by the mere substitution of expectations in place of annuities, observing that all absolute assurances become of course replaced by unity or certainty. Hence the following formulæ for determining these probabilities.

$$\left. \begin{aligned} E_{\frac{1}{xy}} &= \mu_x \bar{e}_{xy} - \frac{d\bar{e}_{xy}}{dx} \\ E_{\frac{1}{xyz}} &= \mu_x \bar{e}_{xyz} - \frac{d\bar{e}_{xyz}}{dx} \\ E_{\frac{2}{xyz}} &= E_{\frac{1}{xy}} + E_{\frac{1}{xz}} - 2E_{\frac{1}{xyz}} \\ E_{\frac{3}{xyz}} &= 1 - E_{\frac{1}{xy}} - E_{\frac{1}{xz}} + E_{\frac{1}{xyz}} \end{aligned} \right\} \dots \dots \dots (49)$$

NOTE.—The fundamental formulæ for assurances are comprehended in three types. The first of these is applicable generally, and may be written in an abbreviated form, by separating the symbols of operation, thus:  $\bar{A} = \left(\mu - \frac{d}{dx}\right) \bar{e}$ , which may be regarded as the radix of the formulæ (27), (31), (32), (40), and (41). The other two forms, which appertain exclusively to absolute assurances, are  $\bar{A} = 1 - \delta \bar{e}$  and  $\bar{A} = \delta \left(\frac{1}{i} - a + \frac{\mu}{12}\right)$ , and they are respectively exhibited in (28), (33), (42) and (29), (34), (43).

There is yet another form, to which special reference has not been made, viz.  $\bar{A}_x = -\frac{1}{l_x} \frac{d(l\bar{a})_x}{dx}$ ,  $\bar{A}_{\frac{1}{xy}} = -\frac{1}{l_{xy}} \frac{d(l_{xy}\bar{a}_{xy})}{dx}$ , which for calculation may be practically represented by

$$\bar{A}_x = \frac{(l\bar{a})_{x-1} - (l\bar{a})_{x+1}}{2l_x}$$

$$\bar{A}_{\frac{1}{xy}} = \frac{l_{x-1}\bar{a}_{x-1,y} - l_{x+1}\bar{a}_{x+1,y}}{2l_{xy}}.$$

The same considerations may be extended to the formula (27), expressing the value of an absolute assurance. When  $i=0$ ,  $\bar{A}$  becomes *unity* and  $a$  becomes  $\bar{e}$ ; thus we have the following remarkable relation between the true expectation of life and the force of mortality, viz.,

$$\left. \begin{aligned} 1 &= \mu \bar{e} - \frac{d\bar{e}}{dx} \\ \therefore \mu &= \frac{1}{\bar{e}} \left( 1 + \frac{d\bar{e}}{dx} \right) \\ &= \frac{1}{\bar{e}} \left\{ 1 - \frac{1}{2} (\bar{e}_{x-1} - \bar{e}_{x+1}) \right\} \end{aligned} \right\} \dots \dots \dots (50)$$

By this means the numerical value of the force of mortality might otherwise be deduced from a table of expectations, and the accuracy of the result would scarcely be affected by using the ordinary tabular expectations.

Enough has now been given in the present paper to establish the practical utility and logical consistency of the Continuous Method of dealing with annuities and assurances, and the general power and efficiency of the leading principles of the new theory as an instrument of investigation. There is, indeed, good ground to expect that the adoption of the method here presented will lead to more extended researches and a higher range of inquiry in this most important and interesting subject.

To bring the various formulæ into immediate numerical operation it is only necessary to prepare tables showing the values of the force of mortality and of continuous annuities on single and joint lives. These are easily deduced from existing tables by the use of the formulæ (30) and (20). As an example of what is requisite, a table is appended, showing the force of mortality, and the continuous single annuities at 3 per cent. according to the Northampton mortality. A second table, containing the force of discount, is in itself complete and independent, and will, of course, serve generally for all tables of mortality.

Force of Mortality, &amp;c., Northampton Table.

Age	$\log \mu$	$\mu$	$\bar{a}$ 3 per Cent.	Age	$\log \mu$	$\mu$	$\bar{a}$ 3 per Cent.
10	7.99422	.00987	21.1600	51	8.46774	.02936	12.6779
11	7.95760	.00907	20.9768	52	8.48342	.03044	12.4253
12	7.95288	.00897	20.7806	53	8.49684	.03139	12.1689
13	7.95680	.00905	20.5782	54	8.51069	.03241	11.9086
14	7.96074	.00914	20.3696	55	8.52500	.03350	11.6447
15	7.96473	.00922	20.1545	56	8.53980	.03466	11.3772
16	7.98159	.00958	19.9325	57	8.55512	.03590	11.1060
17	8.01838	.01043	19.7150	58	8.57100	.03724	10.8313
18	8.06060	.01150	19.5097	59	8.58748	.03868	10.5531
19	8.09699	.01250	19.3173	60	8.60461	.04024	10.2716
20	8.13170	.01354	19.1349	61	8.62244	.04192	9.9869
21	8.16214	.01453	18.9671	62	8.63839	.04349	9.6994
22	8.17740	.01505	18.8075	63	8.65490	.04518	9.4038
23	8.18398	.01527	18.6449	64	8.67229	.04702	9.1051
24	8.19066	.01551	18.4792	65	8.69037	.04902	8.7982
25	8.19745	.01576	18.3106	66	8.71220	.05155	8.4880
26	8.20435	.01601	18.1387	67	8.73518	.05435	8.1751
27	8.21136	.01627	17.9636	68	8.75945	.05747	7.8600
28	8.21848	.01654	17.7852	69	8.78516	.06098	7.5435
29	8.22573	.01682	17.6031	70	8.81248	.06494	7.2263
30	8.23309	.01710	17.4178	71	8.84164	.06944	6.9096
31	8.24058	.01740	17.2287	72	8.87290	.07463	6.5950
32	8.24821	.01771	17.0359	73	8.90658	.08065	6.2847
33	8.25597	.01803	16.8392	74	8.94310	.08772	5.9814
34	8.26387	.01836	16.6385	75	8.98297	.09615	5.6892
35	8.27192	.01870	16.4338	76	9.01865	.10439	5.4142
36	8.28012	.01906	16.2247	77	9.04576	.11111	5.1403
37	8.28847	.01943	16.0113	78	9.06859	.11711	4.8603
38	8.29700	.01982	15.7935	79	9.09528	.12453	4.5644
39	8.30569	.02022	15.5709	80	9.13501	.13646	4.2677
40	8.31744	.02077	15.3434	81	9.18035	.15148	3.9843
41	8.33233	.02149	15.1153	82	9.22808	.16908	3.7128
42	8.34747	.02226	14.8869	83	9.28729	.19377	3.4637
43	8.36011	.02291	14.6582	84	9.34259	.22009	3.2730
44	8.37017	.02345	14.4252	85	9.37885	.23925	3.0978
45	8.38048	.02401	14.1875	86	9.41266	.25862	2.9379
46	8.39104	.02461	13.9453	87	9.44604	.27928	2.7867
47	8.40186	.02523	13.6982	88	9.47009	.29518	2.6581
48	8.41295	.02588	13.4462	89	9.47478	.29839	2.4858
49	8.42711	.02674	13.1890	90	9.48337	.30435	2.2670
50	8.44718	.02800	12.9312				

Table of Force of Discount ( $\delta$ ).

Rate of Interest per cent.	$\delta$	$\log \delta$	Rate of Interest per cent.	$\delta$	$\log \delta$
$\frac{1}{2}$	.00499	7.69789	$5\frac{1}{2}$	.05354	8.72868
1	.00995	7.99784	6	.05827	8.76544
$1\frac{1}{2}$	.01489	8.17285	$6\frac{1}{2}$	.06297	8.79917
2	.01980	8.29672	7	.06766	8.83032
$2\frac{1}{2}$	.02469	8.39257	$7\frac{1}{2}$	.07232	8.85926
3	.02956	8.47069	8	.07696	8.88627
$3\frac{1}{2}$	.03440	8.53658	$8\frac{1}{2}$	.08158	8.91158
4	.03922	8.59352	9	.08618	8.93539
$4\frac{1}{2}$	.04402	8.64362	$9\frac{1}{2}$	.09075	8.95787
5	.04879	8.68833	10	.09531	8.97914

*On the value of Reversionary Annuities payable half-yearly, quarterly, &c., according to the conditions which prevail in practice. By THOMAS BOND SPRAGUE, M.A., Vice-President of the Institute of Actuaries.*

[Read before the Institute, 26th April, 1869.]

THE ordinary formula for the value of a reversionary annuity,  $a_x - a_{xy}$ , is not strictly applicable to the cases which occur in practice. For it assumes that the first payment of the annuity is made at the end of the year in which  $y$  dies, and the last payment at the end of the year before that in which  $x$  dies. But the conditions which prevail in practice are different; for whenever a reversionary annuity is granted, it will run from the death of  $y$ , and the first payment will be made twelve months after that death, i.e., on the average, six months later than is assumed in the above formula. On the other hand, the reversionary annuity will in practice be payable up to the day of the death of  $x$ ; or, on the average, a full year's payment will be made six months later than the last payment is made according to the above formula. Hence we may say that in the reversionary annuity of practice, exactly the same payments are made, on the average, as in the theoretical annuity; but that each of them is made six months later in the one case than in the other; and it follows that the value of the reversionary annuity of practice is, approximately,

$$\frac{a_x - a_{xy}}{\sqrt{1+i}}, \text{ or } (a_x - a_{xy}) \left( 1 - \frac{i}{2} + \frac{3}{8}i^2 - \frac{5}{16}i^3 + \dots \right) \dots (1)$$

Let us now look at the question from a rather different point of view. We may say approximately that in consequence of the reversionary annuity running from the death of  $y$  only, and not from the end of the previous year, the value is diminished by that of  $\mathcal{L}\frac{1}{2}$  payable on the death of  $y$ , in the lifetime of  $x$ , or by  $\frac{1}{2} A_{\frac{1}{x,y}}$ .

Again, we may say in the same way that in consequence of the annuity being payable up to the death of  $x$ , instead of ceasing at the end of the previous year, its value is increased by the value of  $\mathcal{L}\frac{1}{2}$  payable on the death of  $x$  after the death of  $y$ , or by  $\frac{1}{2} A_{\frac{2}{x,y}}$ .

Hence the correction to be applied to the ordinary formula will be

$$\frac{1}{2} A_{\frac{2}{x,y}} - \frac{1}{2} A_{\frac{1}{x,y}}.$$

But since

$$A_{\frac{x}{2}, y} = A_x - A_{\frac{1}{2}, y}, \text{ this becomes}$$

$$\begin{aligned} & \frac{1}{2} \left( A_x - A_{\frac{1}{2}, y} - A_{\frac{1}{2}, y} \right) \\ &= \frac{1}{2} (A_x - A_{xy}) \\ &= -\frac{1}{2} \frac{i(a_x - a_{xy})}{1+i}; \end{aligned}$$

and the value of the reversionary annuity of practice is approximately

$$\begin{aligned} & (a_x - a_{xy}) \left( 1 - \frac{1}{2} \cdot \frac{i}{1+i} \right) \\ &= (a_x - a_{xy}) \left( 1 - \frac{i}{2} + \frac{i^2}{2} - \&c. \right) \dots (2) \end{aligned}$$

which agrees with the former formula as far as the two first terms of the second factor. The conditions here supposed do not agree so closely with those of practice as those previously assumed; and the formula (1) is therefore to be considered preferable.

Next, suppose that the reversionary annuity, instead of being paid yearly, is payable half-yearly; then it is clear that exactly the same amount will be paid to the annuitant as before; but in each year there will be two payments of  $\pounds \frac{1}{2}$ , instead of a single payment of  $\pounds 1$ . Now the value of  $\pounds 1$  due in a year, is  $\frac{1}{1+i}$ ; and the value of the two payments of  $\pounds \frac{1}{2}$ , at the end of half a year and a year respectively, is  $\frac{1}{2\sqrt{1+i}} + \frac{1}{2(1+i)}$ . The latter value bears to the former a ratio equal to  $\frac{\sqrt{1+i}+1}{2}$ .

This being true of each year's payments, the value of the reversionary annuity payable half-yearly will be greater than that of the same annuity payable yearly in the same ratio; and will therefore be approximately

$$\begin{aligned} & \frac{a_x - a_{xy}}{\sqrt{1+i}} \cdot \frac{\sqrt{1+i}+1}{2} \\ &= (a_x - a_{xy}) \cdot \frac{1 + (1+i)^{-\frac{1}{2}}}{2} \\ &= (a_x - a_{xy}) \left( 1 - \frac{i}{4} + \frac{3}{16}i^2 - \frac{5}{32}i^3 + \dots \right) \end{aligned}$$

Next, if the reversionary annuity is payable  $m$  times a year, the

value at the beginning of a year of the payments  $\pounds \frac{1}{m}$ , to be made at  $m$  equal intervals during the year is

$$\begin{aligned} \frac{1}{m} \left( \frac{1}{v^m} + \frac{1}{v^{2m}} + \dots + v \right) \\ = \frac{1}{m} \frac{1-v}{1-v^m} \end{aligned}$$

and the ratio which this bears to the value of  $\pounds 1$  to be paid at the end of the year is

$$\begin{aligned} \frac{\frac{1}{m} \cdot \frac{1-v}{1-v^m} \cdot \frac{1}{v}}{\frac{1}{v^m-1}} &= \frac{1}{m} \cdot \frac{i}{v^{\frac{1}{m}}-1} \\ &= \frac{1}{m} \cdot \frac{i}{(1+i)^{\frac{1}{m}}-1} \\ &= 1 + \frac{i}{2} \left( 1 - \frac{1}{m} \right) - \frac{i^2}{12} \left( 1 - \frac{1}{m^2} \right) + \frac{i^3}{24} \left( 1 - \frac{1}{m^2} \right) \dots \end{aligned}$$

as I have shown in vol. xiii., p. 206.

Hence the value of the reversionary annuity payable  $m$  times a year is approximately

$$\frac{a_x - a_{xy}}{\sqrt{1+i}} \left\{ 1 + \frac{i}{2} \left( 1 - \frac{1}{m} \right) - \frac{i^2}{12} \left( 1 - \frac{1}{m^2} \right) + \frac{i^3}{24} \left( 1 - \frac{1}{m^2} \right) \dots \right\} \dots (3)$$

Here making  $m=2$ , we get the value of the annuity when payable half-yearly

$$\begin{aligned} \frac{a_x - a_{xy}}{\sqrt{1+i}} \left\{ 1 + \frac{i}{4} - \frac{i^2}{16} + \frac{i^3}{32} - \dots \right\} \\ = (a_x - a_{xy}) \left( 1 - \frac{i}{4} + \frac{3}{16} i^2 - \frac{5}{32} i^3 + \dots \right) \dots (4) \end{aligned}$$

as found above.

Again, making  $m=4$  the value becomes

$$\begin{aligned} \frac{a_x - a_{xy}}{\sqrt{1+i}} \left\{ 1 + \frac{3}{8} i - \frac{5}{64} i^2 + \frac{5}{128} i^3 - \dots \right\} \\ = (a_x - a_{xy}) \left\{ 1 - \frac{i}{8} + \frac{7}{64} i^2 - \frac{3}{32} i^3 + \dots \right\} \dots (5) \end{aligned}$$

which is the approximate value of the reversionary annuity when payable quarterly.

Making  $m$  infinite, the value of the annuity supposed to be payable momentarily, becomes

$$\frac{a_x - a_{xy}}{\sqrt{1+i}} \left( 1 + \frac{i}{2} - \frac{i^2}{12} + \frac{i^3}{24} - \dots \right) \\ = (a_x - a_{xy}) \left( 1 + \frac{i^2}{8} + \frac{5}{24} i^3 \dots \right) \dots \dots (6)$$

The treatment of this problem by former writers does not appear to have been happy. In a paper by the late Dr. Young, reprinted in this *Journal* (vol. vii., p. 22), it is stated that since the same correction, viz.  $\frac{1}{2}$ , is applied to each of the terms in  $a_x - a_{xy}$  when the annuity is payable half-yearly, we get the paradoxical result that the value of the reversionary annuity is the same whether payable yearly or half-yearly. His explanation of the paradox is by no means clear; and it appears to be impossible to conclude from his remarks what he believed to be the correct value of either the yearly or half-yearly reversionary annuity.

The following explanation will, I believe, be found satisfactory. The reversionary annuity of which the value is  $a_x - a_{xy}$ , runs, not from the death of  $y$ , but from the end of the previous year, so that a full year's payment is supposed to be made to  $x$  at the end of the year in which  $y$  dies. If, however, the annuity be payable half-yearly, then  $x$  will receive only  $\frac{1}{2}$  at the end of the year in which  $y$  dies; and the other  $\frac{1}{2}$  will be payable in the middle of that year, and will be received by  $y$ , should he be then alive, the chance of which, on the supposition of equal decrements, is  $\frac{1}{2}$ . Thus, by the alteration to half-yearly payments,  $x$  has an even chance of losing  $\frac{1}{2}$ ; or, taking the average, may be said to lose  $\frac{1}{4}$  payable in the middle of the year in which  $y$  dies; and this loss just counterbalances the advantage (also equal to  $\frac{1}{4}$ , when referred to the same epoch) he derives in consequence of the payments in the succeeding years being made half-yearly. The value of the annuity thus remains unaltered under the suppositions made.

In Vol. iv. p. 299, of this *Journal*, the following formula was given by Mr. Holmes Ivory, for the value of a reversionary annuity payable half-yearly for the life of  $x$  after the death of  $y$ ;

$$a_x - a_{xy} - \frac{1}{4} A \frac{1}{xy} \dots \dots \dots (7)$$

The reasoning by which this formula was arrived at does not seem to be satisfactory; and the result appears to be fairly open to the charge brought against it by Mr. Thomas Carr (Vol. vii. p. 110), that it represents the annuity when payable half-yearly as costing *less* than when payable yearly. It will be found that the formula gives approximately the value of a reversionary annuity payable

half-yearly, which runs from the death of  $y$ , but *ceases with the last half-yearly payment prior to the death of  $x$* , not being continued to the day of the death of  $x$ . For, as we have seen above, the value of the reversionary annuity payable yearly, which runs from the death of  $y$ , is

$$a_x - a_{xy} - \frac{1}{2} A \frac{1}{x.y} \dots \dots \dots (8)$$

this annuity not being continued to the death of  $x$ , but ceasing with the last yearly payment which precedes that death. If now this annuity is made payable half-yearly, its value at the time it is entered upon is increased by  $\frac{1}{4}$ ; and consequently its value at the present time is increased by  $\frac{1}{4} A \frac{1}{x.y}$ . Adding this to (8) we get

Mr. Ivory's formula; but there is no reason to suppose that he had in view the annuity here described. If any further proof were required of the accuracy of the views here expressed, it would be furnished by the fact that the formula given by Mr. Ivory for the value of a reversionary annuity payable quarterly, viz.:

$$a_x - a_{xy} - \frac{3}{8} A \frac{1}{x.y} \dots \dots \dots (9)$$

does not admit of any similar interpretation. For it is easily seen that the value of the reversionary annuity, payable quarterly, which runs from the death of  $y$ , but ceases with the last quarterly payment prior to the death of  $x$ , is

$$\begin{aligned} & \left( a_x - a_{xy} - \frac{1}{2} A \frac{1}{x.y} \right) + \frac{3}{8} A \frac{1}{x.y} \\ & = a_x - a_{xy} - \frac{1}{8} A \frac{1}{x.y} \end{aligned}$$

It will be seen that Mr. Ivory's formulæ represent the value of the annuity when payable quarterly as being less than when payable half-yearly.

Not having succeeded in fully understanding the drift of Mr. Carr's remarks, I will make no comment on them, but simply add that he, like Dr. Young, has omitted to give any formulæ for the value of reversionary annuities payable half-yearly, quarterly, &c.

It will be seen that the reasoning by which the formulæ (1) and (3) were established, applies equally to the values of reversionary annuities purchased as an investment. If the value of such an annuity payable yearly during the life of  $x$  after the death of  $y$ —the first payment being made at the end of the year in which  $y$  dies, and the last at the end of the year before that in which  $x$  dies—be  $a$ ; then the value of such an annuity

running from the death of  $y$  to the death of  $x$  is approximately

$$\frac{a}{\sqrt{1+i}} = a \left( 1 - \frac{i}{2} + \frac{3}{8}i^2 - \frac{5}{16}i^3 + \dots \right) \dots \dots \dots (10)$$

If, furthermore, the annuity is payable half-yearly, its value will be

$$a \left( 1 - \frac{i}{4} + \frac{3}{16}i^2 - \frac{5}{32}i^3 + \dots \right); \dots \dots \dots (11)$$

and if quarterly,

$$a \left( 1 - \frac{i}{8} + \frac{7}{64}i^2 - \frac{3}{32}i^3 + \dots \right) \dots \dots \dots (12)$$

These formulæ are applicable in whatever way we determine  $a$ . We may either with Mr. Jellicoe take it equal to

$$\frac{1}{P_x + d_5} - 1 - (a_{xy})_4 \dots \dots \dots (13)$$

or we may take either of the two formulæ I have explained in my paper *On the value of Reversionary Life Interests*, in the Number of the *Journal* for January last. It will be noticed however that

we must not, under the suppositions now made, subtract  $\frac{1}{2} A_{\frac{1}{xy}}$ , as

I have there done, so that instead of the formulæ there given we shall have for  $a$  the formulæ

$$\frac{1}{P_x + d_6} - 1 - (a_{xy})_6 \dots \dots \dots (14)$$

and

$$\frac{1 - (P_x + d_6)(1 + a_{xy})_6}{P_x + d_6} \dots \dots \dots (15)$$

As a first approximation to the truth, which will often be quite sufficient, we may say that interest for half a year is to be deducted from these values when the annuity is payable yearly; and interest for one quarter, and eighth part of a year respectively, when the annuity is payable half-yearly and quarterly; this interest being of course computed at six per cent.

In a similar manner we may deduce the approximate value of *any complete annuity* payable half-yearly or quarterly from the value of the same annuity payable yearly. We have only to multiply this value by

$$1 + \frac{i}{4} - \frac{i^2}{16} + \frac{i^3}{32} - \dots$$

and

$$1 + \frac{3i}{8} - \frac{5i^2}{64} + \frac{5i^3}{128} - \dots$$

respectively, or simply to add interest for a quarter of a year and three-eighths of a year respectively.

This simple rule applies to all annuities, whether whole life, temporary, deferred, or intercepted; and whether on one life or on several; provided only that the annuity is *complete*, or a proportionate part is payable up to the day of the death of the life or lives on which the annuity depends. For, if that is the case, it is very easy to see that whether the annuity be payable yearly, half-yearly, quarterly, or more frequently, the sum of the payments to the annuitant will in all cases be the same, the increase in the value arising solely from the interest on the sums prepaid.

This rule, however, is not strictly correct, even on the supposition of equal decrements. Comparing the cases of yearly and half-yearly payments, we see that in the latter case the annuitant gains half a year's interest on  $\mathcal{L}\frac{1}{2}$  for every year through which the nominee lives. But in the year in which the nominee dies, there is only an even chance of the annuitant obtaining any advantage, since if death occurs in the first half of the year, there is no advantage from the half-yearly payment of the annuity; and the advantage when death does occur in the second half of the year is limited to an average gain of three months' interest on  $\mathcal{L}\frac{1}{4}$ . The value given by the rule is therefore too large, and the error is approximately equal to

$$\left\{ \frac{\sqrt{1+i}-1}{2} - \frac{(1+i)^{\frac{1}{2}}-1}{4} \right\} A_x$$

$$= \left\{ \frac{3}{16}i - \frac{5}{128}i^2 + \frac{9}{512}i^3 - \dots \right\} A_x.$$

The error will therefore generally affect the third place of decimals, unless  $A_x$  be very small; but usually not the second place of decimals.

I will conclude this paper by finding an expression for the value of the annuity here considered, on the supposition of equal decrements, *i.e.* on the supposition that the deaths are distributed uniformly over each year of life.

First suppose that the annuity is payable yearly during the life of  $x$  on the anniversary of the death of  $y$ . Then considering the instant distant from the present time,  $n+t$ , where  $n$  is an integer and  $t < 1$ ; there will be a payment of the annuity due at that instant if (1)  $x$  be then alive and (2)  $y$  died at the time  $t$  in any one of the 1st, 2nd . . .  $n$ th years. Now the probability of  $x$  being alive at the time  $n+t$ , is, on the supposition of equal decrements,

$$(1-t)p_{x, n} + tp_{x, n+1}$$

Also the probabilities of  $y$  having died at the time  $t$  in the 1st, 2nd, . . .  $n$ th, years respectively, are

$$\begin{aligned} & (1-p_{y,1})dt \\ & (p_{y,1}-p_{y,2})dt \\ & (p_{y,2}-p_{y,3})dt \\ & \vdots \\ & (p_{y,n-1}-p_{y,n})dt \end{aligned}$$

And the sum of these, *i.e.* the probability of  $y$  having died at the time  $t$  in some one of the  $n$  years is

$$(1-p_{y,n})dt.$$

It is, in fact, obvious that on the suppositions we have made, assuming  $y$  to have died within the  $n$  years, he is as likely to have died at any time  $t$  ( $< 1$ ) as at any other such time. Hence the probability of a payment of the annuity falling due at the time  $n+t$ , is

$$\{(1-t)p_{x,n}+tp_{x,n+1}\}(1-p_{y,n})dt;$$

and the present value of the payment so made will be found by multiplying this quantity by  $v^{n+t}$ , and will therefore be

$$v^{n+t}\{(1-t)p_{x,n}+tp_{x,n+1}-(1-t)p_{xy,n}-tp_{x,n+1,y}\}dt.$$

The value of the annuity will now be found by integrating this expression with respect to  $t$  between the limits 0 and 1, and summing it with respect to  $n$  for the values 1, 2, 3 . . . to the end of life; and it is immaterial which of these operations we perform first.

Integrating first with respect to  $t$ , we have

$$\int v^t dt = \frac{v^t}{\log_e v} + C. \quad \therefore \int_0^1 v^t dt = -\frac{1-v}{\log_e v} = \frac{1-v}{\delta} = \frac{i}{(1+i)\delta},$$

putting  $\delta$  for  $-\log_e v$ .

Again

$$\begin{aligned} \int_0^1 tv^t dt &= \frac{1}{\delta^2} - \frac{v}{\delta} - \frac{v}{\delta^2} \text{ (vol. xiii., p. 359)} \\ &= \frac{i-\delta}{(1+i)\delta^2}, \end{aligned}$$

$$\text{whence} \quad \int_0^1 (1-t)v^t dt = \frac{i\delta-i+\delta}{(1+i)\delta^2}.$$

Thus the result of the integration becomes

$$v^n \frac{i\delta - i + \delta}{(1+i)\delta^2} (p_{x,n} - p_{xy,n}) + v^n \frac{i - \delta}{(1+i)\delta^2} (p_{x,n+1} - p_x p_{x+1,y,n}).$$

Next summing this with respect to  $n$ , for the values 1, 2, 3 . . . to the end of life, we have

$$\begin{aligned} \Sigma v^n p_{x,n} &= a_x; & \Sigma v^n p_{xy,n} &= a_{xy}; \\ \Sigma v^n p_{x,n+1} &= \frac{a_x}{v} - p_x; & \Sigma v^n p_{x+1,y,n} &= a_{x+1,y}. \end{aligned}$$

So that the result of the summation is

$$\begin{aligned} & \frac{i\delta - i + \delta}{(1+i)\delta^2} (a_x - a_{xy}) + \frac{i - \delta}{(1+i)\delta^2} \{ (1+i)a_x - p_x - p_x a_{x+1,y} \} \\ &= \frac{i^2}{(1+i)\delta^2} a_x - \frac{i\delta - i + \delta}{(1+i)\delta^2} a_{xy} - \frac{i - \delta}{\delta^2} \cdot \frac{a_{xy-1}}{p_{y-1}} \dots \dots (16) \end{aligned}$$

since 
$$a_{x,y-1} = \frac{p_x p_{y-1}}{1+i} (1 + a_{x+1,y}).$$

To test this formula, suppose that the life  $y$  is replaced by certainty; then  $p_{y-1}$  becomes 1; and  $a_{xy}$ ,  $a_{x,y-1}$  both become  $a_x$ ; and the formula becomes

$$\frac{i^2 - i\delta + i - \delta - (1+i)(i - \delta)}{(1+i)\delta^2} \cdot a_x, \text{ which } = 0.$$

This result is clearly correct, as in the case supposed the reversionary annuity is indefinitely postponed and will never be entered upon.

Again, suppose the life  $x$  replaced by certainty; then  $a_x$  becomes  $v + v^2 + \dots ad\ inf. = \frac{v}{1-v} = \frac{1}{i}$ . Also  $a_{xy}$  is replaced by  $a_y$ ; and  $\frac{a_{xy-1}}{p_{y-1}}$  by  $\frac{a_{y-1}}{p_{y-1}}$ , which is equal to  $\frac{1+a_y}{1+i}$ . Hence the formula becomes

$$\begin{aligned} & \frac{i}{(1+i)\delta^2} - \frac{i\delta - i + \delta}{(1+i)\delta^2} \cdot a_y - \frac{i - \delta}{(1+i)\delta^2} (1 + a_y) \\ &= \frac{1}{(1+i)\delta} - \frac{i}{(1+i)\delta} \cdot a_y \\ &= \frac{1 - ia_y}{(1+i)\delta}. \end{aligned}$$

Now by the suppositions we have made, the reversionary annuity has been replaced by a perpetual annuity certain running from the death of  $y$ , the value of which is that of  $\frac{1}{i}$  payable at the instant of  $y$ 's death or  $\frac{\bar{A}_y}{i}$ . But on the supposition of equal decrements we have (as is shown by Baily)

$$\bar{A}_y = \frac{i}{\delta} A_y = \frac{i}{\delta} \frac{1 - ia_y}{1 + i}.$$

So that the value of the reversionary annuity certain is  $\frac{1 - ia_y}{(1 + i)\delta}$ , which agrees with the value already found.

Let us next consider the case when the annuity is "complete"; and investigate the value of the addition that is to be made to (16) on account of the payment up to the day of death. Suppose that  $y$  dies in some year at the time  $t$ , which, as before, is  $< 1$ ; then the annuity is payable at the time  $t$  in each subsequent year during the life of  $x$ ; and when  $x$  dies, a proportionate part is payable from the date of the last payment to the death of  $x$ . Suppose that  $x$  dies at the time  $\tau$ , reckoned (like  $t$ ) from the beginning of the year; then there are two cases according as  $\tau$  is  $>$  or  $< t$ . If  $\tau > t$ , or  $x$  die after the payment of the annuity has been made in any year, there will be  $\tau - t$  payable; but if  $\tau < t$ , or  $x$  die before the payment of the annuity is made, there will be  $1 - t + \tau$  payable.

First suppose  $\tau > t$ ; then the value of the correction, supposing that  $x$  dies in the  $n$ th year, and therefore at the time  $n - 1 + \tau$  reckoned from the present time, is

$$(\tau - t)v^{n-1+\tau}(p_{x,n-1} - p_{x,n})(1 - p_{y,n})dt d\tau$$

This has to be integrated with respect to  $\tau$ , between the limits  $t$  and 1; then it has to be integrated with respect to  $t$  between the limits 0 and 1; and lastly it has to be summed with respect to  $n$  giving it the values 1, 2, . . . to the end of life. Now

$$\begin{aligned} \int (\tau - t)v^{\tau} d\tau &= -\frac{(\tau - t)v^{\tau}}{\delta} + \int \frac{v^{\tau}}{\delta} d\tau \\ &= -\frac{\tau - t}{\delta} \cdot v^{\tau} - \frac{v^{\tau}}{\delta^2} + C \end{aligned}$$

$$\therefore \int_t^1 (\tau - t)v^{\tau} d\tau = \frac{v^t}{\delta^2} - \frac{1 - t}{\delta} v - \frac{v}{\delta^2} = H, \text{ suppose.}$$

Next integrating with respect to  $t$

$$\int H dt = -\frac{v^t}{\delta^3} - \left(t - \frac{t^2}{2}\right) \frac{v}{\delta} - \frac{vt}{\delta^2} + C$$

$$\begin{aligned} \therefore \int_0^1 H dt &= \frac{1}{\delta^3} - \frac{v}{\delta^3} - \frac{v}{2\delta} - \frac{v}{\delta^3} \\ &= \frac{i}{(1+i)\delta^3} - \frac{1}{2(1+i)\delta} - \frac{1}{(1+i)\delta^3} \\ &= \frac{2i - \delta^2 - 2\delta}{2(1+i)\delta^3}. \end{aligned}$$

This has now to be multiplied into

$$\begin{aligned}
 & \Sigma v^{n-1} (p_{x,n-1} - p_{x,n}) (1 - p_{y,n}) \\
 &= \Sigma v^{n-1} \left( p_{x,n-1} - p_{x,n} - \frac{p_{x-1,y,n}}{p_{x-1}} + p_{xy,n} \right) \\
 &= 1 + a_x - \frac{a_x}{v} - \frac{a_{x-1,y}}{vp_{x-1}} + \frac{a_{xy}}{v} \\
 &= 1 - ia_x - (1+i) \left( \frac{a_{x-1,y}}{p_{x-1}} - a_{xy} \right).
 \end{aligned}$$

Thus the value of the correction when  $\tau > t$  becomes

$$\frac{2i - \delta^2 - 2\delta}{2(1+i)\delta^3} \left\{ 1 - ia_x - (1+i) \left( \frac{a_{x-1,y}}{p_{x-1}} - a_{xy} \right) \right\} \dots (17)$$

Next suppose  $\tau < t$ ; then the value of the correction on the supposition that  $x$  dies in the  $(n+1)$ th year is

$$(1-t+\tau)v^{n+\tau}(p_{x,n}-p_{x,n+1})(1-p_{y,n})dtd\tau.$$

This has to be integrated with respect to  $\tau$  between 0 and  $t$ ; then to be integrated with respect to  $t$  between 0 and 1; and lastly to be summed with respect to  $n$  giving it the values 1, 2, . . . to the end of life.

$$\begin{aligned}
 \text{First, } \int (1-t+\tau)v^{\tau}d\tau &= -(1-t+\tau)\frac{v^{\tau}}{\delta} + \int \frac{v^{\tau}d\tau}{\delta} \\
 &= -(1-t+\tau)\frac{v^{\tau}}{\delta} - \frac{v^{\tau}}{\delta^2} + C
 \end{aligned}$$

$$\therefore \int_0^t (1-t+\tau)v^{\tau}d\tau = \frac{1-t}{\delta} + \frac{1}{\delta^2} - \frac{v^t}{\delta} - \frac{v^t}{\delta^2} = K, \text{ suppose.}$$

Next integrating with respect to  $t$

$$\begin{aligned}
 \int Kdt &= \frac{1}{\delta} \left( t - \frac{t^2}{2} \right) + \frac{t}{\delta^2} + \frac{v^t}{\delta^2} + \frac{v^t}{\delta^3} + C \\
 \therefore \int_0^1 Kdt &= \frac{1}{2\delta} + \frac{1}{\delta^2} + \frac{v}{\delta^2} + \frac{v}{\delta^3} - \frac{1}{\delta^2} - \frac{1}{\delta^3} \\
 &= \frac{1}{2\delta} + \frac{1}{(1+i)\delta^2} - \frac{i}{(1+i)\delta^3} \\
 &= \frac{(1+i)\delta^2 + 2\delta - 2i}{2(1+i)\delta^3}.
 \end{aligned}$$

And this has to be multiplied into

$$\begin{aligned}
 & \Sigma v^n (p_{x,n} - p_{x,n+1})(1 - p_{y,n}) \\
 &= \Sigma v^n (p_{x,n} - p_{x,n+1} - p_{xy,n} + p_{x,p_{x+1,y,n}}) \\
 &= a_x - \left( \frac{a_x}{v} - p_x \right) - a_{xy} + p_x a_{x+1,y} \\
 &= -ia_x - a_{xy} + (1+i) \frac{a_{x,y-1}}{p_{y-1}};
 \end{aligned}$$

so that the value of the correction when  $\tau < t$  is

$$\frac{(1+i)\delta^2 + 2\delta - 2i}{2(1+i)\delta^3} \left\{ -ia_x - a_{xy} + (1+i) \frac{a_{x-y-1}}{p_{y-1}} \right\} \dots (18)$$

The total value of the correction is therefore, adding together (17) and (18),

$$\begin{aligned} & \frac{2i - \delta^2 - 2\delta}{2(1+i)\delta^3} \left\{ 1 - ia_x - (1+i) \frac{a_{x-1-y}}{p_{x-1}} + (1+i)a_{xy} \right\} \\ & + \frac{(1+i)\delta^2 + 2\delta - 2i}{2(1+i)\delta^3} \left\{ -ia_x - a_{xy} + (1+i) \frac{a_{x-y-1}}{p_{y-1}} \right\} \\ & = \frac{2i - \delta^2 - 2\delta}{2(1+i)\delta^3} - \frac{i^2}{2(1+i)\delta} \cdot a_x + \frac{(2i - \delta^2 - 2\delta)(2+i) - i\delta^3}{2(1+i)\delta^3} \cdot a_{xy} \\ & - \frac{2i - \delta^2 - 2\delta}{2\delta^3} \cdot \frac{a_{x-1-y}}{p_{x-1}} + \frac{(1+i)\delta^2 + 2\delta - 2i}{2\delta^3} \cdot \frac{a_{x-y-1}}{p_{y-1}} \dots (19) \end{aligned}$$

To test this formula, suppose  $x$  replaced by certainty, then we get

$$\begin{aligned} & \frac{2i - \delta^2 - 2\delta}{2(1+i)\delta^3} - \frac{i}{2(1+i)\delta} + \frac{(2i - \delta^2 - 2\delta)(2+i) - i\delta^3}{2(1+i)\delta^3} \cdot a_y \\ & - \frac{2i - \delta^2 - 2\delta}{2\delta^3} \cdot a_y + \frac{(1+i)\delta^2 + 2\delta - 2i}{2\delta^3(1+i)} (1 + a_y) \end{aligned}$$

which vanishes identically, as it should.

Next suppose  $y$  replaced by certainty, then we get

$$\begin{aligned} & \frac{2i - \delta^2 - 2\delta}{2(1+i)\delta^3} - \frac{i^2}{2(1+i)\delta} \cdot a_x + \frac{(2i - \delta^2 - 2\delta)(2+i) - i\delta^3}{2(1+i)\delta^3} \cdot a_x \\ & - \frac{2i - \delta^2 - 2\delta}{2\delta^3(1+i)} (1 + a_x) + \frac{(1+i)\delta^2 + 2\delta - 2i}{2\delta^3} \cdot a_x \end{aligned}$$

which also vanishes identically, as it should.

Adding (19) to (16) we should get a formula for the value of the complete reversionary annuity payable yearly; but this is so complicated as to be practically useless; and no good purpose would be gained by investigating the corresponding formulæ for the values of the annuity when payable half-yearly, &c.

By a process similar to the above it may be shown that the value of a contingent assurance payable at the instant of the death of  $x$  provided  $y$  survive, or  $\bar{A}_{xy}^1$ , is, on the supposition of equal decrements,

$$\frac{i\delta - i + \delta}{(1+i)\delta^2} + \frac{2i\delta - 2i + 2\delta - i^2}{(1+i)\delta^2} a_{xy} - \frac{i\delta - i + \delta}{\delta^2} \cdot \frac{a_{x-y-1}}{p_{y-1}} + \frac{i - \delta}{\delta^2} \cdot \frac{a_{x-1-y}}{p_{x-1}} \dots (20)$$

Also that the value of an assurance payable at the instant of the death of the first of the two joint lives  $x$  and  $y$ , or  $\bar{A}_{xy}$ , is, on the same supposition,

$$2 \frac{i\delta - i + \delta}{(1+i)\delta^2} + 2 \frac{2i\delta - 2i + 2\delta - i^2}{(1+i)\delta^2} a_{xy} + \frac{2i - 2\delta - i\delta}{\delta^2} \left( \frac{a_{x-1 \cdot y}}{p_{x-1}} + \frac{a_{x \cdot y-1}}{p_{y-1}} \right). \dots (21)$$

*On the Value of Reversionary Life Interests. By the late GRIFFITH DAVIES, ESQ., F.R.S. Communicated by GRIFFITH DAVIES, ESQ., Actuary of the Law Life Assurance Society.*

TO find the present value of a Survivorship Annuity of £1 on the life of A to be entered on at the death of B, supposing the purchaser to effect an increasing insurance on A's life so that the sum insured may correspond with the amount of his outlay (of purchase money, premiums, and interest) at the end of each year up to the average duration of the joint lives of A and B, and to be at the end of that period of the exact amount which the annuity purchased would cover the annual interest thereon, beside providing for the premiums to keep it insured during the remainder of A's life.

Let  $S$  = sum advanced as purchase money for proposed annuity.

$P$  = equated annual premium for requisite insurance.

$p$  = annual premium for the insurance of £1 on A's life.

$e$  = expectation of joint lives.

$i$  = rate of interest allowed the purchaser.

$r = 1 + i$  = amount of £1.

$v = \frac{1}{r}$  = present worth of £1.

$d = 1 - v$ .

D, N, M, &c., as in Table XIII. (Davies' "Tables of Life Contingencies," published in 1825).

Then it is manifest that the improved amount of the purchase money at the end of  $x$  years =  $Sr^x$ ; the improved amount of the premiums at the end of the same period =  $\frac{Pr(r^x - 1)}{i}$ , and consequently the sum of both which then constitutes the aggregate outlay of the purchaser =  $Sr^x + \frac{Pr(r^x - 1)}{i}$ , and if in this expression we substitute  $e$  for  $x$  we have the amount of the purchaser's outlay at the end of the period which A and B may be expected to

live together  $= Sr^e + \frac{Pr(r^e-1)}{i}$ , the annual interest on which  $= Sir^e + Pr^{e+1} - Pr$ , and if to this we add the annual premium  $P$  we have  $Sir^e + Pr^{e+1} - Pr + P = Sir^e + Pr^{e+1} - Pi = 1$  by the conditions of the problem.

$$\therefore P = \frac{1 - Sir^e}{r^{e+1} - i} = \frac{v^e - Si}{r - iv^e}$$

Referring again to the foregoing expression, we have the amount of the purchaser's outlay

$$\text{at the end of } x \text{ years} \quad . \quad . \quad . \quad Sr^x + \frac{Pr(r^x-1)}{i}$$

$$\text{at the end of } (x-1) \text{ years} \quad . \quad Sr^{x-1} + \frac{Pr(r^{x-1}-1)}{i}$$

The increment during the  $x$ th year  $= Sr^{x-1}(r-1) + \frac{Pr^x(r-1)}{i}$   
 $= Sir^{x-1} + Pr^x$ , and if in this expression we make  $x$  successively  $= 1, 2, 3, \dots e$  we find that

the increment during the 1st year  $= Si + Pr$

Do. 2nd „  $= (Si + Pr)r$

Do. 3rd „  $= (Si + Pr)r^2$

. . . . .

Do. eth „  $= (Si + Pr)r^{e-1}$

Hence assuming that the sum assured cannot be claimed from the Insurance Office until the end of the year in which  $A$  may die it is manifest, in order to satisfy the conditions of the problem, the following insurances must be effected, viz.:—the purchase money and the first year's increment as an immediate insurance on  $A$ 's life; and the second year's increment as a further insurance on the same life deferred for one year, the third year's as a still further insurance deferred for two years—and so on to the average duration of the joint lives of  $A$  and  $B$ .

It therefore follows, from the construction of the Table referred to, that the single premium for the

$$\text{1st of the said insurances} = \frac{M}{D} \{S + (Si + Pr)\}$$

$$\text{2nd do.} = \frac{{}^1Mr}{D} \{Si + Pr\}$$

$$\text{3rd do.} = \frac{{}^2Mr^2}{D} \{Si + Pr\}$$

. . . . .

$$\text{eth do.} = \frac{{}^{e-1}Mr^{e-1}}{D} \{Si + Pr\}$$

The sum of which

$$= \frac{SM}{D} + \frac{Si + Pr}{D} (M + {}^1Mr + {}^2Mr^2 + \dots + {}^{e-1}Mr^{e-1})$$

and dividing this sum by  $(1 + A)$  or its equivalent  $\frac{{}_1N}{D}$  we have the equated annual premium  $P$ .

$$P = \frac{SM}{{}_1N} + \frac{Si + Pr}{{}_1N} (M + {}^1Mr + {}^2Mr^2 + \dots + {}^{e-1}Mr^{e-1})$$

But  $\frac{M}{{}_1N}$  = annual premium for the assurance of £1, which put  $= p$ ; and if we put  $\frac{1}{{}_1N} (M + {}^1Mr + {}^2Mr^2 + \dots + \&c.) = Q$ , leaving the method of determining the value of  $Q$  to be shown hereafter, we have

$$P = Sp + SiQ + PrQ, \text{ or } P = \frac{S(p + iQ)}{1 - rQ}.$$

Hence equating this expression with the value of  $P$  already determined in terms of  $S$ , we have

$$\frac{S(p + iQ)}{1 - rQ} = \frac{v^e - Si}{r - iv^e}, \text{ from which,}$$

$$S = \frac{v - Q}{p(r^e - d) + d(r^e - iQ)}, \text{ and } P = \frac{v(p + iQ)}{p(r^e - d) + d(r^e - iQ)}.$$

Having now determined the values of  $S$  and  $P$  in terms of  $Q$  and known quantities, it only remains to point out the method of finding the value of  $Q$ . For this purpose, let  $n$  represent the age of the life  $A$ , on which the annuity depends; and let

$$L = Mr^n + {}^1Mr^{n+1} + {}^2Mr^{n+2} + \&c.$$

$${}^eL = {}^eMr^{n+e} + {}^{e+1}Mr^{n+e+1} + \&c.$$

$$\text{and } K = {}_1Nr^n, \text{ then } Q = \frac{L - {}^eL}{K}.$$

The annexed Table has been constructed from these values of  $L$  and  $K$ .

The foregoing is a copy of an investigation made, at least 40 years ago, by the late Mr. Griffith Davies; and the formula was used by him in cases relating to Survivorship Annuities.

*Survivorship Annuities, &c., to pay 5 per Cent beside the Premium charged for Assurances by the Northampton 3 per Cent.*

Age.	K.	L.	Age.	K.	L.	Age.	K.	L.
14	149532.0	263082.7	42	120195.3	162521.2	70	36615.9	40158.6
15	149486.3	260274.0	43	118005.4	158212.7	71	33475.5	36490.8
16	149362.5	257391.6	44	115733.9	153867.1	72	30410.8	32953.0
17	149156.4	254433.1	45	113380.9	149486.0	73	27436.4	29557.8
18	148868.0	251400.2	46	110946.9	145071.2	74	24567.7	26318.5
19	148500.8	248297.6	47	108432.1	140624.6	75	21822.0	23249.2
20	148059.3	245130.7	48	105837.5	136148.2	76	19217.3	20364.9
21	147545.9	241903.8	49	103163.4	131644.3	77	16772.8	17681.4
22	146966.6	238623.5	50	100411.4	127115.4	78	14495.5	15202.3
23	146323.9	235293.6	51	97585.0	122566.7	79	12387.3	12926.4
24	145616.6	231914.0	52	94691.6	118006.5	80	10444.8	10847.4
25	144842.9	228484.4	53	91736.6	113441.2	81	8673.4	8967.2
26	144001.7	225004.6	54	88723.4	108874.9	82	7083.0	7292.1
27	143091.0	221474.4	55	85655.2	104311.9	83	5678.6	5823.7
28	142109.8	217893.8	56	82535.5	99756.9	84	4465.3	4563.1
29	141056.6	214262.7	57	79369.0	95214.9	85	3452.6	3516.1
30	139929.9	210581.0	58	76160.4	90691.2	86	2623.9	2662.9
31	138728.1	206848.8	59	72914.6	86191.5	87	1959.0	1981.4
32	137449.9	203066.1	60	69637.2	81721.8	88	1436.1	1447.0
33	136094.2	199233.1	61	66334.6	77288.6	89	1034.5	1037.9
34	134659.5	195349.9	62	63013.4	72898.8	90	725.7	724.7
35	133144.3	191416.7	63	59680.9	68559.7	91	489.2	486.2
36	131547.7	187433.9	64	56341.4	64275.7	92	310.1	304.8
37	129868.3	183401.8	65	53003.5	60054.9	93	175.9	173.0
38	128105.0	179320.9	66	49672.5	55902.3	94	84.2	82.5
39	126256.8	175191.7	67	46357.5	51826.7	95	30.9	30.1
40	124322.8	171014.8	68	43069.0	47837.5	96	6.3	6.1
41	122301.8	166791.0	69	39817.9	43944.6			

*On a Method of obtaining De Moivre's formula in the simplest terms.* By M. CHARLON, Manager of the "Spanish Phoenix," Madrid. Communicated by M. HENRIQUEZ PIMENTEL, Professor of Mathematics at The Hague, Holland.\*

WHEN we consider in a table of mortality the consecutive terms which express the number living at each age, we remark that the differences between those numbers are almost constant in the middle ages of life; they change afterwards, remaining however uniform for another series of terms; so that the table can be considered as a series of decreasing arithmetical progressions. Let  $c$  be the difference between  $l_x$  and  $l_{x+1}$ ; and let us suppose this difference constant for  $n$  years. We shall then have for the value of the life annuity during that time

\* The communication of M. Pimentel is written in French; but we have thought it more useful to give a translation. We have also altered the notation in some respects. —ED. J. I. A.

$$\frac{1}{l_x} \{ (l_x - c)v + (l_x - 2c)v^2 + (l_x - 3c)v^3 + \dots + (l_x - nc)v^n \};$$

which we can write

$$v + v^2 + v^3 + \dots + v^n - \frac{c}{l_x} (v + 2v^2 + 3v^3 + \dots + nv^n)$$

The first series is equal to  $\frac{1-v^n}{i}$ , or  $I_n$ , suppose. If we put

$I'_n$  for the second series, neglecting the common factor  $\frac{c}{l_x}$ , and subtract from it the preceding, we shall find

$$I'_n - I_n = v^2 + 2v^3 + \dots + (n-1)v^n.$$

Multiplying the two members of this equation by  $1+i$ , or  $\frac{1}{v}$ , we shall have

$$(I'_n - I_n)(1+i) = v + 2v^2 + \dots + (n-1)v^{n-1}.$$

Comparing this expression with the value of  $I'_n$ , we find

$$(I'_n - I_n)(1+i) = I'_n - nv^n;$$

whence 
$$I'_n = \frac{I_n(1+i) - nv^n}{i}.$$

Multiplying by  $\frac{c}{l_x}$ , which we have neglected, and adding the first term, the total value of the life annuity for  $n$  years will be

$$I_n - \frac{c}{l_x} \cdot \frac{I_n(1+i) - nv^n}{i}.$$

If after  $n$  years, *i.e.* from the age  $x+n$ , the difference changes and becomes  $c'$ , which remains constant for  $n'$  years, we must add a like quantity, discounted to the age  $x$ , that is to say, multiplied by  $\frac{l_{x+n}}{l_x} \cdot v^n$ . This will be

$$\left\{ I_n - \frac{c'}{l_{x+n}} \cdot \frac{I_n(1+i) - n'v^{n'}}{i} \right\} \frac{l_{x+n}}{l_x} \cdot v^n.$$

If after  $n'$  years, we suppose a third difference  $c''$  constant for  $n''$  years, the third series of the annuity will be expressed by

$$\left\{ I_{n''} - \frac{c''}{l_{x+n+n'}} \cdot \frac{I_{n''}(1+i) - n''v^{n''}}{i} \right\} \frac{l_{x+n+n'}}{l_x} \cdot v^{n+n'};$$

and we shall thus have as many terms as there are differences in the table of mortality which we use, from the age  $x$  to the end of life. Let us suppose, to fix the ideas, that, in this interval, there are five constant differences  $c, c', c'', c''', c''''$ , for  $n, n', n'', n''', n''''$  years;

and, for brevity put,  $n+n'=\ell'$ ,  $n+n'+n''=\ell''$ ,  $n+n'+n''+n'''=\ell'''$ , &c. We shall then have for the value of the annuity multiplied by  $i l_x$ , putting for  $I_n$ ,  $I_{n'}$ ,  $I_{n''}$ , . . . , in the first terms their values, and multiplying out,

$$\begin{aligned} & l_x - l_x v^n - c I_n (1+i) + n c v^n \\ & + l_{x+n} v^n - l_{x+n} v^{n'} - c' I_{n'} v^{n-1} + n' c' v^{n'} \\ & + l_{x+n'} v^{n'} - l_{x+n'} v^{n''} - c'' I_{n''} v^{n'-1} + n'' c'' v^{n''} \\ & + l_{x+n''} v^{n''} - l_{x+n''} v^{n'''} - c''' I_{n'''} v^{n''-1} + n''' c''' v^{n'''} \\ & + l_{x+n'''} v^{n'''} - l_{x+n'''} v^{n''''} - c'''' I_{n''''} v^{n'''-1} + n'''' c'''' v^{n''''} \end{aligned}$$

But, since  $l_x - l_{x+n} = n c$ ,  $l_{x+n} - l_{x+n'} = n' c'$ , &c., and lastly,  $l_{x+n'''} = n'''' c''''$ , we see that the terms  $n c v^n - l_x v^n + l_{x+n} v^n$  in the above expression cancel one another. In the same way

$$n' c' v^{n'} - l_{x+n} v^{n'} + l_{x+n'} v^{n'} = 0, \text{ \&c.}$$

The value of the annuity will thus be reduced to

$$\frac{1}{i l_x} \{ l_x - (1+i) (c I_n + c' I_{n'} v^n + c'' I_{n''} v^{n'} + c''' I_{n'''} v^{n''} + c'''' I_{n''''} v^{n'''} ) \}.$$

If we replace  $I_n$ ,  $I_{n'}$ , &c., in this formula, by their equivalents, we shall find for the value of the annuity

$$\frac{1}{i} - \frac{1+i}{i^2 l_x} \{ c - (c-c') v^n - (c'-c'') v^{n'} - (c''-c''') v^{n''} - (c'''-c''') v^{n'''} - c'''' v^{n''''} \}.$$

The law of the terms here is evident, and we can at once write down the value of the annuity whatever the number of hypotheses that we choose to make.

If we suppose that there is only one difference, and that it is equal to 1, all the terms  $c-c'$ ,  $c'-c''$ , &c., vanish;  $l_x$  becomes equal to  $n$ , and the formula reduces to

$$\frac{1}{i} - \frac{1+i}{i^2 n} (1-v^n)$$

or, replacing  $1-v^n$  by its equivalent  $i I_n$ ,

$$\frac{1}{i} - \frac{1+i}{n i} I_n$$

which is just De Moivre's formula in the simplest form.

*On Assurances against the Risk of "Invalidity" or Permanent Inability to Work.*

LIFE Assurance provides for the family of the deceased in case of premature death; deferred Annuities provide for old age; but both institutions leave uncovered the risk of premature inability to work.

Invalidity Assurance, including the benefits of a deferred Annuity, would be the real complement to Life Assurance. This truth is so deeply felt in Germany, that a good many institutions, employing a large number of officers, workmen, and labourers; many mills, and particularly the Railway Companies, long since directed their attention to the providing for their officers in case of their being invalided. How were they to calculate the annual contribution, how to make the valuation of their liabilities?

There are no data, or at least very insufficient ones, upon which to calculate for different ages the probability of "becoming an invalid" during the next year; and this want induced Dr. Heym in Leipzig in 1851, when he had to make the computation of Invalidity Annuities for the Leipzig and Dresden Railway Society, to establish the hypothesis that the probability of becoming an invalid during the next year is  $\cdot 0001$  at the age of 20, and 1 at the age of 79, and that it forms a geometrical progression in the interval. The results which he derived from this hypothesis agreed with the observations which Professor Hülse had made on the Invalidity of miners in Saxony.

Dr. Wiegand accepted this hypothesis as a basis for his calculation of Invalidity Annuities for the Thuringian Railway Societies, published 1859, as "*Mathematical foundation of Railway Officers' Invalidity Annuities*" (*Mathematische Grundlagen für Eisenbahn Pensionscassen*), with the single modification that he made the geometrical progression cease at the age of 68, supposing the probability of becoming an invalid equal to 1 at the age of 69. The same hypothesis served as a basis for the Invalidity Annuities of Physicians and for the calculations in Wiegand's work, "*Invalidity Assurance*" (*Versicherung gegen Erwerbsunfähigkeit*. Halle, 1865).

But Dr. Wiegand did not stop here. Invalidity Assurance should not remain based upon a mere hypothesis. He made it his object to open the source of experience, which is to furnish sufficiently trustworthy data for the computation of the probability of becoming an invalid. Such data are in possession of the Railway Companies; and at last Dr. Wiegand sees himself, after numerous exertions, in sight of his aim. He avails himself of a moment of repose in his labours to report on what has been achieved and on what is to follow. His survey is contained in a very valuable and interesting work "*On Mortality and Invalidity Statistics for Railway Officials*" (*Die Mortalitäts- und Invaliditäts-Statistik bei Eisenbahn-Beamten Actenmässige Darstellung der darauf bezüglichen Operationen*).

After a short historical introduction, relating the facts just mentioned, which induced him to take up the question of the probability of becoming an Invalid, Dr. Wiegand places before the reader all the measures he had to take before he at last succeeded in obtaining the full support of the different Railway Companies which are now ready to furnish the data of their experience. It will be impossible here to enumerate all his steps. I must refer the readers of this Magazine to the just-named work; and I am sure they will not withhold their admiration from the man who by his never flagging zeal and energy, by enormous labour, and by untiring perseverance, at last persuaded the leading members of the Railway Companies of the expediency of his requests, so that they not only furnished some data just at hand, but elected a Committee, of which Dr. Wiegand is a Member, for the full examination of the Invalidity of Railway Officials.

The object of Dr. Wiegand is by no means confined to the law of Invalidity amongst Railway Officials. He is well aware of the importance of Invalidity Assurance in general, and he has no doubt that Life Assurance Offices will willingly cultivate this branch, as soon as they possess the necessary data for its computation. An Invalidity Table of Railway Officials will, in Dr. Wiegand's opinion, enable them to grant Invalidity Assurance to persons in any occupation, as the Invalidity to be expected in other pursuits is less than with Railway Officials, the Railway Service requiring so vigorous a constitution, that it must pension off its Officials, as invalids, in a state of health which would not entitle them to be considered as invalid in another occupation.

It is scarcely necessary to add that Dr. Wiegand, while he puts at the head of his researches the Invalidity, has not neglected the Mortality of Railway Officials. He says (page 4), "It is nearly an axiom with Life Assurance Societies that railway Officials are subject to a very high rate of mortality. Constant reports of railway accidents, where so and so many officials have perished, must create the persuasion that this class has a very low expectation of life. That is the reason why some Offices entirely refuse to assure Railway Officials, and others only accept them with a very considerable extra premium. But there exists no real measure for the extra risk founded upon experience, and each Office is guided only by the vague feelings of its manager. This state of things is unreasonable. If there exists an extra risk, its true value must be found out by observation."

With reference to Mortality, Dr. Wiegand has examined the materials hitherto furnished to him by the Railway Societies; and although the data are by no means numerous enough to establish a general truth, the results of his examination are important and interesting.

Among 10952 Railway Officials under observation during a year, of whom 2193 belonged to the train officials, the mortality has been 124 and 25 respectively. According to the Experience table it ought to have been 134·2179 and 24·528.

Among 78379 Railway Officials under observation 415 cases of Invalidity have occurred, while, according to Dr. Wiegand's Hypothesis, there would have been 401·789 invalids.

The coincidence is surprising, but only an examination based on more numerous facts will decide about the real law.

*Hamburg, May, 1869.*

WILHELM LAZARUS.

*Notes on Newton's Formulæ for Interpolation.*

I.

NEWTON, in a celebrated lemma (*Princ. Phil. Nat. Math.*, Lib. iii., Lemma v.), proposed and solved the fundamental problem of interpolation by finite differences; and thereby, to say the least, gave us the foundation of the theory of differences.

In the *Methodus Differentialis*, printed in 1715, he treats the same subject more at length; but after a careful examination, I cannot but think that this little treatise was written many years before the said lemma, which bears all appearance of being the ripe fruit of his researches on this matter.

On the other hand, the *Methodus Differentialis* has a great interest of its own, as showing how he dealt with such problems, beginning in the most direct and elementary way, and then—when he had overcome the first difficulties—going on so fast and with such large steps, that it requires no slight attention to follow him.

His solution of the problem is certainly as general, as elegant, and as simple, as we might expect from *summus*\* Newton; but alas,—he has given no proof of it! At least his commentators tell us so, and among them were men of great knowledge and high genius; even Stirling gives it as his opinion that Newton has not chosen the best manner of treating the problem. Nevertheless I am inclined not only to think that Newton's way indeed is the easiest and best, but even to suppose that he regarded his solution as all but self-evident; at least the proof, when undertaken in the right way, is very easy, as we shall see.

Instead of giving a copy of the latin text or a verbatim translation of the said lemma, I have preferred to give a sort of paraphrase. The only alterations made are: 1° that I have omitted the first case (equidistant arguments) as only special, 2° that instead of Newton's *geometrical* language and notation I have used the now commonly used analytical expressions, and a notation in which the arguments are denoted by small letters, the corresponding values of the functions by capitals, and the *divided differences* by a  $\delta$  with dashes indicating their order, and followed by the letters (in brackets) indicating the arguments on which the difference depends. I hope that these alterations may be found immaterial as to the meaning, and of some assistance to the reader. The paraphrase runs thus:

#### LEMMA V.

"Of an unknown function we know  $(n+1)$  values, A, B, C, D, E . . . . , corresponding to the arguments  $a, b, c, d, e$  . . . . ; let it be required to represent these values by an integral and rational algebraical function, so that we shall be able for any argument  $x$  to calculate directly the corresponding value X of the said algebraical function."

"*Solution.* Form the complete system of divided differences in the following manner:

a	A	$\delta'(a, b)$			
b	B	$\delta''(a, b, c)$			
c	C	$\delta'(b, c)$	$\delta'''(a, b, c, d)$		
d	D	$\delta'(c, d)$	$\delta''(b, c, d, e)$	$\delta^{IV}(a, b, c, d, e)$	
e	E	$\delta'(d, e)$	$\delta''(c, d, e)$	$\delta'''(b, c, d, e)$	$\delta^{IV}(a, b, c, d, e)$
		.....	.....	.....	.....
		.....	.....	.....	.....
		.....	.....	.....	.....
		.....	.....	.....	.....

"The divided first differences are found by taking the difference of any two adjacent values of the function and dividing it by the difference of

\* This is the *epitheton ornans*, by which Gauss distinguishes Newton, and, as far as I know, nobody else.

"the corresponding arguments;  $\delta(a, b) = \frac{A-B}{a-b}$ ,  $\delta(b, c) = \frac{B-C}{b-c}$ , and  
 "so on.

"The divided differences of any *higher* order are found by taking the  
 "difference of any two adjacent differences of the preceding order, and  
 "dividing it by the difference of the two arguments which are not common  
 "to the said two differences: thus

$$\delta''(a, b, c) = \frac{\delta(a, b) - \delta(b, c)}{a - c}, \quad \delta''(b, c, d) = \frac{\delta(b, c) - \delta(c, d)}{b - d}, \quad \&c.$$

$$\delta'''(a, b, c, d) = \frac{\delta''(a, b, c) - \delta''(b, c, d)}{a - d}, \quad \delta'''(b, c, d, e) = \frac{\delta''(b, c, d) - \delta''(c, d, e)}{b - e}, \quad \&c.$$

"This done, we shall have

$$X = A + (x - a) \times$$

$$\{ \delta'(a, b) + (x - b) [\delta''(a, b, c) + (x - c) [\delta'''(a, b, c, d) + (x - d) [\delta^{(4)}(a, b, c, d, e) + \dots]]] \}$$

Before I give the proof, I think it well (though it may be unnecessary)  
 to draw the attention of the younger readers to the following points:—

1°. It matters nothing in what way we make the subtraction, provided  
 only that this way is the same in the dividend and in the divisor: (of course,  
 $\frac{A-B}{a-b} = \frac{B-A}{b-a}$ ).

2°. If  $F$  denotes an integral and finite algebraical function of one  
 argument or of more arguments, then not only is  $F(p) - F(q)$  divisible  
 by  $(p - q)$  and the quotient of a lower degree than that of  $F$ , but  
 $F(p, r, s, t, \dots) - F(q, r, s, t, \dots)$  is also divisible by  $(p - q)$ , and the  
 quotient of a lower degree than that of  $F$ ; this is easily shown by putting  
 $F(p, r, s, t, \dots) = k_0 + k_1 p + k_2 p^2 + k_3 p^3 + \dots$  and  $F(q, r, s, t, \dots) = k_0 + k_1 q + k_2 q^2 + k_3 q^3 + \dots$ , which is always permitted, ( $k_0, k_1, k_2, k_3, \dots$   
 being functions of  $r, s, t, \dots$ ).

3°. Hence it follows, since  $A, B, C, \dots$  are functions of the  $n$ th  
 degree, that the divided first differences are of the  $(n-1)$ th degree, the  
 divided second differences are of the  $(n-2)$ th degree, and so on, until the  
 divided  $n$ th difference, which being of the  $(n-n)$ th degree, must be a  
 constant, independent of the arguments. Of course all differences of a  
 degree higher than the  $n$ th, must vanish.

To prove that the value above assigned to  $X$  is correct, we have only  
 to write  $x, X$ , and the several divided differences, at the top of the scheme  
 given above; when we have

$$\begin{aligned} \delta'(x, a) &= \frac{X - A}{x - a}, \quad \delta''(x, a, b) = \frac{\delta'(x, a) - \delta'(a, b)}{x - b}, \\ \delta'''(x, a, b, c) &= \frac{\delta''(x, a, b) - \delta''(a, b, c)}{x - c}, \quad \&c. \end{aligned}$$

Hence

$$\begin{aligned} X &= A + (x - a) \delta'(x, a), \\ \delta'(x, a) &= \delta'(a, b) + (x - b) \delta''(x, a, b), \\ \delta''(x, a, b) &= \delta''(a, b, c) + (x - c) \delta'''(x, a, b, c), \end{aligned}$$

and so on until  $\delta^{(n)}(x, a, b, c \dots) = \delta^{(n)}(a, b, c, d \dots)$

$\therefore$  (by substitution)

$$X = A + (x - a) \times$$

$$\{\delta(a, b) + (x - b)[\delta''(a, b, c) + (x - c)[\delta'''(a, b, c, d) + (x - d)[\delta^{IV}(a, b, c, d, e) + \dots]]]\}$$

4th April, 1869.

LUDV. OPPERMANN.

\* \* We have reason to expect that this is only the first of a series of contributions from the distinguished Corresponding Member of the Institute at Copenhagen.—ED. J. I. A.

## CORRESPONDENCE.

### ON A TABLE FOR FACILITATING THE VALUATION OF ABSOLUTE REVERSIONS.

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—The accompanying Table, suggested by Mr. Sprague's Table of the Value of Life Interests (contained in the 8th volume of your *Journal*) will, I think, be found useful. Its title is sufficient explanation of the purpose for which it is intended.

The values indicated are based upon the well known formula  $v - (1 - v)a$ , in which  $v$  is the present value of £1 due a year hence, and  $a$  the price of a whole-life annuity of £1; and it is evident that, whilst dealing with the same rate of interest, the difference between the results for any two given annuity prices consists of the difference between such prices multiplied by  $(1 - v)$ . If a series of annuity prices be taken in arithmetical progression, the second term of the formula will form a series in like progression, causing the results of the whole expression to diminish by a constant quantity. When, therefore, the values for two prices are known, the value for any intermediate price can be found by the most simple method of interpolation.

If  $a$  be successively increased by one shilling, the series of values, starting from that corresponding to an annuity costing  $x$  pounds, will be—

£	s.	d.	
for $x$	0	0	$v - (1 - v)x$
„ $x$	1	0	$v - (1 - v)x - (1 - v)\frac{1}{20}$
„ $x$	2	0	$v - (1 - v)x - (1 - v)\frac{2}{20}$
„ $x$	3	0	$v - (1 - v)x - (1 - v)\frac{3}{20}$
.....	.....	.....	.....

And similarly, if the successive increase be one penny, we shall have

	£	s.	d.	
for	$x$	0	0	$v - (1-v)x$
„	$x$	0	1	$v - (1-v)x - (1-v)\frac{1}{240}$
„	$x$	0	2	$v - (1-v)x - (1-v)\frac{2}{240}$
„	$x$	0	3	$v - (1-v)x - (1-v)\frac{3}{240}$
.....				.....

from which it is seen that  $y$  shillings added to the annuity price diminish the value sought by  $(1-v)\frac{y}{20}$ , and that  $z$  pence diminish it by  $(1-v)\frac{z}{240}$ . Consequently the value corresponding with  $\begin{smallmatrix} £ & s. & d. \\ x & y & z \end{smallmatrix}$  is

$$v - (1-v)x - (1-v)\frac{y}{20} - (1-v)\frac{z}{240}, \text{ or}$$

$$v - (1-v)\left(x + \frac{y}{20} + \frac{z}{240}\right)$$

In the Table are given, the value for each whole number of pounds which occurs in practice, and the quantity to be deducted for each possible number of shillings, and of pence; so that the means are afforded for readily obtaining the value arising from a combination. To prevent any misunderstanding in the matter, it may be well to give an example. Suppose that an absolute reversion to £1,000 cash has to be dealt with; that the Life Tenant is a male, presently aged 65; and that we desire to know what sum will just secure a purchaser 5 per cent upon the total sum which he must invest. Taking the government price of an annuity, £8. 17s. 10d., we have

for the £8 . . .	571.4286
for the 17s., to deduct,	40.4762
	<hr/>
	530.9524
for the 10d., to deduct,	1.9841
	<hr/>
	528.9683 = £528. 19s. 4d.

I need hardly remark that, in some cases, the Table is applicable where two or more persons are enjoying the proceeds of a Trust Fund.

I am, Sir,

Your obedient servant,

No. 1, Moorgate Street, London,  
7th December, 1868.

HENRY MOUNTCASTLE.

*Table for ascertaining the Value of an Absolute Reversion; so as to allow the Purchaser a given rate of interest upon his outlay; according to the price of an Annuity payable during the life of the person at whose death the Reversion will fall in.*

Sum required for the purchase of an Annuity of £1 at the present Age of the Life Tenant.	Value of a Reversion of £1, assuming Interest at			
	4 per Cent.	4½ per Cent.	5 per Cent.	5½ per Cent.
£4	·807692308	·784688995	·761904762	·739336493
5	·769230769	·741626794	·714285714	·687203791
6	·730769231	·698564593	·666666667	·635071090
7	·692307692	·655502392	·619047619	·582938389
8	·653846154	·612440191	·571428571	·530805687
9	·615384615	·569377990	·523809524	·478672986
10	·576923077	·526315789	·476190476	·426540284
11	·538461538	·483253589	·428571429	·374407583
12	·500000000	·440191388	·380952381	·322274882
13	·461538462	·397129187	·333333333	·270142180
14	·423076923	·354066986	·285714286	·218009479
15	·384615385	·311004785	·238095238	·165876777
16	·346153846	·267942584	·190476190	·113744076
17	·307692308	·224880383	·142857143	·061611374
18	·269230769	·181818182	·095238095	·009478673
19	·230769231	·138755981	·047619048	
20	·192307692	·095693780		
21	·153846154	·052631579		
22	·115384615	·009569378		

Quantities to be *deducted* on account of *shillings* occurring in the price of the annuity.

1s.	·001923077	·002153110	·002380952	·002606635
2	·003846154	·004306220	·004761905	·005213270
3	·005769231	·006459330	·007142857	·007819905
4	·007692308	·008612440	·009523810	·010426540
5	·009615385	·010765550	·011904762	·013033175
6	·011538462	·012918660	·014285714	·015639810
7	·013461538	·015071770	·016666667	·018246445
8	·015384615	·017224880	·019047619	·020853081
9	·017307692	·019377990	·021428571	·023459716
10	·019230769	·021531100	·023809524	·026066351
11	·021153846	·023684211	·026190476	·028672986
12	·023076923	·025837321	·028571429	·031279621
13	·025000000	·027990431	·030952381	·033886256
14	·026923077	·030143541	·033333333	·036492891
15	·028846154	·032296651	·035714286	·039099526
16	·030769231	·034449761	·038095238	·041706161
17	·032692308	·036602871	·040476190	·044312796
18	·034615385	·038755981	·042857143	·046919431
19	·036538462	·040909091	·045238095	·049526066

Quantities to be *deducted* on account of *pence* occurring in the price of the annuity.

1d.	·000160256	·000179426	·000198413	·000217220
2	·000320513	·000358852	·000396825	·000434439
3	·000480769	·000538278	·000595238	·000651659
4	·000641026	·000717703	·000793651	·000868878
5	·000801282	·000897129	·000992063	·001086098
6	·000961538	·001076555	·001190476	·001303318
7	·001121795	·001255981	·001388889	·001520537
8	·001282051	·001435407	·001587302	·001737757
9	·001442308	·001614833	·001785714	·001954976
10	·001602564	·001794258	·001984127	·002172196
11	·001762821	·001973684	·002182540	·002389415

\*\*\* We print this modification of Orchard's tables in the form adopted by our correspondent; but we think no good purpose is served by giving more than five decimal places. In practice, it would probably be better to make the corrections *additive*: thus, taking the above example,

$$£8. 17s. 10d. = £9 - 2s. 2d.$$

Value for the £9 = 523·809524

Add for the 2s. 4·761905

„ „ 2d. 396825

Value, as above, 528·968254

ED. J. I. A.

#### ON "TEN YEAR NON-FORFEITURE POLICIES."

To the Editor of the *Journal of the Institute of Actuaries*.

SIR,—If leisure had permitted I intended to have given in the last Number of the *Journal* a development of the suggestion contained in your foot note to my letter in the January Number, and to have looked at the American ten year non-forfeiture policies from the surrender point of view. I now propose to do this, and as all numerical results given in the present communication are based upon the Experience rate of mortality and three per cent interest, it will be advisable first to give the following recomputed values, on the same basis, of the numerical illustrations contained in my last letter.

Age at Entry.	Law of Surrender.				Law of Surrender.			
	$p=1, p=\frac{1}{2}=p=\frac{1}{4} \dots =p(=p).$				$p=1, p=\frac{1}{2}=p=\frac{1}{4}=p(=p), p=\frac{1}{8}=p=\frac{1}{16}=p=\frac{1}{32}=1.$			
	$p=0.$	$p=\frac{1}{2}.$	$p=\frac{1}{4}.$	$p=1.$	$p=0.$	$p=\frac{1}{2}.$	$p=\frac{1}{4}.$	$p=1.$
30	4·674	4·690	4·686	4·691	4·674	4·689	4·683	4·691
40	5·636	5·633	5·637	5·630	5·636	5·632	5·635	5·630
50	7·002	7·091	7·080	7·088	7·002	7·088	7·056	7·088

Each of these results denotes the annual premium per cent.

If, now, we call  $V_n$  the true cash surrender value of a policy at the end of the  $n$ th year, just before the  $(n+1)$ th premium becomes due, and

$V_n$  the corresponding value given by the American plan, we shall have the following formulæ for the computation of  $V_1, V_2, V_3$ , &c., the annual premium payable being  $\omega$ , and  $p$  denoting as before the probability at the time the  $n$ th renewal premium becomes due, that it will be paid, supposing the life assured to be then in existence.

$$\begin{aligned}
 V_1 &= \frac{1}{D_{x+1}} \left\{ \begin{aligned} &p(M_{x+1} - M_{x+2}) + (1-p)D_{x+1}V'_1 \\ &+ pp(M_{x+2} - M_{x+3}) + p(1-p)D_{x+2}V'_2 \\ &+ ppp(M_{x+3} - M_{x+4}) + pp(1-p)D_{x+3}V'_3 \\ &\vdots \\ &+ ppp \dots p(M_{x+8} - M_{x+9}) + ppp \dots p(1-p)D_{x+8}V'_8 \\ &+ ppp \dots pM_{x+9} + ppp \dots p(1-p)D_{x+9}V'_9 \\ &- \omega p(D_{x+1} + pD_{x+2} + ppD_{x+3} \dots + pp \dots pD_{x+9}) \end{aligned} \right\} \\
 V_2 &= \frac{1}{D_{x+2}} \left\{ \begin{aligned} &p(M_{x+2} - M_{x+3}) + (1-p)D_{x+2}V'_2 \\ &+ pp(M_{x+3} - M_{x+4}) + p(1-p)D_{x+3}V'_3 \\ &+ ppp(M_{x+4} - M_{x+5}) + pp(1-p)D_{x+4}V'_4 \\ &\vdots \\ &+ pp \dots p(M_{x+8} - M_{x+9}) + pp \dots p(1-p)D_{x+8}V'_8 \\ &+ pp \dots pM_{x+9} + pp \dots p(1-p)D_{x+9}V'_9 \\ &- \omega p(D_{x+2} + pD_{x+3} + ppD_{x+4} \dots + pp \dots pD_{x+9}) \end{aligned} \right\} \quad (A)
 \end{aligned}$$

It is not necessary to give the expressions for  $V_3, V_4$ , &c., the law of their formation being sufficiently obvious from the above formulæ. The concluding values of the series are

$$\begin{aligned}
 V_8 &= \frac{1}{D_{x+8}} \left\{ \begin{aligned} &p(M_{x+8} - M_{x+9}) + (1-p)D_{x+8}V'_8 \\ &+ ppM_{x+9} + p(1-p)D_{x+9}V'_9 \\ &- \omega p(D_{x+8} + pD_{x+9}) \end{aligned} \right\} \\
 V_9 &= \frac{1}{D_{x+9}} \left\{ \begin{aligned} &pM_{x+9} + (1-p)D_{x+9}V'_9 - \omega pD_{x+9} \end{aligned} \right\}
 \end{aligned} \quad \left. \begin{array}{l} (A) \\ \text{continued.} \end{array} \right\}$$

According to American practice  $p=1$  and  $V'_2 = \frac{2}{10} \frac{M_{x+2}}{D_{x+2}}$ ,

$V'_2 = \frac{8}{10} \frac{M_{x+3}}{D_{x+3}}, \dots V'_9 = \frac{9}{10} \frac{M_{x+9}}{D_{x+9}}$ . These values being substituted in the above, and the further supposition made that  $\frac{p}{2} = \frac{p}{3} = \frac{p}{4} = \dots = \frac{p}{9} (=p)$ , we get, after a little reduction,

$$V_1 = \frac{1}{D_{x+1}} \left\{ M_{x+1} - (1-p) \left( \frac{8}{10} M_{x+2} + \frac{7}{10} p M_{x+3} + \frac{6}{10} p^2 M_{x+4} + \dots + \frac{1}{10} p^7 M_{x+9} \right) - \omega (D_{x+1} + p D_{x+2} + p^2 D_{x+3} + \dots + p^8 D_{x+9}) \right\}$$

$$V_2 = \frac{1}{D_{x+2}} \left\{ M_{x+2} - (1-p) \left( \frac{8}{10} M_{x+3} + \frac{7}{10} p M_{x+4} + \frac{6}{10} p^2 M_{x+5} + \dots + \frac{1}{10} p^7 M_{x+9} \right) - \omega p (D_{x+2} + p D_{x+3} + p^2 D_{x+4} + \dots + p^7 D_{x+9}) \right\}$$

$$V_3 = \frac{1}{D_{x+3}} \left\{ M_{x+3} - (1-p) \left( \frac{7}{10} M_{x+4} + \frac{6}{10} p M_{x+5} + \frac{5}{10} p^2 M_{x+6} + \dots + \frac{1}{10} p^6 M_{x+9} \right) - \omega p (D_{x+3} + p D_{x+4} + p^2 D_{x+5} + \dots + p^6 D_{x+9}) \right\}$$

$\vdots$

$\vdots$

$\vdots$

$$V_8 = \frac{1}{D_{x+8}} \left\{ M_{x+8} - (1-p) \left( \frac{2}{10} M_{x+9} + \frac{1}{10} p M_{x+9} \right) - \omega p (D_{x+8} + p D_{x+9}) \right\}$$

$$V_9 = \frac{1}{D_{x+9}} \left\{ M_{x+9} - (1-p) \frac{1}{10} M_{x+9} - \omega p D_{x+9} \right\}$$

In writing down the values of  $V_4, V_5, V_6, V_7$ , the law indicated by the expressions for  $V_2$  and  $V_3$  must be followed.  $V_1$  is not included in that law owing to the exceptional value of  $p$  as compared with  $p, p, \&c.$

If we use the same values of  $V'_2, V'_3, \&c.$ , as above, and suppose  $\frac{p}{1} = 1, \frac{p}{2} = \frac{p}{3} = \frac{p}{4} = \frac{p}{5} (=p)$  and  $\frac{p}{6} = \frac{p}{7} = \frac{p}{8} = \frac{p}{9} = 1$  we shall obtain from the equations (A) the following series of surrender values.

$$V_1 = \frac{1}{D_{x+1}} \left[ M_{x+1} - (1-p) \left( \frac{8}{10} M_{x+2} + \frac{7}{10} p M_{x+3} + \frac{6}{10} p^2 M_{x+4} + \frac{5}{10} p^3 M_{x+5} \right) - \omega \{ D_{x+1} + p D_{x+2} + p^2 D_{x+3} + p^3 D_{x+4} + p^4 (N_{x+4} - N_{x+9}) \} \right]$$

$$V_2 = \frac{1}{D_{x+2}} \left[ M_{x+2} - (1-p) \left( \frac{8}{10} M_{x+3} + \frac{7}{10} p M_{x+4} + \frac{6}{10} p^2 M_{x+5} + \frac{5}{10} p^3 M_{x+6} \right) - \omega p \{ D_{x+2} + p D_{x+3} + p^2 D_{x+4} + p^3 (N_{x+4} - N_{x+9}) \} \right]$$

$$V_3 = \frac{1}{D_{x+3}} \left[ M_{x+3} - (1-p) \left( \frac{7}{10} M_{x+4} + \frac{6}{10} p M_{x+5} + \frac{5}{10} p^2 M_{x+6} \right) - \omega p \{ D_{x+3} + p D_{x+4} + p^2 (N_{x+4} - N_{x+9}) \} \right]$$

$$V_4 = \frac{1}{D_{x+4}} \left[ M_{x+4} - (1-p) \left( \frac{6}{10} M_{x+5} + \frac{5}{10} p M_{x+6} \right) - \omega p \{ D_{x+4} + p (N_{x+4} - N_{x+9}) \} \right]$$

$$V_5 = \frac{1}{D_{x+5}} \left\{ M_{x+5} - (1-p) \left( \frac{5}{10} M_{x+5} \right) - \omega p (N_{x+4} - N_{x+9}) \right\}$$

$$V_6 = \frac{1}{D_{x+6}} \{ M_{x+6} - \omega (N_{x+5} - N_{x+9}) \}$$

$$V_7 = \frac{1}{D_{x+7}} \{ M_{x+7} - \omega (N_{x+6} - N_{x+9}) \}$$

$$V_8 = \frac{1}{D_{x+8}} \{ M_{x+8} - \omega (N_{x+7} - N_{x+9}) \}$$

$$V_9 = \frac{1}{D_{x+9}} \{ M_{x+9} - \omega (N_{x+8} - N_{x+9}) \}$$

The numerical values of  $V_2$ ,  $V_3$ , &c., for a policy of £100 deduced from this last set of formulæ on the supposition that  $p = \frac{1}{3}$  are set forth in the following table, the age at entry being successively taken at 30, 40, and 50. The values of  $V_1$  are omitted as being unnecessary, the American regulations not allowing a surrender until two annual premiums have been paid. The values of  $\omega$  for ages 30, 40, and 50, as given in the table at the commencement of this letter are .04689, .05632, and .07088 respectively.

Age at Entry.	$V_2$ .	$V_3$ .	$V_4$ .	$V_5$ .	$V_6$ .	$V_7$ .	$V_8$ .	$V_9$ .
30	8.185	12.489	16.938	21.524	26.216	31.179	36.328	41.669
40	9.804	14.983	20.353	25.913	31.657	37.607	43.776	50.180
50	11.782	17.964	24.307	30.727	36.945	44.061	51.486	59.256

These *cash* values are given to enable the reader to convert them into reversionary sums by any table of single premiums he may prefer. For the purpose of illustration I have formed a table of single premiums upon the Experience rate of mortality and 3 per cent interest, with an addition of  $7\frac{1}{2}$  per cent throughout,\* and by this table the cash sums above given would purchase paid-up policies of the following amounts,  $P_n$  being the amount of such policy at the end of the  $n$ th year.

Age at Entry.	$P_2$ .	$P_3$ .	$P_4$ .	$P_5$ .	$P_6$ .	$P_7$ .	$P_8$ .	$P_9$ .
30	18.600	27.891	37.171	46.409	55.529	64.871	74.233	83.614
40	18.614	27.921	37.224	46.512	55.769	65.030	74.313	83.636
50	18.604	27.870	37.058	46.046	54.432	63.838	73.373	83.085

It will be seen that the amount is in every case below that given by the American Companies. It is likely however that the tables of single premiums adopted by some London Offices would give larger values for  $P_7$ ,  $P_8$ ,  $P_9$ , from the same cash values  $V_7$ ,  $V_8$ ,  $V_9$ , used in forming this table, but in no instance is it probable that the paid-up policy would reach the round numbers held out by the Americans.

\* Would it not suffice, considering all the circumstances of the problem, to use the net single premiums, as is virtually the case when all the premiums have been paid?—  
Ed. J. I. A.

As appertaining to the general subject in hand it will be well to examine the effect of assuming  $V'_1=V_1, V'_2=V_2, \dots V'_9=V_9$  in the formulæ (A). The ten premiums will not now be necessarily equal, therefore we will denote them, in the order in which they are paid, by  $\omega_0, \omega_1, \omega_2, \omega_3, \dots \omega_9$  respectively. We shall then have

$$\left. \begin{aligned} V_9 &= p \frac{M_{x+9}}{D_{x+9}} + (1-p)V_9 - p\omega_9 \\ \therefore V_9 &= \frac{M_{x+9}}{D_{x+9}} - \omega_9 \\ \text{Also } V_8 &= p \frac{M_{x+8}-M_{x+9}}{D_{x+8}} + (1-p)V_8 + pp \frac{M_{x+9}}{D_{x+8}} \\ &\quad + p(1-p) \frac{D_{x+9}}{D_{x+8}} V_9 - p\omega_8 - pp \frac{D_{x+9}}{D_{x+8}} \omega_9 \end{aligned} \right\} \quad (B)$$

and if we substitute for  $V_9$  its value just found, this equation will reduce to

$$\left. \begin{aligned} V_8 &= \frac{M_{x+8}}{D_{x+8}} - \omega_8 - \frac{D_{x+9}}{D_{x+8}} \omega_9 \\ \text{Proceeding in the same way we get} \\ V_7 &= \frac{M_{x+7}}{D_{x+7}} - \omega_7 - \frac{D_{x+8}}{D_{x+7}} \omega_8 - \frac{D_{x+9}}{D_{x+7}} \omega_9 \\ &\vdots \\ V_1 &= \frac{M_{x+1}}{D_{x+1}} - \omega_1 - \frac{D_{x+2}}{D_{x+1}} \omega_2 \dots - \frac{D_{x+9}}{D_{x+1}} \omega_9 \\ \text{and } V_0 &= \frac{M_x}{D_x} - \omega_0 - \frac{D_{x+1}}{D_x} \omega_1 \dots - \frac{D_{x+9}}{D_x} \omega_9 \end{aligned} \right\} \quad (B) \text{ continued.}$$

It will be observed that the quantities  $p, p, \dots p$ , have now disappeared entirely from the equations, and therefore if we give to  $V_9, V_8, V_7$ , &c., any values we please, the ten premiums determined by these ten equations will be true for *all* laws of surrender.

The premiums expressed in terms of  $V_9, V_8, V_7$ , &c., are

$$\left. \begin{aligned} \omega_9 &= \frac{1}{D_{x+9}} (M_{x+9} - D_{x+9} V_9) \\ \omega_8 &= \frac{1}{D_{x+8}} (M_{x+8} - M_{x+9} - D_{x+8} V_8 + D_{x+9} V_9) \\ \omega_7 &= \frac{1}{D_{x+7}} (M_{x+7} - M_{x+8} - D_{x+7} V_7 + D_{x+8} V_8) \\ &\vdots \\ \omega_1 &= \frac{1}{D_{x+1}} (M_{x+1} - M_{x+2} - D_{x+1} V_1 + D_{x+2} V_2) \\ \omega_0 &= \frac{1}{D_x} (M_x - M_{x+1} - D_x V_0 + D_{x+1} V_1) \end{aligned} \right\} \quad (C)$$

It will be interesting to see what these premiums are, on the supposition that the various surrender values are those given by the American scheme.

It is only necessary to put  $V_9 = \frac{9}{10} \frac{M_{x+9}}{D_{x+9}}$ ,  $V_8 = \frac{8}{10} \frac{M_{x+8}}{D_{x+8}}$ , . . . .

$V_2 = \frac{2}{10} \frac{M_{x+2}}{D_{x+2}}$ ,  $V_1=0$ , and  $V_0=0$ , in the equations (C), and we shall obtain the following for the true premium values, taking £100 as the amount of the policy.

Age at Entry.	$\omega_0$ .	$\omega_1$ .	$\omega_2$ .	$\omega_3$ .	$\omega_4$ .	$\omega_5$ .	$\omega_6$ .	$\omega_7$ .	$\omega_8$ .	$\omega_9$ .
30	0.818	8.713	4.688	4.685	4.680	4.675	4.668	4.659	4.649	4.636
40	1.006	10.443	5.640	5.647	5.654	5.657	5.654	5.642	5.618	5.581
50	1.547	12.886	7.111	7.108	7.088	7.049	6.987	6.900	6.784	6.634

These premiums would, as I have already intimated, give the assurance company an exact equivalent for the risk undertaken, whatever were the law according to which surrenders might happen to take place. The supposition  $V_1=0$ , made above, causes the value of  $\omega_0$  to express merely the assurance risk of the first year, leaving  $\omega_1$  to provide for the assurance risk of the second year, and for the whole of the surrender value at the end of that year; but, as no surrender value is allowed the first year, we may equalize the first two premiums without disturbing the accuracy of the table just given.

Let  $\omega'$  be the annual payment for the first and second year equivalent to the premiums  $\omega_0$  and  $\omega_1$ , then

$$\omega' \left( 1 + \frac{D_{x+1}}{D_x} \right) = \omega_0 + \omega_1 \frac{D_{x+1}}{D_x} \quad \therefore \quad \omega' = \frac{\omega_0 D_x + \omega_1 D_{x+1}}{D_x + D_{x+1}}.$$

This expression for  $\omega'$ , however, may be simplified for calculation. By adding together the two last of equations (C), remembering that  $V_0=0$  and  $V_1=0$ , we get

$$\omega_0 D_x + \omega_1 D_{x+1} = M_x - M_{x+2} + D_{x+2} V_2 = M_x - \frac{4}{5} M_{x+2}$$

and this being the numerator of  $\omega'$  we have  $\omega' = \frac{M_x - \frac{4}{5} M_{x+2}}{D_x + D_{x+1}}$ . When

$x=30$  we shall find  $\omega'=4.691$ ; therefore, in the place of each of the values 0.818 and 8.713 in the table, we are at liberty to put 4.691, and this substitution brings the whole series of premiums for age 30 within much nearer limits of equality. At age 40 we find  $\omega'=5.630$ , and at 50,  $\omega'=7.088$ , which may be substituted in the same manner for the tabular values of  $\omega_0$  and  $\omega_1$  at those ages.

By examining the equations (C) it appears that the expression for  $\omega_0$ , namely,  $\frac{M_x - M_{x+1}}{D_x} + \frac{D_{x+1}}{D_x} V_1$ , is composed of the value, at the commencement of the first year, of that year's risk and of the cash surrender, payable at the end of the year. It is therefore evident that whatever be the number of surrenders in the first year (supposing any to be then allowed) the premium  $\omega_0$ , thus calculated, would be sufficient to provide

for them all. Next take the case of a policy upon which the second year's premium  $\omega_1$  has just been paid. Here the sum  $V_1$  not having been taken, stands to the credit of the policyholder when he enters upon the second year, and therefore when he pays the premium  $\omega_1$  the office holds  $V_1 + \omega_1$ . Now, from the equation expressing the value of  $\omega_1$  in (C), we find that  $V_1 + \omega_1 = \frac{M_{x+1} - M_{x+2}}{D_{x+1}} + \frac{D_{x+2}}{D_{x+1}} V_2$ , which shows that the sum in the hands of the Society at the commencement of the second year is exactly sufficient to provide for the second year's risk and the surrender  $V_2$  at the end of that year, hence the Society cannot suffer loss however many surrenders take place in the second year. The same reasoning applied to the subsequent years will be found to lead to similar results.

Suppose we now proceed to find what *uniform* annual premium ( $\omega'$ ) is equivalent to the series of premiums  $\omega_0, \omega_1, \omega_2, \omega_3, \dots, \omega_9$  as determined by (C). We must then have an equation satisfied, which, after multiplying both sides by  $D_x$ , becomes

$$\omega'(D_x + pD_{x+1} + ppD_{x+2} + \dots + pp \dots pD_{x+9}) = \omega_0 D_x + pD_{x+1}\omega_1 + ppD_{x+2}\omega_2 + \dots + pp \dots pD_{x+9}\omega_9$$

and if we substitute for  $\omega_0, \omega_1, \&c.$ , their values given by (C), previously putting  $V_0=0, V_1=\frac{1}{10}\frac{M_{x+1}}{D_{x+1}}, V_2=\frac{2}{10}\frac{M_{x+2}}{D_{x+2}}, \&c.$ , the above equation will be found to give for  $\omega'$  precisely the same expression as that which was obtained for  $\omega$  by the formulæ (1) and (2) in my last letter. From this we see that it is solely on account of charging a *uniform* premium that it becomes necessary to introduce the probabilities of surrender, and thus, in the absence of the knowledge of what those probabilities are, to bring a speculative element into the contract.

There is yet one other case to be glanced at. We may suppose the premiums to be all equal, or  $\omega_0=\omega_1=\omega_2=\dots=\omega_9(=\omega)$ , the values of  $V_9, V_8, V_7, \&c.$ , then become,

$$\begin{aligned} V_9 &= \frac{M_{x+9}}{D_{x+9}} - \omega \\ V_8 &= \frac{M_{x+8}}{D_{x+8}} - \omega \frac{N_{x+7} - N_{x+9}}{D_{x+8}} \\ V_7 &= \frac{M_{x+7}}{D_{x+7}} - \omega \frac{N_{x+6} - N_{x+9}}{D_{x+7}} \\ &\vdots \\ V_1 &= \frac{M_{x+1}}{D_{x+1}} - \omega \frac{N_x - N_{x+9}}{D_{x+1}} \\ V_0 &= \frac{M_x}{D_x} - \omega \frac{N_{x-1} - N_{x+9}}{D_x} \end{aligned}$$

Since  $V_0=0$  in practice, the last equation gives  $\omega = \frac{M_x}{N_{x-1} - N_{x+9}}$ , which determines the premium, and, substituting this value in each of the other

equations, the nine surrender values become known. We have here the familiar case of a whole-life assurance, all the premiums for which are comprised in ten equal annual payments, one at the commencement of each of the first ten years, but where no surrender values of given amounts form any part of the contract, and it is evident from what precedes that this is the only possible instance—for the same description of policy—in which a *uniform* premium will exactly provide (under any law of surrender) for a set of surrender values, the amounts of which *might* be specified beforehand or at the time the assurance was effected.

The actual amounts of the surrenders which might be thus held out to the assurer, without introducing any uncertainty or speculation into the transaction, are the values of  $V_9$ ,  $V_8$ , &c., given by the last set of formulæ, and these are specified for ages 30, 40, and 50, at entry, in the following table. On comparing them with the results previously obtained for  $V_9$ ,  $V_8$ , &c., on another hypothesis, it will be seen that the difference is only in the decimal in each case.

Age at Entry	$\omega$ .	$V_2$ .	$V_3$ .	$V_4$ .	$V_5$ .	$V_6$ .	$V_7$ .	$V_8$ .	$V_9$ .
30	4.674	8.152	12.445	16.891	21.498	26.273	31.223	36.357	41.684
40	5.636	9.812	14.986	20.345	25.894	31.641	37.594	43.768	50.176
50	7.002	11.600	17.683	23.974	30.494	37.262	44.304	51.653	59.342

The figures here given for  $\omega$  are derived, of course, from  $\omega = \frac{M_x}{N_{x-1} - N_{x+9}}$ .

Sufficient materials have now probably been given in this and my former letter to enable any one interested in the subject to form an opinion as to the merits of the American system of ten year nonforfeiture policies. Its simplicity of statement is its one recommendation, and no doubt a great and important one, but it is plain that if a Company issued a considerable number of such policies, some care would be necessary at each periodical valuation in determining the reserve required for the risks, in order to attain that degree of exactness and certainty in the results to which most English Actuaries are accustomed.

I am,  
Sir,

Your most obedient Servant,

SAMUEL YOUNGER.

17, Waterloo Place,  
Pall Mall, London,  
31st May, 1869.

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—In Mr. Higham's paper on the value of "selection," in vol. i., in discussing the effect of taking the lives in quinquennial groups, he says in a foot-note, page 186, that if the numbers living at ages  $m$ ,  $m+1$ ,  $m+2$ ,  $m+3$ ,  $m+4$ , respectively, be represented by 10, 9, 8, 7, 6, then, if the probability of living a year diminish by second differences, the probability

for the quinquennial combination is  $= 1\text{st term} + \frac{7}{4}d_1 + \frac{52}{32}d_2$ ,

$d_1$ ,  $d_2$ , being the 1st and 2nd orders of differences of  $p_m$ .

This result is not quite self-evident, so I have ventured to send you a demonstration of it.

We have

$$\begin{array}{lll}
 p_m = p_m & \therefore & 10p_m = 10p_m \\
 p_{m+1} = p_m + d_1 & \therefore & 9p_{m+1} = 9p_m + 9d_1 \\
 p_{m+2} = p_m + 2d_1 + d_2 & \therefore & 8p_{m+2} = 8p_m + 16d_1 + 8d_2 \\
 p_{m+3} = p_m + 3d_1 + 3d_2 & \therefore & 7p_{m+3} = 7p_m + 21d_1 + 21d_2 \\
 p_{m+4} = p_m + 4d_1 + 6d_2 & \therefore & 6p_{m+4} = 6p_m + 24d_1 + 36d_2.
 \end{array}$$

Adding together, and dividing by 40, ( $=10+9+8+7+6$ ), we get

$$\frac{10p_m + 9p_{m+1} + 8p_{m+2} + 7p_{m+3} + 6p_{m+4}}{40} = \frac{1}{40}(40p_m + 70d_1 + 65d_2)$$

$$\text{or, probability of combination} = p_m + \frac{7}{4}d_1 + \frac{13}{8}d_2$$

$$= p_m + \frac{7}{4}d_1 + \frac{52}{32}d_2,$$

which is the result given by Mr. Higham.

I am, Sir,

Your obedient servant,

June 3rd, 1869.

W. SUTTON.

#### "EVILLY-DISPOSED."

*To the Editor of the Assurance Magazine.*

SIR,—Mr. Bunyon having misquoted the word to which I objected, has not unnaturally failed to understand the objection itself.

In his "Law of Fire Insurance," he wrote "evilly-disposed" as one word, with the hyphen; not as two words, "evilly disposed," as they stand in his letter to you of the 6th March. In the latter case, the word *evilly* is rightly used as an adverb, as it is in the quotations which Mr. Bunyon gives, and as it is also by Shakespeare in *Timon of Athens*, where there occurs the phrase, "Good deeds evilly bestowed." So used, I have no objection to it, archaic or other: my objection is to its being linked, though an adverb, to the neutral word "disposed," to be employed when so compounded as an adjective,—an "evilly-disposed" person. It will be noticed that the word *disposed* fails of itself to qualify "person," and needs an adjectival prefix as a sort of grammatical co-efficient to give it the force and meaning of a true adjective.

Mr. Bunyon's quotations wholly fail to justify his use of the word, nor can I find any that will justify it. There are, on the other hand, numerous examples among the old writers—the Fathers of our language—of the word "evil" forming part of a compound adjective. Thus, Sterling speaks of "evil-conquered states"; Shelton, of an "evil-favored countenance"; Spenser, of an "evil-gotten mass" and an "evil-ordered train"; Sir Philip Sidney, of "evil-wishing states"; and Lansdown, of an "evil-fated line." Daniel, in his "History of the Civil Wars," has a similar word—"evil-minded"—which is still in every day use. Without multiplying these

examples, I submit that Mr. Bunyon should have written "evil-disposed," not "evilily-disposed."

Though I have avoided as much as possible all grammatical technicalities, and have purposely confined myself to a broad, general defence of my objection, I feel that an apology is due to you for intruding into your columns a discussion for which they are hardly suited and certainly not intended.

Yours faithfully,

THE REVIEWER OF MR. BUNYON'S BOOK.

London, June 9, 1869.

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## INSTITUTE OF ACTUARIES.

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### PROCEEDINGS OF THE INSTITUTE.—SESSION 1868–69.

*First Ordinary Meeting, Monday, 30th November, 1868,*

The President in the Chair.

Read and confirmed the minutes of the anniversary meeting, held on the 6th June, 1868.

Mr. T. B. Sprague, M.A., read a paper "On the Value of Reversionary Life Interests."

Thanks having been voted to Mr. Sprague, the meeting adjourned to Monday, 21st December, 1868.

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*Second Ordinary Meeting, Monday, 21st December, 1868.*

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected, viz. :—

*Fellow.*

Edward Samuel Barnes.

*Associates.*

Thomas Steadman Aldis.

Henry Collins.

John Demden, B.A.

David Drimmie.

Henry William Eaton.

John Andrew Greig.

John Harrison, Jr.

Estévan Hallet Louis Hartwig.

Thomas Emley Young.

Sidney T. Jewsbury.

Edward Lichfield.

George Anderson Meaden.

Clement Hugh Oldham.

Oliver Selby.

William George Spens.

Alfred Sprules.

Vere George Webb.

Mr. J. Coles read a paper "On Railway Debenture Stock considered as an Investment for the funds of a Life Assurance Company."

Thanks having been voted to Mr. Coles, the meeting adjourned to Monday, 25th January, 1869.

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*Third Ordinary Meeting, Monday, 25th January, 1869.*

The President, and afterwards Mr. Sprague, M.A., Vice-President, in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected, viz. :—

*Fellow.*

Morrice Alexander Black.

*Associates.*

Charles Edward Brown.

Tyvon Richard Eccles.

Aubyn Francis Margary.

Theodore William Peacock.

John Charles Proctor.

Thomas Northcote Toller.

Frederick William Womersly.

The following was announced to be the result of the Examinations for 1868.

## MATRICULATION EXAMINATION.

Eleven gentlemen presented themselves for this Examination, of whom five passed in the following order of merit:—

1. { T. S. Aldis.
- { T. N. Toller.
3. L. H. Greaves.
4. J. A. Greig.
5. F. W. Womersly.

## SECOND YEAR'S EXAMINATION.

Five gentlemen appeared for this Examination, and four passed in the following order of merit:—

1. David Carment.
2. William Hughes.
3. Felix Bassett.
4. T. Hyde Johnson.

## THIRD YEAR'S EXAMINATION.

Two gentlemen presented themselves for this Examination, and passed in the following order, viz.:—

1. William Sutton.
2. Edwin Justican.

Thanks were voted to the Examiners.

The President read a paper "On the Mortality Experience of Life Assurance Companies, collected by the Institute of Actuaries."

Thanks having been voted to the President, the meeting adjourned to Monday, 22nd February, 1869.

*Fourth Ordinary Meeting, Monday, 22nd February, 1869.*

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz.:—

Francis Homan Berry.  
Edward Clifton Griffith.  
Charles Edward Mason.

Mr. A. H. Bailey read a paper "On Rates of Premium for Foreign Travelling and Residence."

Thanks having been voted to Mr. Bailey, the meeting adjourned to Monday, 29th March, 1869.

*Fifth Ordinary Meeting, Monday, March 29th, 1869.*

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz.:—

William Edward Bools.  
 Samuel Kirkness.  
 Pembroke Marshall.  
 Joseph Whittall.

Mr. W. S. B. Woolhouse read a paper "On an improved theory of Annuities and Assurances."

Thanks having been voted to Mr. Woolhouse, the meeting adjourned to Monday, 26th April, 1869.

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*Sixth Ordinary Meeting, Monday, 26th April, 1869.*

The President in the chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentleman was elected an Associate, viz. :—

John Mathew Dove.

Mr. H. W. Manly read a paper "On different modes of constructing Tables of the Value of Policies."

Mr. T. B. Sprague, M.A., read a paper "On the Value of Reversionary Annuities payable half yearly, quarterly, &c., according to the conditions which prevail in practice."

Thanks having been voted to Mr. Sprague and Mr. Manly, the meeting adjourned to Monday, 29th November, 1869.

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*The Twenty-second Annual General Meeting, Saturday, 5th June, 1869.*

SAMUEL BROWN, Esq., the President, in the Chair.

Mr. ARCHIBALD DAY (Hon. Secretary) read the notice convening the meeting, and the following Report and Statement of Accounts :—

"The Council have to report that the number of members of the Institute on the 31st March last was 234, of whom 102 were Fellows and 132 Associates. During the course of the year 46 new members were admitted, and 33 taken off the books from deaths and withdrawals, showing an increase of 13 in the number over the corresponding period of last year.

"The accounts, which have been examined by the Auditors, are presented in the same form as last year. It will be noticed that the annual subscriptions amount to £489. 6s., while the ordinary expenditure is £568. 12s. 9d. The increase in the expenditure arises almost entirely from an unusually heavy outlay for the *Journal* which is not likely to recur, the cost of six numbers being comprehended therein, and the Council do not doubt that the outgoings of the current year will not exceed the ordinary annual income. The funds of the Institute amount to £1,228. 17s. 2d., and deducting those appropriated to specific purposes, the General Fund (including £189 contributed by eight members who have compounded for their subscriptions) amounts to £755. 16s. 2d., so that the members will doubtless be of opinion that the finances of the Institute are in a sound condition.

"The Council have at length the pleasure to announce that the laborious Mortality Experience investigation is at last completed. The introduction and the whole of the tables are in print and will very shortly be published. And it is a remarkable proof of the utility of the Institute, that through its instrumentality by far the most complete and authentic information that has yet been collected on the subject of human mortality will thus have been given to the world.

"The following papers have been read during the past Session :—

30th Nov., 1868.—Mr. T. B. Sprague, M.A., 'On the Value of Reversionary Life Interests.'

21st Dec., 1868.—Mr. J. Coles. 'On Railway Debenture Stock considered as an Investment for the funds of a Life Assurance Company.'

25th Jan., 1869.—Mr. S. Brown. ‘On the Mortality Experience of Life Assurance Companies, collected by the Institute of Actuaries.’

22nd Feb., 1869.—Mr. A. H. Bailey. ‘On Rates of Premiums for Foreign Travelling and Residence.’

29th March, 1869.—Mr. W. S. B. Woolhouse. ‘On an improved theory of annuities and assurances.’

26th April, 1869.—Mr. H. W. Manly. ‘On different modes of constructing Tables of the Value of Policies.’

” —Mr. T. B. Sprague, M.A. ‘On the Value of Reversionary Annuities, payable half-yearly, quarterly, &c., according to the conditions which prevail in practice.’

“During the present Session of Parliament Bills have been introduced by the Right Hon. Stephen Cave to amend the law relating to Life Insurance Companies, and by the Government, to reduce the limit of the Life Assurance contracts made through the Post Office to £5.

The first of these measures appeared to the Council to be so important that they thought it advisable to take steps to summon a meeting of the representatives of all the life offices in the kingdom, in order to consider the Bill. This meeting was accordingly held, resolutions were passed which have been communicated to Mr. Cave; and the Council have reason to believe that the Bill will be materially improved in consequence.

“The Government Bill does not appear to call for remark.

“The Council have resolved again to offer a Messenger Prize of the value of Ten Guineas, for the best essay, to be written by a member of the Institute, not being a member of the Council. The Prize will be given in the form of books, to be selected by the successful competitor from works relating to Mathematical, Statistical, or Economical Science.

“The subject proposed on the present occasion is as follows:—

“‘Legislation in reference to Life Assurance and Life Assurance Companies.’

“(Signed) SAMUEL BROWN,  
“President.”

The PRESIDENT said—“Gentlemen, in rising to move ‘That this report be adopted, entered on the minutes, and printed in the *Journal*,’ I need say but very few words. I think you must all admit that the financial condition of our Institute appears to be very flourishing indeed; and, as far we can see, the original object of the founders has been most thoroughly carried out, and is every year progressing more successfully. (Hear, hear.) I think there has been no falling off in any department of the Institute. If we look, for instance, at the papers which have been read at our sessional meetings, we find they are of the most practical character. We have those by Mr. Sprague, which are of the highest value, as scientific papers; and we have also one by Mr. Coles on Railway Debenture Stock, a very important question regarding the investments of assurance companies. (Hear, hear.) We have, then, as eminently practical, Mr. Bailey’s paper ‘On the Rates of Premium for Foreign Travelling and Residence;’ and, above all, Mr. Woolhouse’s very important paper upon a new theory of annuities and reversions, which will facilitate in a most remarkable manner the valuation of difficult and complicated contingencies. (Hear, hear.) All of these are either of the most useful character or of the highest value for the student in his further researches in our science. (Hear, hear.) We may then come to the consideration of the way in which this Institute has lately acted on the question of legislation. Formerly we should probably have had considerable difficulty in bringing together so large and influential a body as the Institute was the means of collecting, to discuss the important measure now passing before Parliament; and I believe the effect of the harmony and good feeling shown by all the members of our

profession on that occasion will be, that the bill will be considerably improved in the course of its progress through the House of Commons, if it should finally pass into a law. (Hear, hear.) I need not detain you further, except to remark that the Messenger Prize, which, most appropriately, on this occasion has been proposed to be on the subject of legislation in reference to assurance companies, will, I trust, enable several gentlemen connected with our profession to throw much light on the effect of special legislation for life assurance companies in different countries, and so enable us to see whether we are better with it, or, as I believe, better without it. But independently of that, the papers and discussions, the *Journal*, and the movement in connection with legislation—all these stamp the value of the Institute, and enable us to recognise the great professional advantages which it has conferred upon us. (Hear, hear.) There is only one source of regret connected with the past year, and that is our having to record the loss by death of two or three eminent men—men who had long been associated with us, and of whose character we always thought so highly. I allude, first of all, to the death of Mr. Spens, who was one of the most eminent of the Scottish managers, who had always taken great interest in our Institute, and who contributed to our *Journal* a very valuable and carefully-written paper on the experience of his own Office. He was a man eminently respected by every one both in Scotland and England who was at all acquainted with his character. (Hear, hear.) Every one, too, will admit the great ability of Mr. Percy M. Dove, whose loss we have also to lament,—the marvellous administrative power which he showed in his conduct of the company with which he was associated, and by which he was enabled to raise it from its small original condition, when he first took charge of it, to be one of the most powerful and enterprising offices in the kingdom, and which, in fire business at any rate, is competing on the grandest scale with the leading companies both in this country and abroad. He was a man, too, who was very much respected in private life; and on his visits to London we always found him taking a most active interest in all the concerns of the Institute, and endeavouring by every means in his power to conciliate the friendship of its members. (Hear, hear.) And, last of all, we have lost a very esteemed and dear friend in Mr. Lodge, one of those who at the commencement of our Institute took so warm and active a part in its success, whose talents and ability were of the highest order. He had a wonderful knowledge of mankind, and, owing to his great courtesy, kindness, and cordiality of manner, and to his high integrity and warmth of heart, he became endeared to every one who had the pleasure of knowing him. These are drawbacks upon the otherwise pleasing history of the past year. In other respects we have very much to congratulate the members of the Institute upon, and so long as the high qualities which distinguished those eminent men who have passed from us shall continue amongst us, so long as their devotion to duty, their energy, their determination to take and maintain a foremost place in their profession shall still inspire us, I have no fear that there will be any deterioration in the character of the Institute, or any falling back from that high position which it has attained in public estimation. (Cheers.) I beg to move, gentlemen, 'That the report be adopted, entered on the minutes, and printed in the *Journal*.' If any member wishes for any explanation I am sure I shall be most happy to give it to the best of my ability."

Mr. A. BADEN—"I beg leave to second that motion. I am sure it must have been a source of great satisfaction to every one present to have heard this excellent report read, and it must have been equally satisfactory to have listened to the admirable address of our President on the subjects therein mentioned." (Hear, hear.)

The motion was at once agreed to unanimously.

Mr. Manly and Mr. Stark having been appointed scrutineers, a ballot was taken for the election of a President, Vice-Presidents, Council, and Officers for the ensuing year, which resulted as follows:—

*President.*

SAMUEL BROWN.

*Vice Presidents.*ALEXANDER GLEN FINLAISON.  
WILLIAM BARWICK HODGE.THOMAS BOND SPRAGUE, M.A.  
J. HILL WILLIAMS.*Council.*

MARCUS N. ADLER, M.A.  
 ANDREW BADEN.  
 ARTHUR HUTCHESON BAILEY.  
 SAMUEL BROWN.  
 CHARLES JOHN BUNYON, M.A.  
 \* EDWARD BUTLER.  
 GEORGE CUTCLIFFE.  
 ARCHIBALD DAY.  
 \* HENRY DEVEREUX DAVENPORT.  
 ALEXANDER GLEN FINLAISON.  
 ALEXANDER PEARSON FLETCHER.  
 WILLIAM JOHN HANCOCK.  
 AUGUSTUS HENDRIKS.  
 WILLIAM BARWICK HODGE.  
 CHARLES JELlicoe.

CHARLES TERRELL LEWIS.  
 WILLIAM MATTHEW MAKEHAM.  
 JAMES MEIKLE.  
 JOHN MESSENT.  
 \* BENJAMIN NEWBATT.  
 EDWARD A. NEWTON, M.A.  
 WILLIAM P. PATTISON.  
 HENRY WILLIAM PORTER, B.A.  
 HENRY AMBROSE SMITH.  
 COL. JOHN THOS. SMITH.  
 THOMAS BOND SPRAGUE, M.A.  
 \* JOHN STOTT.  
 ROBERT TUCKER.  
 JOHN HILL WILLIAMS.  
 W. S. B. WOOLHOUSE.

*Treasurer.*

GEORGE CUTCLIFFE.

*Honorary Secretaries.*

ARTHUR H. BAILEY.

ARCHIBALD DAY.

A vote of thanks was accorded to the scrutineers, and acknowledged by Mr. Manly.

The PRESIDENT said that as two of the auditors—Mr. Mountcastle and Mr. Haycraft—were disqualified through having become fellows of the Institute, he would propose Mr. Emmens and Mr. Hopkinson to fill their places for the ensuing year. Mr. Manly remained, and he trusted he would again render them his services.

Mr. HODGE seconded the motion, which was carried unanimously.

Mr. HODGE proposed a resolution, of which notice had been given, with regard to a slight alteration in clause 4 of the Constitution and Laws of the Institute, which was considered desirable by the Council. The original clause declared that—"Every candidate for future admission as an associate must be approved by the Council and proposed by two or more members, who shall certify, from their acquaintance with him or his works, that he is a fit person to be admitted. The certificate to that effect, having been read, shall be suspended in some conspicuous place in the meeting room of the Institute during an ordinary meeting, there to remain until the following ordinary meeting, at which a ballot shall be taken among all the members there present; and in the event of not less than three-fourths of the votes given being in the candidate's favour, he shall be declared duly elected; otherwise, not; and in the latter case he shall remain ineligible for re-nomination till the next session." The alteration proposed was to substitute after the word "admitted" in this clause, the following words, in lieu of those above,—“The certificate to that effect, having been read at the council, shall be suspended in some conspicuous place in the meeting room of the Institute until the following ordinary meeting, at which a ballot shall be taken among all the members there present.” It was necessary that the certificate should be suspended during the ordinary meeting, which was a mere matter of form, and it was intended to give the Council the ability to suspend it at once, provided they saw no objection to the candidate. He apprehended there was not likely to

\* New Members.

be any opposition to this alteration, the object of which was to save time in the election of members of the Institute, and therefore he should not detain them with any remarks upon it.

Mr. MESSENT, in seconding the resolution, said he thought they had all felt there was a little inconvenience about this rule as it stood at present, and that the proposed amendment would be an improvement.

The alteration was agreed to unanimously.

Mr. HODGE said there was one more amendment proposed, and that was with regard to clause five, which was rather more important than the preceding. This clause was as follows:—"Any associate of the Institute who shall, for a period of three years, have been actuary to the Government, or actuary, principal officer, manager, secretary, or assistant-actuary to any Life Assurance, Annuity, or Reversionary Interest Society; or, lastly, any associate who shall have obtained the certificate of competency, shall be entitled, upon application, to be at once admitted a fellow, without ballot, subject to such regulations as may from time to time be appointed." Some little inconvenience and difference of opinion had arisen as to the rights of members under this clause; and it was thought advisable by a gentleman who had proposed it to the Council, and who he was sorry to say was not present to explain his views more fully to the meeting, that it should be altered. The Council, who readily adopted the suggestion made to them, thought it desirable that the offices mentioned in the clause should be held "three consecutive years" instead of merely "three years." He (Mr. Hodge) rather thought that that was the proper construction and original intention of the rule; but in order to remove any doubt they proposed to insert the word "consecutive," and also that the right should not be absolute, but subject to the approval of the Council. Discussions had sometimes arisen upon points of a rather delicate nature, which it was difficult for the Council to bring forward, although they felt that they were right in acting upon them; and therefore they had thought it advisable to recommend to the members that the approval of the Council should be necessary to enable a person to exercise the right of becoming a fellow of the Institute without ballot. This was no doubt asking of the members some amount of confidence in the Council, and it would be for them to say whether the management had been such in the past, as to induce them to place that confidence in them in the future. He was satisfied that the working of the proposed clause would be an advantage, and that it would obviate such little inconveniences as those which had occasionally arisen. He, therefore, begged to move the substitution of the following clause for clause 5 in the Laws of the Institute, viz:—"Any associate of the Institute who shall, for a period of three consecutive years, have been actuary to the Government, or actuary, principal officer, manager, secretary, or assistant actuary to any Life Assurance, Annuity, or Reversionary Interest Society; or, lastly, any associate who shall have obtained the certificate of competency, shall, upon application, be admitted a fellow, without ballot, subject to the approval of the Council, and to such regulations as may from time to time be appointed."

Mr. CUTCLIFFE had much pleasure in seconding the motion.

Mr. COLLES asked whether the officer must have been engaged for three consecutive years in one Company, or whether three years' service in different companies was sufficient?

Mr. HODGE said it meant any company or companies, provided there was no break.

The motion was then carried.

Mr. R. P. HARDY said that no one could have listened to the remarks of the President without feeling thoroughly satisfied with the results of the past year, and also thankful to the Vice-Presidents, and Council, not merely for the able support which they had rendered him, but for having so worthily sustained the dignity of the Institute in the eyes of the profession, and for having so eminently forwarded the views of its founders. (Hear, hear.) He had no doubt he had the sympathies of all present in proposing "That the best thanks of the members be given to the President, Vice-President, Council,

and other officers of the Institute, for their services during the past year." (Cheers.)

Mr. MANLY seconded the resolution, which was cordially agreed to.

The PRESIDENT, in reply, said—"Gentlemen, in returning thanks to you for the honour you have done us, I need hardly say that it has been a great pleasure to see the success which our endeavours to promote the prosperity of the Institute have met with. It is also very gratifying to me gratefully to record the support which I have received from those around me, and which has been the principal cause of the prosperity of the Institute during the past year. It might seem invidious to mention names, but there are several gentlemen who I am sure will immediately recur to you as having been of the greatest service to the Institute. I may venture, however, to refer especially to Mr. Sprague—(hear, hear)—who, as editor of the *Journal*, has certainly contributed to keep up its character as one of the most important publications in connection with any profession of the present day. (Hear, hear.) We must also recognize the great assistance he has rendered in carrying out those two important works which we have lately had in view. First, let me speak of the Mortality Experience of the Life Assurance Offices, collected by this Institute, of which I have the pleasure to have a printed copy, just completed, lying before me, and which, consequently, will soon be in the hands of the profession. This is a work of the highest interest, for it is evident that the law of mortality founded upon the actual experience of the Assurance Companies is one of the most important questions which it can fall within our province to determine. The labour has been very great, the collection and compilation of the facts have been the work of some years; and the Institute is to be congratulated upon having been the means of accomplishing so great a task. (Hear, hear.) The other question is also one in which Mr. Sprague has rendered us very efficient service, and that is the question of a common system of notation for Life Assurance problems. This is beginning to excite the greatest interest in many parts abroad. In America, in the State of New York, the Government Superintendent of the Insurance Department, Mr. Barnes, has endeavoured to collect the opinions of various persons, actuaries and mathematicians, as to having a common notation there, and if possible the same as may be agreed upon by joint action here. I had the pleasure a few days ago of receiving a visit from an eminent American gentleman—Professor Newton of Yale College, Connecticut—who is anxious to co-operate in any way in his power to promote a system of common notation to be in use in both countries. The same attempt is being made by the highest scientific men, actuaries and mathematicians, in Germany, who have formed themselves into an Institute something like our own, and are pursuing nearly the same objects, and they wish as far as they can to decide upon a common notation for Germany, and which it would be a great advantage to frame in unison with what may be adopted here. This is truly a very important question. I am sure that Mr. Sprague is one of the fittest men to take a leading part in it here, and so far as I have been enabled to examine his work, which is very much advanced, he has shown his usual scientific ability, and afforded proof of his thorough acquaintance with the practical details of the subject, in making it apply to questions of daily official occurrence, so as to impart to the system a popularly useful as well as scientific character. (Hear, hear.) On this account, gentlemen, I beg to say that, whilst I thank you on my own behalf, I must be permitted to add that it is owing to the cordial and kind support I have received from your late distinguished President, Mr. Jellicoe, and from all those around me, that I am enabled to congratulate you on the continued success of the Institute. I tender you my best thanks in their name and my own for this appreciation of our services. (Cheers.) It only remains for me to announce that the Library will, as usual, be closed during the month of September."

The proceedings then terminated.\*

\* The above report of the proceedings of the Annual Meeting is extracted from the *Insurance Record*.



JOURNAL  
OF THE  
INSTITUTE OF ACTUARIES  
AND  
ASSURANCE MAGAZINE.

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*On Different Modes of constructing Tables of the Values of Policies.*  
By HENRY W. MANLY, of the London and Provincial Law  
Assurance Society.

[Read before the Institute, 26th April, 1869.]

IN the construction of the Tables of Policy Values for the Messenger Prize Essay (1868), the operations were considerably facilitated by the use of different modes of working applicable to the different data at hand; an explanation of which forms the subject of the present paper.

In the formation of Tables the great objects to be attained are accuracy and facility of working. These may be acquired in a high degree if it be possible by means of a simple operation to deduce each value in succession from the one which immediately precedes it, in which case the chances of error are reduced to a minimum, and the construction facilitated by the work not requiring to be checked except at distant intervals. Where, however, the operations cannot be easily performed except by the aid of logarithms, errors may sometimes occur in finding the corresponding natural numbers, which will of course require checking. But although there is no way of shortening this portion of the work, still greater accuracy may be secured by the use of

two different forms of tables, that is, by using a table of Anti-logarithms first and checking the work by a table of Logarithms used inversely, or *vice versé*.

Now there exists a relation between the values of policies which will be found very useful in constructing a table, namely: that

$$\frac{1 - V_{x|n+1}}{1 - V_{x|n}} = \frac{a_{x+n+1}}{a_{x+n}} = 1 - V_{x+n|1}$$

or, using logarithms,

$$\lambda(1 - V_{x|n+1}) - \lambda(1 - V_{x|n}) = \Delta\lambda a_{x+n};$$

and consequently

$$\lambda(1 - V_{x|n+1}) = \lambda(1 - V_{x|n}) + \Delta\lambda a_{x+n}$$

from which it will be easily seen that a table of the values of  $\lambda(1 - V_{x|n})$  can be formed by continuously adding the successive values of  $\Delta\lambda a_x$ . This operation is best performed by means of a perforated card, cut so as to allow of 3 lines being left between each value, as represented in the annexed diagram; where the values are taken, as they will be throughout this paper, from Dr. Farr's English Life Tables No. 3 (Males, 3 per cent), because they contain most of the information required.

$$\lambda(1 - V_{15|1})$$

$$\lambda(1 - V_{15|2})$$

$$\lambda(1 - V_{15|3})$$

$$\lambda(1 - V_{15|4})$$

$\Delta\lambda a_x$	
15	·9958737
	·9958737
16	·9958864
	·9917601
17	·9959474
	·9877075
18	·9960447

$$\cdot 9887522$$

An easier process than the above would perhaps be to commence at the oldest age and add the successive values of  $-\Delta\lambda a_x$ , since  $\lambda(1 - V_{x|n}) = \lambda(1 - V_{x|n+1}) - \Delta\lambda a_{x+n}$ ; thus,

$$\lambda(1-V_{15|5})$$

$$\cdot 9799198$$

$$\lambda(1-V_{15|4})$$

$$\text{Ar. Co. } \Delta\lambda_a \ 19 \quad \cdot 0088824$$

$$\cdot 9887522$$

$$18 \quad \cdot 0089553$$

$$\lambda(1-V_{15|3})$$

$$\cdot 9877075$$

$$17 \quad \cdot 0040526$$

$$\lambda(1-V_{15|2})$$

$$\cdot 9917601$$

$$16 \quad \cdot 0041136$$

$$\lambda(1-V_{15|1})$$

$$\cdot 9958787$$

It will thus be seen that with a set of perforated cards containing the values of  $\Delta\lambda a_x$  (or  $-\Delta\lambda a_x$ ) the principal portion of the work will be easily performed.

The cards, however, take some little time to make and are rather awkward to use, so that something both in time and convenience would be gained if a method could be devised which would obviate the necessity of their use and at the same time be equally efficacious.

In the present instance this can be easily done, and, I think, with great advantage, from another relation of the Policy Values. Thus, we have

$$\lambda(1-V_{x|n}) = \lambda a_{x+n} - \lambda a_x$$

and

$$\begin{aligned} \lambda(1-V_{x+1|n-1}) &= \lambda a_{x+n} - \lambda a_{x+1} \\ &= \lambda(1-V_{x|n}) - \Delta\lambda a_x \end{aligned}$$

so that when the first column containing the values of  $\lambda(1-V_{x|n})$  is formed for all values of  $n$ , the next column, which will consist of the values of  $\lambda(1-V_{x+1|n-1})$ , will be formed by the simple addition of the arithmetical complement of  $\Delta\lambda a_x$  to all the values in the first column, the third column would in the same manner be formed by the constant addition of ar. co.  $\Delta\lambda a_{x+1}$  to each value in the

second, and so on throughout the Table. By this method the work will check itself in horizontal lines, the top value in each column becoming zero in the next, if all the values in that line are correct, since

$$\lambda(1 - V_{x+1}) - \Delta\lambda a_x = \lambda(1 - V_{x+1|0}) = 0.$$

The whole operation will however be best explained by an example. A preliminary Table will have to be formed of the values of  $\lambda a_x$  and their differences. Let us take the following from the English Life Table No. 3 (Males, 3 per cent).

Age (x).	$\log a_x$	$\Delta \log a_x$	Ar. Co. $\Delta \log a_x$
15	1.3637102	.9958737	.0041263
16	1.3595839	.9958864	.0041136
17	1.3554703	.9959474	.0040526
18	1.3514177	.9960447	.0039553
19	1.3474624	.9961676	.0038324
20	1.3436300		

The first column containing the values of  $\lambda(1 - V_{15|x})$  will be found by placing the successive values of  $\Delta\lambda a_x$  on the paper, leaving 3 lines between each, and continuously adding as explained by the card method; but it will be quite sufficient to place the value of  $\lambda a_{15}$  at the top of a card, and, sliding it down, subtract that value from the successive values of  $\lambda a_x$ , placing the results down in a column on ordinary cross-ruled paper, with 3 lines interval between each. It would perhaps be better to check these values at first, but it is not necessary, since, if there be any mistake made, it is sure to appear afterwards, as will be directly seen from the example, in which I have purposely made an error. We next place the value ar. co.  $\Delta\lambda a_{15}$  upon the top of a card, and add it to each value in the first column, placing the results immediately opposite in the second column. The first result being zero shows that the top value in the first column is correct. The third column is formed in the same manner by adding ar. co.  $\Delta\lambda a_{16}$  to each value in the second column; and so on throughout the Table. I may here mention what perhaps has occurred to many before me, that when we have a series of values to add in this manner, the best way is to place them upon a slip of cross-ruled paper, on every other line, so that as soon as you have finished with one value you can double it under, and the next will be immediately ready for use. The annexed diagram will, I think, fully explain

my meaning:—in which the lines represent where the paper is doubled under.

It will be observed that this method is not attended with the difficulties of adding two quantities placed horizontally to each other, for here the values to be added are placed one under the other, and their sum only withdrawn horizontally.

We will now suppose that the work has proceeded very steadily until column ( $x=18$ ) is finished, when we find on proceeding to the formation of the next that the addition of ar. co.  $\Delta\lambda a_{18}$  to the first value (in the preceding column) does not make 0,—that there is in fact a difference of unity in the sixth place. We know that the error must be in the same horizontal line, because each value is formed from the one adjacent in the preceding column; and a little reflection will show that the values in that line are the differences of  $\log a_{15}$ ,  $\log a_{16}$ , &c. subtracted from  $\log a_{19}$ ; so in order to find the error, we place  $\lambda a_{19}$  upon the bottom of a card and proceeding backwards subtract the values of  $\lambda a_{18}$ ,  $\lambda a_{17}$ , &c. from it until the error is found, which in this case is in the first column.

When this part of the work is finished, all that is required is to find the natural numbers corresponding to the logarithms and subtract them from unity.

$x$	$\Delta\lambda a_x$
15	0041263
16	0041136
17	0040526
18	0039553
19	0088324

$n$	$x=15$	$n$	$x=16$	$n$	$x=17$	$n$	$x=18$	$n$	$x=19$
1	9958737 990544 009456								
2	9917601 981206 018794	1	9958864 990573 009427						
3	9877075 972092 027908	2	9918338 981372 018628	1	9959474 990712 009288				
4	2 983752 963279 036721	3	8 9878795 972475 027525	2	2 9919921 981730 018270	1	4 9960457 990934 009066		
5	9799198 954816 045184	4	9840461 963931 036069	3	9881597 973105 026895	2	9922123 982228 017772	1	9961676 991214 008786

Whenever a Table of Policy Values is required the annuity values will generally be calculated, and the above method will, I think, be found the best. But sometimes it may happen that the premiums only are given, as for example, when we wish to form a table from the Office premiums. In such a case the process will be very similar, since

$$\lambda(1 - V_{x|n}) = \lambda(P_x + d) - \lambda(P_{x+n} + d)$$

and

$$\begin{aligned} \lambda(1 - V_{x+1|n-1}) &= \lambda(P_{x+1} + d) - \lambda(P_{x+n} + d) \\ &= \lambda(1 - V_{x|n}) + \Delta\lambda(P_x + d) \end{aligned}$$

The preliminary Table to be constructed would be as follows:

Age (x)	$P_x$	$P_x + d$	$\lambda(P_x + d)$	$\Delta\lambda(P_x + d)$
15	·01415407	·04328028	·6362901	·0041256
16	·01456717	·04369338	·6404157	·0041141
17	·01498305	·04410926	·6445298	·0040531
18	·01539663	·04452284	·6485829	·0039553
19	·01580397	·04493018	·6525362	·0038322
20	·01620219	·04532840	·6563704	

The first column containing the values of  $\lambda(1 - V_{15|n})$  would be formed by placing  $\lambda(P_{15} + d)$  at the bottom of a card and, sliding it down, subtracting the successive values of  $\lambda(P_x + d)$  from it. The second column would be formed by the addition of  $\Delta\lambda(P_{15} + d)$  to the values in the first column, and so on throughout the table as explained with the annuities-due. The check upon the work will be the same, and the mode of finding any error very similar; that is to say, if it occurred in the same place as above, we should place the value  $\lambda(P_{19} - d)$  on the top of a card and proceeding backwards deduct it from the values  $\lambda(P_{18} - d)$ ,  $\lambda(P_{17} - d)$ , &c. until the mistake was found. The results if worked out from the above values would be very similar to the above table formed from the annuities.

It will no doubt be noticed that I have here taken the values of  $P_x$  to a great many decimal places, but that was simply in order to show how closely the values by the two different methods might be made to approach. In fact the results, theoretically speaking, should be exactly the same, but, if only five or even six places had been taken instead of eight, there would have been a discrepancy between the final results by the two methods, varying to the extent sometimes of two in the fifth place.

The third and last method is one which will perhaps be the

least needed of all, and has reference more especially to the case where the Office in valuing the policies credits itself with the present value of the loaded premiums, or with premiums other than the pure premiums of the Table of Mortality used in the valuation.

Here we are reduced to the formation of our Table from the familiar formula

$$V_{x|n} = A_{x+n} - P_x a_{x+n};$$

but even here we shall be able to facilitate the operations very considerably, since

$$\lambda(A_{x+n} - V_{x|n}) = \lambda P_x + \lambda a_{x+n}$$

and

$$\begin{aligned} \lambda(A_{x+n} - V_{x+1|n-1}) &= \lambda P_{x+1} + \lambda a_{x+n} \\ &= \lambda(A_{x+n} - V_{x|n}) + \Delta \lambda P_x \end{aligned}$$

so that the successive columns of primary values will be formed by the simple addition of  $\Delta \lambda P_x$ .

The process to be pursued under this method will be best explained by expressing the Table of values symbolically, thus,

$n$	$x=15$	$n$	$x=16$	$n$	$x=17$
1	$A_{16} - P_{15}a_{16}$				
2	$A_{17} - P_{15}a_{17}$	1	$A_{17} - P_{16}a_{17}$		
3	$A_{18} - P_{15}a_{18}$	2	$A_{18} - P_{16}a_{18}$	1	$A_{18} - P_{17}a_{18}$

where it will be seen that the premiums are the same throughout each column, and that the assurance and annuity-due values are the same throughout each horizontal line.

The preliminary table required by this method to form the same values as before, will be

Age ( $x$ )	$A_x$	$\lambda a_x$	$\lambda P_x$	$\Delta \lambda P_x$
15	327033	13637102	21508814	0124938
16	333395	3595839	1633752	0122250
17	339680	3554703	1756002	0118253
18	345815	3514177	1874255	0113408
19	351746	3474624	1987663	0108077
20	357441	3436300	2095740	

The first column containing the values of  $\lambda P_{15}a_x$  is formed by placing  $\lambda P_{15}$  upon the top of a card, and, commencing with  $\lambda a_{16}$ , adding it to the successive values of  $\lambda a_x$ , placing the results

down with an interval of three lines space between each. The second column is formed by adding  $\Delta\lambda P_{15}$  to the values in the first, and the third by adding  $\Delta\lambda P_{16}$  to those in the second, and so on throughout the Table. The check upon the work by this method is not quite so simple as by the other methods, since the first values in each column must be checked by the actual addition of  $\lambda P_x$  to  $\lambda a_{x+1}$ ; but the error, when there is one, will be found as before in the horizontal line.

The next process is to find the corresponding natural numbers; which, having been checked, have to be subtracted from the respective values of  $A_x$ . As each line treats with only one value of  $A_x$  at a time, we require the means of moving each value across the paper in such a manner that it will serve the purpose of a minuend to all the quantities to be subtracted from it. This is most conveniently accomplished by placing upon a slip of cross-ruled paper, on every alternate line, the values of  $A_x$ , commencing at the oldest age, so that as each value is finished with, it may be doubled under, and the next be immediately ready for use. Each of the results will of course require checking, but I need hardly say that this will be accomplished by adding the last value found to the previous subtrahend, which should agree with the respective values of  $A_x$ .

$n$	$x=15$	$n$	$x=16$	$n$	$x=17$	$n$	$x=18$	$n$	$x=19$
1	5104653 328941 009454								
2	5063517 320887 018793	1	5188455 330252 009428						
3	5022991 317906 027909	2	5147929 327185 018630	1	5270179 336525 009290				
4	4983438 315024 036722	3	5108376 324218 027528	2	5230626 333474 018272	1	5348879 342679 009067		
5	4945114 312256 045185	4	5070052 321370 036071	3	5192302 330545 026896	2	5310555 339609 017772	1	5423963 348655 008786

I trust I have been able to make my explanations intelligible.

A Table of Policy Values may sometimes be of great utility, such as when it is desired in the investigations to value each policy

separately, or when the profits have to be distributed according to the values of the policies, or even for the purpose of calculating the surrender values; and should such a table ever be required, I think I have shown that, far from the computations being difficult and laborious, they can be constructed with the greatest ease and accuracy.

The above paper was written before I had seen Dr. Zillmer's ingenious application of M. Thomas' Arithmometer to the construction of Tables of Policy Values. By means of that instrument such tables are more easily constructed and are less liable to error; but without it, his formula must be reduced to that given above before it is applicable for ordinary working.

*Notes on Newton's Formulae for Interpolation.* By PROFESSOR  
LUDVIG OFFERMANN, of Copenhagen.

## II.

IN the first note I gave the demonstration of Newton's formulae of Interpolation.

Now the question arises, "How has Newton found this general solution of the problem," or rather, "how has this general form of the problem and the consequent general solution been suggested to Newton?"

The answer is given in the *Methodus Differentialis*.

There Newton proceeds as follows (in the notation I have taken the liberty of making some slight alterations):

Given, the  $(n+1)$  arguments  $a, b, c, d, e, f, g \dots$ ; and the corresponding  $(n+1)$  values  $A, B, C, D, E, F, G \dots$

Proposed, to determine the algebraic integer and rational function of the  $n$ th degree, which assigns the given values to the given arguments.

*Solution.* Let the function in question be denoted by

$$X = \kappa_0 + \kappa_1 x + \kappa_2 x^2 + \kappa_3 x^3 + \dots + \kappa_n x^n$$

then the  $(n+1)$  quantities  $\kappa_0, \kappa_1, \kappa_2, \kappa_3 \dots \kappa_n$  are to be determined by  $(n+1)$  linear equations of the form

$$A = \kappa_0 + \kappa_1 a + \kappa_2 a^2 + \kappa_3 a^3 + \kappa_4 a^4 + \dots$$

$$B = \kappa_0 + \kappa_1 b + \kappa_2 b^2 + \kappa_3 b^3 + \kappa_4 b^4 + \dots$$

$$C = \kappa_0 + \kappa_1 c + \kappa_2 c^2 + \kappa_3 c^3 + \kappa_4 c^4 + \dots$$

$$\dots \dots \dots \dots \dots \dots \dots \dots$$

By successive eliminations we obtain the following systems of equations:

$$\frac{A-B}{a-b} = \delta'(a, b) = \kappa_1 + \kappa_2(a+b) + \kappa_3(a^2+ab+b^2) \\ + \kappa_4(a^3+a^2b+ab^2+b^3) + \dots$$

$$\frac{B-C}{b-c} = \delta'(b, c) = \kappa_1 + \kappa_2(b+c) + \kappa_3(b^2+bc+c^2) \\ + \kappa_4(b^3+b^2c+bc^2+c^3) + \dots$$

$$\frac{C-D}{c-d} = \delta'(c, d) = \kappa_1 + \kappa_2(c+d) + \kappa_3(c^2+cd+d^2) \\ + \kappa_4(c^3+c^2d+cd^2+d^3) + \dots$$

$$\dots \dots \dots$$

$$\frac{\delta'(a, b) - \delta'(b, c)}{a-c} = \delta''(a, b, c) = \kappa_2 + \kappa_3(a+b+c) \\ + \kappa_4(a^2+ab+ac+b^2+bc+c^2) + \dots$$

$$\frac{\delta'(b, c) - \delta'(c, d)}{b-d} = \delta''(b, c, d) = \kappa_2 + \kappa_3(b+c+d) \\ + \kappa_4(b^2+bc+bd+c^2+cd+d^2) + \dots$$

$$\dots \dots \dots$$

$$\frac{\delta''(a, b, c) - \delta''(b, c, d)}{a-d} = \delta'''(a, b, c, d) = \kappa_3 + \kappa_4(a+b+c+d) + \dots$$

$$\frac{\delta''(b, c, d) - \delta''(c, d, e)}{b-e} = \delta'''(b, c, d, e) = \kappa_3 + \kappa_4(b+c+d+e) + \dots$$

$$\frac{\delta'''(a, b, c, d) - \delta'''(b, c, d, e)}{a-e} = \delta^{IV}(a, b, c, d, e) = \kappa_4 + \dots$$

&c.

This process must of course end with

$$\delta^n(a, b, c, \dots) = \kappa_n$$

and then  $\kappa_{n-1}$ ,  $\kappa_{n-2}$ , ...,  $\kappa_2$ ,  $\kappa_1$ ,  $\kappa_0$ , may be found by substitution from below.

The above is the substance of the two first Propositions in the *Meth. Diff.* In PROP. I. it is asserted that the divisions by which the divided differences are obtained, may always be performed without giving rise to fractional quotients, and this is actually shown as far as the fourth difference. It is to be noted, that in this Proposition the function  $X$  is not limited to a finite number of terms. In PROP. II. the number is supposed finite; and it is asserted that the highest divided difference is the coefficient of the

highest term of  $X$ , and that from this and the other divided differences we may find the coefficients in the other terms of  $X$ , as is actually shown in the example in PROP. I.

Newton then adds simply, *From these Propositions the following may easily be obtained.* ("Ex his Propositionibus quæ sequuntur facile colligi possunt"), and this is perfectly true.

In the two next Propositions are given the formulæ for Interpolation by *central* divided differences, in PROP. III. for equidistant arguments with the constant difference 1, in PROP. IV. for arguments not equidistant. In both Propositions, the two cases of an odd and of an even number of given values, are distinguished.

Then it is shown that the preceding may be applied to *Approximate Interpolation* (PROP. V.) and *Integration* (PROP. VI.) of any function of which a number of values are known.

After these six Propositions comes a SCHOLIUM, in which Newton—after pointing out how useful the preceding theory is in the calculation of tables and in solving problems depending on integrations—concludes his important little tract with this theorem: "Through any number of points may be drawn not only a parabolic curve, but also an infinite number of other curves," or in modern analytical language: "The condition, that to any given number of arguments ( $a, b, c \dots$ ) correspond as many given values of an unknown function ( $A, B, C \dots$ ), may be satisfied not only by an integral and rational algebraical function, but also by an infinite number of other functions." This is actually proved in a very ingenious manner, without making use of periodic functions.

*On the Theory of Probabilities.* By SIR JOHN F. W. HERSCHEL, BART., K.H., M.A., D.C.L., &c. &c. *Being extracts from a review of "Quetelet on Probabilities," which appeared in the "Edinburgh Review" for July, 1850.\**

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THE theory of Probabilities has been characterized by Laplace, one of those who have contributed most largely to its advance,—as "good sense reduced to a system of calculation;" and such, no

\* This essay contains so much that is of permanent interest to all students of the Theory of Probabilities that we believe we are doing our readers a real service in reproducing the greater part of it. We would gladly have reprinted the whole, but for the limited amount of space at our disposal; but those who may wish to read the essay in its entirety, will find it in the collected volume of Sir John Herschel's essays published by Messrs. Longmans in 1857.—ED. J. I. A.

doubt, it is. But it must be especially noticed that there is hardly any subject to which thought can be applied, which calls for so continuous an application of that excellent quality, or in which it is easier to make mistakes from simple want of circumspection. And, moreover, that its reduction to calculation is attended with difficulties of a very peculiar nature, such as occur in no other application of mathematical analysis to practical subjects, arising out of the great magnitudes of the numbers concerned, which defeat the ordinary processes of arithmetical and logarithmic calculation, by exhausting the patience of the computer, and require special methods of *approximate* evaluation to bring them within the compass of human industry. These methods form a conspicuous feature of the general subject, and have furnished scope for very extraordinary displays of mathematical talent and invention. That very large numbers will inevitably be concerned in questions where numerous and independent contingencies may take place, and in any order or mode of combination, will be apparent to any one who considers the astonishing *fecundity* of such combinations numerically estimated, when the combining elements are many. For example, the number of possible "hands" at whist (regard being had to the trump) is 1,270,027,119,200.

The calculus of Probabilities, under the less creditable name of the doctrine of Chances, originated at the gaming table; and was for a long time confined to estimating the chances of success and failure in throws of dice, combinations of cards, and drawings of lotteries. It has since effectually obliterated the stain of its cradle, as there is no monitor more severe, no lecture which can be delivered on the certain ruin which attends habitual gambling more emphatic than may be found in its demonstrations. Questions of this kind, it is true, are still retained in treatises on the subject; nor indeed can they be conveniently dispensed with, since they furnish the simplest and readiest illustrations of the combination of independent events, and the superposition of contingencies arising out of them, which belong essentially to its principles. They, however, form a very insignificant part of its applications, in comparison with the problems which its scope at present takes in, and which its modern developments have enabled it to handle.

Its first advances towards the dignity of a distinct branch of Mathematics are attributable to the celebrated Blaise Pascal, and his no less celebrated contemporary and correspondent Fermat,—

both reasoners of extraordinary acuteness, and who seem to have been specially attracted (like many of their followers) by the close reasoning and careful analysis its problems demand for their successful issue. Subsequent to these, but still among its earlier contributors, we find the distinguished names of Huyghens (to whom we owe the first treatise on the subject), those of the Grand Pensionary De Witt, Hudde, and Halley (with whom originated its application to the probabilities of life and the construction of tables of mortality), and that of James Bernouilli, who may be considered the first philosophical writer on the subject. To him we owe the demonstration of two great fundamental theorems or laws of Probability, as applied to the results of very numerous trials of any proposed species of contingency: viz., 1st, that in any vast number of trials there is a demonstrably greater probability that the events will happen in numbers proportional to *their respective chances in a single trial*, than in any other *specified* proportion; and, 2dly, that a number of trials may always be assigned so great, as to make the probability of the events happening in numbers falling within any assigned limits of deviation from that proportion, however narrow, approach to certainty as nearly as we please. The first of these propositions has the air of a truism, when the meaning of its terms is not nicely weighed. But the second is obviously of paramount importance; since it goes to take the totality of results obtained in any sufficiently extensive series of trials, almost out of the domain of chance, and to place in evidence the influence of any "cause" or circumstantial condition common to the whole series, which may give even a trifling preponderance of facility to any one of the classes of events contemplated over the rest.

Common sense, it may perhaps be said, would tell us as much as this. No doubt it might suggest some such propositions as likely enough to be true; and the usual course of inductive reasoning up to causes tacitly assumes their truth. But when we come to demand what number of trials may reasonably be expected to bring out into prominence a very small given preponderance of facility? or to declare within what limits of accuracy such preponderance may reasonably be expected to be represented on the upshot or final average of a given number of trials?—or, lastly, what is the probability that on a given number of trials such an average will represent the preponderant facility in question within given limits of exactness? all of them, and especially the last, evidently practical questions of much interest; we find ourselves

forced to appeal from the unaided judgment of simple good sense, to strict numerical calculation,—taking for its basis not a mere *aperçu* but a rigorous demonstration of the truth of the propositions above stated. This is very much the case with all the more important conclusions of this theory; when generally enunciated, they are almost universally seen to be pretty plainly conformable to ordinary clear-judging apprehension of their relations. Even the apparently paradoxical conclusions by which we are occasionally startled, lose that aspect when their exact wording is duly attended to, and all the conditions implied in it clearly apprehended. It is their applicability to exact computation, and the handle they afford thereby for precise determinations useful in practice, which give them all their value.

Problems of the class above mentioned were first successfully treated by De Moivre, to whom also we owe the happy idea of applying Stirling's theorem to approximate to the ratio of the high numbers which enter into such calculations, without which they would be impracticable. From these it would appear but a small step to pass to what may be deemed the *inverse calculus* of Probabilities, which applies the knowledge gained by the observation of past events to the prediction of future, by concluding from the succession of facts observed the respective degrees of probability of the existence of each out of several equipossible determining conditions, and thence starting as it were anew, and ascertaining from the knowledge thus acquired the probability of an event or events similarly determined *in futuro*. It was reserved, however, for another member of the gifted family of Bernouilli to make this step, which has in some respects changed the whole aspect of the subject, and given to it that degree of importance it possesses as an auxiliary of the inductive philosophy.

It may perhaps be doubted whether subsequent writers have added very materially to the intrinsic philosophy of the subject, though there can be no hesitation as to the value of the improvements they have made in its methods of procedure, whether in point of elegance or power; the extension given to its formulæ; or the numerous and important applications made of its principles, especially in those cases (which comprise almost all the really interesting ones) where the transition has to be made from the finite to the infinite, from the limited though often large number of possible combinations which its simple and more elementary problems offer, to the *literally infinite* multitude which the gradation of natural causes and influences obliges us to consider,

and which calls for the perpetual employment of the most refined theories, and the most delicate and abstruse applications of the integral calculus. In all these respects the great work of Laplace ("Théorie Analytique des Probabilités") stands deservedly pre-eminent; occupying in this department of science the same rank and position which the "Mécanique Analytique" of his illustrious rival Lagrange holds in that of force and motion, and marking (we had almost said) the *ne plus ultra* of mathematical skill and power. So completely has this sublime work been held to embody the subject in its utmost extent, and to satisfy every want of the theorist, that an interval of a quarter of a century elapsed from the date of its appearance (1812) before any further original contribution of moment was made to the theory. The valuable memoir of Poisson, published in 1837, on the probability of judicial decisions\* (which contains a *résumé* of the whole theory of Probabilities), though admirable for its clear exposition of principles and elegant analysis, can hardly be said to have carried the general subject much beyond the point where Laplace left it.

It may easily be imagined that a work like this of Laplace, followed at a short interval by an admirable *exposé* of its contents by himself ("Essai Philosophique sur les Prob."), could not fail to make a lively impression and to excite general attention. Laplace possessed in an eminent degree the talent of stating the most profound results of his own geometry in a style at once philosophical, luminous, and pleasing. Few works have been more extensively read or more generally appreciated than this Essay and that on the "Système du Monde" by the same author. There is in both a breadth and simple dignity corresponding to the greatness of the subjects treated of, a loftiness of style, the direct result of generality of conception, and which is felt as adding to rather than detracting from clearness of statement, and a masterly treatment which fascinates the attention of every reader. Nowhere can be found so great a body of important discoveries, so consecutively linked together, and so distinctly and impressively announced. It is not, perhaps, too much to say, that were all the literature of Europe, these two Essays excepted, to perish, they would suffice to convey to the latest posterity an impression of the intellectual greatness of the age which could produce them, surpassing that afforded by all the monuments antiquity has left us.

\* Recherches sur la Probabilité des Jugemens en Matière Criminelle et en Matière Civile; précédées des Règles Générales du Calcul des Probabilités. Paris, 1837.

Previous to the publication of the "Essai Philosophique," few except professed mathematicians, or persons conversant with insurances and similar commercial risks, possessed any knowledge of the principles of this calculus, or troubled themselves about its conclusions,—regarding them as merely curious, and perhaps not altogether harmless speculations. Thenceforward, however, apathy was speedily exchanged for a lively and increasing desire to know something of a system of reasoning which for the first time seemed to afford a handle for some kind of exact inquiry into matters no one had ever expected to see reduced to calculation and bearing on the most important concerns of life. Men began to hear with surprise, not unmingled with some vague hope of ultimate benefit, that not only births, deaths, and marriages, but the decisions of tribunals, the results of popular elections, the influence of punishments in checking crime—the comparative value of medical remedies, and different modes of treatment of diseases—the probable limits of error in numerical results in every department of physical inquiry—the detection of causes physical, social, and moral,—nay, even the weight of evidence, and the validity of logical argument—might come to be surveyed with that lynx-eyed scrutiny of a dispassionate analysis, which, if not at once leading to the discovery of positive truth, would at least secure the detection and proscription of many mischievous and besetting fallacies. Hence a demand for elementary treatises and popular exposition of principles, which has been liberally answered.

Among the valuable works of this kind in the French and English languages which have appeared since the epoch in question, we may notice more especially Lacroix's "*Traité Élémentaire du Calcul des Probabilités*; Paris, 1822," and the several encyclopædic essays and articles on the subject by Sir John Lubbock and Mr. Drinkwater (Bethune), in the *Library of Useful Knowledge*, by Mr. Galloway in the *Encyclopædia Britannica* (since published separately in a small and compendious form—a work of great merit and utility), and by Mr. De Morgan in the *Encyclopædia Metropolitana*. To the last-mentioned treatise, as well as to two admirable chapters on the subject in the recent elaborate work by the same author on the *Formal Logic*, we may refer as containing, *par excellence*, the clearest views of the *métaphysique* of the subject, and the most satisfactory analysis of the state of the mind as to belief or disbelief, and the degree of assurance afforded by the conclusions of the calculus in cases where the data themselves are vague and uncertain, which can any

where be found. All or any of these works will afford the English student a perfect insight into the mathematical treatment and reasonings of the subject, and consequently serve as an abundant preparation for the study and mastery of Laplace's great work; but we would caution all who desire to enter upon the more general and intricate parts of the theory, never for an instant to lose sight of special examples and numerical particulars, since nothing can exceed the bewilderment of ideas experienced by the tyro in this department of mathematics, who trusts himself *with both feet off the ground* to the whirl of symbols and notations in which those who are accustomed to ride these storms know how to guide their course, and even seem to feel a wild and fierce delight in the turmoil.

There is, however, a very large portion of those who desire to know something of the results at which thinking men have arrived in this as in all other departments of knowledge, to whom a book full of mere algebraic formulæ and calculations must remain for ever sealed. These are not necessarily or generally persons of despicable acquirements or intellect; nor is this their curiosity to be slighted as devoid of a reasonable object or motive. They desire to understand with a view to apply. Mathematicians, in common with men of high science in all departments, have long since begun to perceive that they have to address a mixed audience of a highly important and respectable character—an audience by no means disposed to treat them with derision or distrust, but, on the contrary, to regard them as their fitting instructors in matters within the scope of their legitimate pretensions, if only they will condescend to make themselves intelligible. Learned jargon such an audience will not endure. Charlatanerie of every description it can detect and chastise. Common-sense statement driven home by pointed illustration, and an earnest endeavour to inform, are what it eagerly desires, and in such a spirit is assuredly entitled to receive at the hands of those able to afford it.

The work now before us is conceived on these principles, and on this view of the duty devolving on those who have advanced beyond the ordinary limits of knowledge, to pause occasionally in their onward career, and inform the world, in plain terms and without exaggeration, whither they have got, and what they see beyond, which may make it worth while either for themselves to continue in the track, or for others to follow in it; as well as to render easy and intelligible to all whom it may concern the practical application of the information acquired. Its author is a teacher

well worth listening to, and may claim attention on the excellent ground that he has himself approached his subject in a practical manner, through a long and severe apprenticeship to the actual collection of data in a great variety of departments, and to the deduction from them of definite results of unmistakeable value and import, by the rules and principles he professes to teach.

\* \* \* \* \*

A comparatively small portion of the work, the first and least extensive only of four divisions into which it is broken, and an appendix in the form of notes containing tables and formulæ, are devoted to the theory of Probabilities in the abstract, and to the illustration of its fundamental axioms and propositions; all which have been so repeatedly and so well laid down and elucidated in the various treatises we already possess, that it is hardly possible to place them in any very new and more than usually striking light. The distinction between mathematical and moral expectation belongs to this part of the subject, and can hardly be put more pointedly than it was originally done by Buffon, who first called attention to it.

“If two men were to determine to play for their whole property [supposed equal, and with equal risks], what would be the effect of the agreement? The one would only double his fortune, and the other would reduce his to nought. What proportion is there between the loss and the gain? The same that there is between all and nothing. The gain of the one is but a moderate sum; the loss of the other is numerically infinite, and morally so great that the labour of his whole life may not suffice to restore his property.”

It was on such considerations that Daniel Bernouilli was led to propose, as a rule for estimating the value of a very small pecuniary or other material advantage, its *relative* value as compared with the total fortune of the party benefited, and for the moral as distinguished from the mathematical expectation of such advantage, that relative value multiplied by the probability of its accruing. On this or some equivalent mode of estimation is founded the principle of the subdivision of risks, which, rightly understood, so as to preserve their absolute independence while multiplying their number, is the best guarantee of commercial security. It is by such subdivision carried to an extreme point,\* that insurance and annuity offices thrive, and that benefit societies might do so, were it not for the single great risk which the dishonesty of entrusted agents throws in their way as a fearful stumbling-block.

\* This phrase appears to us rather too strong. While the same Office will grant insurances for £100 and for £10,000 at its own risk, it can scarcely be said that the subdivision of risks is carried to an extreme point.—ED. J. I. A.

In the case of savings' banks, this is, in fact, the only risk; and, as experience has too recently\* and abundantly shown, a most imminent and fatal one. To annihilate this risk by a perpetual and searching superintendence, carried even to the utmost stretch of suspicious vigilance, obnoxious as it may appear, is the paramount duty of all who connect themselves with them as managers or trustees. Of the general benefit of such institutions, which, by guaranteeing the security of the produce of successful exertions, tend to cherish habits of industry, prudence, and frugality, no one can entertain a doubt. It is in this point of view that a certain considerable amount of national indebtedness, so far from meriting denunciation as an evil, ought to be regarded as an indispensable element and engine of civilization. In its practical working it resolves itself into the establishment of a savings' bank on a vast scale, administered with what may be considered a perfect exemption from the consequences of dishonesty in its officials, and subject only to the inconvenience (no doubt a considerable one), of its deposits being withdrawable only at a market value,—but that market the fairest, readiest, and openest which can anywhere exist. Yet it is too commonly forgotten by those who deprecate taxation, while insisting on the objects for which taxation is instituted, and which alone it can secure, that the interest on savings' bank deposits is derivable only from that source, and that every depositor is as truly (and in some respects even more emphatically) a tax-holder—as the proprietor of consols.

To render the consequences of our actions certain and calculable as far as the conditions of humanity will allow, and narrow the domain of chance, as well in practice as in knowledge, is so thoroughly involved in the very conception of law and order as to make it a primary object in every attempt at the improvement of social arrangements. Extensive and unexpected fluctuation of every description, as it is opposed to the principle of divided and independent risks, so it also, by consequence, stands opposed to the most immediate objects of social institutions, and forms the element in which the violent and rapacious find their opportunities. Nothing, therefore, can be more contrary to sound legislative principle than to throw direct obstacles in the way of provident proceedings on the part of individuals (as, for instance, by the exorbitant taxation of insurances), or to encourage a spirit of general and reckless speculation, by riding unreservedly over established laws of property, for the avowed purpose of affording a

\* 1850.

clear area for the development of such a spirit on a scale of vast and simultaneous action. The sobering influence of an upper legislative assembly, refusing its sanction to the measures demanded, or spreading it over time, can alone repress or moderate these epidemic outbreaks of human cupidity: and its mission is abandoned, and its functions *pro tanto* abdicated, if it retreat from the performance of this duty.

The first and most important application of the calculus of Probabilities (since it applies to all departments of science, and affords a measure of the degree of precision attained in all numerical determinations) is that which relates to means and limits, and forms the second division of M. Quetelet's work. A general idea of the sort of questions contemplated in this department of the theory, and the kind of relations they involve, may be conveyed by the following simple case. Suppose a man to throw stones at random, and without any aim. From the marks left by any given number of them, however great, on a wall, we could obtain no impression, or a fallacious one, of his intention. All that we could conclude from their evidence would be, that, if he aimed at anything, it was not a point in the surface of the wall, and that only stray missiles had struck it. But, suppose he had been practising with a rifle at a wafer on the wall; which being subsequently removed, we were required to indicate at once the situation it had occupied, and his skill as a marksman. It is obvious enough that, from the evidence of a great number of shot-marks, both might be determined, at least with a certain degree of approximation, and with a probability of error less in proportion to their number. The theory of Probabilities affords a ready and precise rule, applicable not only to this, but to far more intricate cases: it is this: that the most probable determination of one or more invariable elements from observation is that in which the sum of the squares of the individual errors or aberrations from exactness which the observations imply, shall be the least possible. In the case before us the "errors" are the distances of the shot-marks from the point where the centre of the wafer was fixed; to ascertain which we have, therefore, to resolve the geometrical problem (a very elementary one)—"to find a point such that the sum of the squares of its distances from a certain number of given points shall be a minimum,"—a problem which is, in effect, identical with that of finding their centre of gravity. As to the skill of the marksman, it may be estimated in two different ways:—1st, by ascertaining what is the probability that he will place a single shot within a given distance: this may be done by

counting the number of marks within that distance of the point ascertained as above, and dividing it by the total number: or, 2ndly, by ascertaining within what distance of the mark he would probably (*i.e.* more probably than the contrary, or with a probability exceeding one half) place it: this may be done by describing circles about the wafer's place (found as above) for a centre, and measuring the radius of that which just includes half the total number of marks. For it is obvious that, so far as the evidence before us goes, and judging only from the numbers of instances favourable or unfavourable, there is just as great a presumption that he will shoot within as without that circle; and, if it be ever so little enlarged, the scale will turn in his favour.

Suppose the rifle replaced by a telescope duly mounted; the wafer by a star on the concave surface of the heavens, always observed for a succession of days at the same sidereal time; the marks on the wall by the degrees, minutes, and seconds, read off on divided circles; and the marksman by an observer; and we have the case of all direct astronomical observation where the place of a heavenly body is the thing to be determined. Or we may substitute for the wall the floor of a lofty building or deep mine, and for the marksman an experimenter dropping, with all possible care, smooth and perfectly spherical leaden balls from a fixed point at the summit of the building or the mouth of the mine, with intent to determine, by the means of a great number of trials, the true point of incidence of a falling body,—a physical experiment of great interest. We might, if we pleased, instance more complicated cases, in which the elements to be determined are numerous and not *directly* given by observation, but with such we shall not trouble our readers: suffice it to say that the rule above stated, or, as it is technically called the "Principle of Least Squares," furnishes, in all cases, a system of geometrical relations characteristic of the *most probable* values of the magnitudes sought, and which, duly handled, suffice for their numerical determination.

This important principle was first promulgated, rather as a convenient and impartial mode of procedure than as a demonstrable theorem, by Legendre. Its demonstration was first attempted by Gauss,—but his proof is in fact no proof at all, since it takes for granted that in the case of a single element, variously determined by *any finite number of observations however small* the arithmetical mean is the most probable value,—a thing to be demonstrated, not assumed, not to mention other objections. Laplace has given a rigorous demonstration, resting on the comparison of equipossible

combinations, infinite in number. His analysis is, however, exceedingly complicated, and, although presented more neatly by Poisson, and in this work stripped by M. Quetelet of all superfluous difficulties and reduced to the most simple and elementary form we have yet seen, yet must of necessity be incomprehensible to all whose knowledge of the higher analysis has not perfectly familiarized them with those delicate considerations involved in the transition from finite differences to ordinary differentials. Perhaps, therefore, our non-mathematical readers will pardon us if we devote a single page to what appears to us a simple, general, and perfectly elementary proof of the principle in question, requiring no further acquaintance with the transcendental analysis than suffices for understanding the nature of logarithms.

We set out from three postulates. 1st, that the probability of a compound event, or of the concurrence of two or more independent simple events, is the product of the probabilities of its constituents considered singly; 2dly, that there exists a relation or numerical law of connexion (at present unknown) between the amount of error committed in any numerical determination and the probability of committing it, such that the greater the error the less its probability, according to some regular LAW of progression, *which must necessarily be general and apply alike to all cases, since the causes of error are supposed alike unknown in all*; and it is on this ignorance, and not upon any peculiarity in cases, that the idea of probability in the abstract is founded; 3dly, that the errors are equally probable if equal in numerical amount, whether in excess, or in defect of, or in any way beside the truth. This latter postulate necessitates our assuming the function of probability to be what is called in mathematical language *an even function*, or a function of the square of the error, so as to be alike for positive and negative values; and the postulate itself is nothing more than the expression of our state of *complete* ignorance of the causes of error, and their mode of action. To determine the form of this function, we will consider a case in which the relations of space are concerned.

Suppose a ball dropped from a given height, with the intention that it shall fall on a given mark. Fall as it may, its deviation from the mark is *error*, and the probability of that error is the unknown function of its square, *i.e.* of the sum of the squares of its deviations in any two rectangular directions. Now, the probability of any deviation depending solely on its magnitude, and not on its direction, it follows that the probability of each of these

rectangular deviations must be the same function of *its* square. And since the observed oblique deviation is equivalent to the two rectangular ones; supposed concurrent, and which are essentially independent of one another,\* and is, therefore, a compound event of which they are the simple independent constituents, therefore its probability will be the product of their separate probabilities. Thus the form of our unknown function comes to be determined from this condition, viz., that the product of such functions of two independent elements is equal to the same function of their sum. But it is shown in every work on algebra that this property is the peculiar characteristic of, and belongs only to, the exponential or antilogarithmic function. This, then, is the function of the square of the error, which expresses the probability of committing that error. That probability decreases, therefore, in geometrical progression, as the square of the error increases in arithmetical. And hence it further follows, that the probability of successively committing any given system of errors on repetition of the trial, being, by postulate I., the product of their separate probabilities, must be expressed by the same exponential function of the sum of their squares however numerous, and is, therefore, a maximum when that sum is a minimum.

Probabilities become certainties when the number of trials is infinite, and approach to practical certainty when very numerous. Hence this remarkable conclusion, viz., that if an exceedingly large number of measures, weights, or other numerical determinations of any constant magnitude, be taken,—supposing no bias, or any cause of error acting preferably in one direction, to exist—not only will the number of small errors vastly exceed that of large ones,† but the results will be found to group themselves about the mean of the whole, always according to one invariable law of numbers (that just announced), and *that* the more precisely the greater the total number of determinations.

\* That is, the increase or diminution in one of which may take place without increasing or diminishing the other. On this, the whole force of the proof turns. (H. 1857.)

† Sir Joshua Reynolds, in his celebrated Lectures to the Royal Academy, has laid it down as the fundamental principle of the pictorial art, that beauty of form and feature consists in their close approximation to the mean or average conformation of the human model. Were this the case, ugliness ought to be extremely rare, and the highest degrees of beauty those of the most ordinary occurrence, a conclusion contrary to all experience. (H. 1857.) Another consequence follows, viz., that in designing the original prototype of the human form and face, the designer *had not in view especially* the production of what men call beauty, but some one or more objects of greater importance to the well-being of the total organism. The *animus* of making the *beautiful* thing, in that sense, was absent. The *capability* of beauty having been secured in the plan of the organization, it seems as if it were intended that the perfection of personal beauty like the highest genius or the most exalted goodness should occur but rarely in our species.

Such being the case, and the law of distribution of errors over the whole range of possible error being known, it becomes practicable to assign the relative numbers of cases in which the errors will fall respectively within and beyond any proposed limit on the average of an infinite number of trials, and thence to assign, *a priori*, the probability of committing in any single future trial—not a given specific amount of error, but an error *not exceeding that limit*, provided only the probable error of a single trial be known; which, as we have seen, can always be ascertained on the evidence of foregone experience, if very extensive. Computations of this sort are rendered exceeding easy by a table, originally calculated by Kramp, with a widely different object, which is given in the notes to M. Quetelet's book, and more *in extenso*, with differences, at the end of Mr. Galloway's work above noticed.

What is yet more remarkable is, that the skill with which the trials are performed is absolutely of no importance so far as the *law* of distribution of the errors over their total range is concerned. An important consequence follows from this: viz., that rude and unskilful measurements of any kind, if accumulated in very great numbers, are competent to afford precise mean results. The only conditions are the continual *animus mensurandi*, the absence of bias, the correctness of the scale with which the measures are compared, and the assurance that we have the *entire range of error* at least in one direction within the record.

In a matter so abstract, and on which, at first sight, human reason would appear to have so little hold, it is assuredly satisfactory to find the same conclusion, and *that* one so positive and definite, reached by different roads and from different starting points. It is not easy to imagine a principle of demonstration having less in common than that given above with those of Laplace, Poisson, and Quetelet. Yet the conclusions are identical, and the verifications afforded by experience in all cases where the trials have been sufficiently numerous, and care taken to guard against bias, have been of the most unequivocal character.

Some of these verifications, adduced by M. Quetelet as instances of the practical application of his rules of calculation in the theory of means and limits, have an interest independent of their value as such. They form part of a series of researches in which he has engaged extensively on the normal condition, physical and moral, of the human species, and, *inter alia*, as regards its physical development, in respect of stature, weight, strength, &c. By the assemblage of data collected from the experience of others, as well

as his own, he has arrived at a variety of interesting conclusions as to the law of progressive increase and decay in all these respects, of the *typical* individual, of either sex, during the period of life, which are given at large in his work "*Essai de Physique Sociale*."\* We shall offer no apology for placing one or two of these before our readers.

From the 13th volume of the "Edinburgh Medical Journal," M. Quetelet extracts a record of the measurement of the circumference of the chests of 5738 Scotch soldiers of different regiments. The measures are given in inches, and are grouped in order of magnitude, proceeding by differences of 1 inch, each group containing of course (we presume) all that differ by less than half an inch in excess or defect from its nominal value. The extreme groups are those of 33 and 48 inches, and the respective numbers in the several groups stand arranged as in the table below.† Supposing each measure exactly performed, these, therefore, may be taken as the results of nature's own measurements of her own model; and the question whether she recognizes such a model? is at once decided by inspection of the groups, in which the *animus mensurandi* is broadly apparent. It is equally so that such model would fall within the group of 40 inches. An exact calculation of the mean, allowing to each group a weight in proportion to the number it contains, assigns 39·830 inches as the circumference of the chest of this model.

Now this result, be it observed, is a *mean* as distinguished from an *average*. The distinction is one of much importance, and is very properly insisted on by M. Quetelet, who proposes to use the word mean only for the former, and to speak of the latter as the "arithmetical" mean. We prefer the term average, not only because both are truly arithmetical means, but because the term *average* carries already with it that vitiated and vulgar association which renders it less fit for exact and philosophical use. An average may exist of the most different objects, as of the heights of

\* Sur l'Homme et sur la Développement de ses Facultés; ou Essai de Physique Sociale. Paris, Bachelier, 1835.

† Inches .....	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	
Groups as per Observations }	3	18	81	185	420	749	1073	1079	934	658	370	92	50	21	4	1	Totals.
M. Quetelet ....	4	17	63	185	419	765	1054	1140	961	629	321	125	40	9	2	1	5738
Our calculation ..	6	21	72	200	433	746	1024	1103	943	639	341	145	50	12	2	1	5738

houses in a town, or the sizes of books in a library. It may be convenient, to convey a general notion of the things averaged; but involves no conception of a natural and recognizable central magnitude, all differences from which ought to be regarded as deviations from a standard. The notion of a mean, on the other hand, does imply such a conception, standing distinguished from an average by this very feature, viz., the regular march of the groups, increasing to a maximum, and thence again diminishing.\* An average gives us no assurance that the future will be like the past. A mean may be reckoned on with the most implicit confidence. All the philosophical value of statistical results depends on a due apprehension of this distinction, and acceptance of its consequences.

The recognition of a *mean*, as thus distinguished from a mere average, among a series of results so grouped in order, depends on the observance of a conformity between the law of progression in the magnitude of the groups, and the abstract law of probability above stated, from which every consideration has been excluded, but the reality of *some* central truth, and an intention of arriving at it, liable to be baffled by none but purely casual causes of error. And the test to be applied, in this and all similar cases, is this. Is it possible to assign such a mean value, and such a probable error as shall alone, by the simple application of the table of probabilities, reproduce the numbers under the several groups in order with no greater deviations than shall be fairly attributable to a want of observations numerous enough to bring out the truth? In the instance before us, the answer to this inquiry is contained in the results of calculation as compared with fact in the table above referred to. The mean we have used is 39·830 inches, and our probable error 1·381 inches. Those of M. Quetelet differ somewhat from these values, which accounts for the trifling discrepancy of the results.

The coincidence admits of being placed in even a more striking light. In the complete expression, by theory, of all the groups in a statement of this kind, three elements are involved—the mean value—the maximum group *having that mean for its centre*—and the probable error. And to determine these, it ought to suffice to have before us three terms of the series. Suppose then we take for our data the numbers corresponding to 35, 39, and 43 inches, viz., 81, 1073, and 370, given by observation. Then, by a com-

\* Adopting this distinction, which appears to us not only scientifically correct, but practically convenient, it follows that we must speak of the "*average* duration of life at a given age," instead of the "*mean* duration"—the phrase introduced by Dr. Farr.—Ed. J. I. A.

putation of no great difficulty, there will result for the mean value, 39·834 inches, and for the probable error 1·413 inches, both agreeing almost precisely with those already stated. For the greatest possible group of an inch in amplitude the same calculation gives 1161, which is in obvious accord with observation. No doubt, then, can remain, as to the reality of a typical form, from which all deviations are to be regarded as irregularities. On this M. Quetelet observes,—

“I now ask if it would be exaggerating to make an even wager that a person little practised in measuring the human body would make a mistake of an inch in measuring a chest of more than 40 inches in circumference. Well! admitting this probable error, 5738 measurements made on the same individual, would certainly not group themselves with more regularity as to the order of magnitude than these 5738 measurements made on the Scotch soldiers; and if the two series were given us without their being particularly designated, we should be much embarrassed to state which series was taken from 5738 different soldiers, and which was obtained from one individual with less skill and ruder means of appretiation. (*Transl.* p. 92.)

This is assuredly an over-statement. So far from less skill being supposed in the measurements of the individual, the probable error of nature is nearly half as much more than that assumed here for the term of comparison (1 inch); and it is clearly beyond the bounds of any supposable negligence or rudeness of practice, to commit such errors as the extreme registered deviations (7 inches one way, and 9 the other), in a series of such measurements however multiplied, or even half those amounts.

We are thus led to the important and somewhat delicate question,—What we are to consider as reasonable limits, in such determinations—beyond which, if deviations from the central type be recorded, they are either to be referred to exaggeration, or regarded as monstrosities.

The answer to this question must evidently depend, first, on the “probable” deviation from the mean or typical value; secondly, on the number of cases experience has offered, or within which we agree to limit our range of speculation. It results from the tables above cited that 20,000 might be betted against 1, that an observed deviation, one way or other from the type, will not exceed sixfold its “probable” value; and therefore we shall have double that amount of chances against such a deviation in either direction separately. Among 40,000 individuals, therefore, we are entitled to expect to find one so far deviating from the mean type in excess, and one in defect. Beyond this the probabilities decrease with extreme rapidity. Thus, for a 7-fold deviation, we must seek

our specimen among 263,000 ; and, for an 8, 9, 10-fold, among 4,760,000, 250,000,000, and 25,000,000,000 respectively.

\* \* \* \* \*

Practically speaking, nothing can be simpler or more easily stated than the rules for handling any given series of determinations of a single *quæsitum* supposed to be arranged to our hands in regular progressive groups, with a view to derive from it numerically the only things which it is really important to know, viz., the *most probable value*, the *probable error* of a single determination, and the *weight* of the result as compared with that similarly derived from a different and independent series. But when the data are otherwise grouped, which is a case by no means of unfrequent occurrence, or when a portion only is regularly arranged in groups, and all above or below certain limits massed together in the gross without regard to grouping, much delicacy subsists in deciding, according to just principles, on the exact amount of all these elements ; and it would have added much to the practical utility and value of M. Quetelet's work had he given some examples of this nature, with plain and brief rules or formulæ for their working. This is the more to be regretted, because we are actually left at a loss to decide by what numerical process his mean results, where stated, have been arrived at in some of the examples set down. For instance, in that of the Scotch soldiers, where all the groups are regular and all stated, we find it merely mentioned incidentally that the mean is "a little more than 40 inches, whereas the really most probable mean is 39·830, while that which the course of the figures in the tabulated working of the example would appear to indicate as resulting from an equipartition of the numbers of cases in excess and defect is 39·525. Again, in the example of the conscripts, where the extreme groups are massed undistinguishably, the rule of equipartition, according to its simplest and most obvious application to the tabulated figures, would place the mean at 63·939 inches, whereas we find it indicated rather than stated, as follows : " *If it be observed* that the mean height is about 63·947 inches." The difference, it is true, is trifling in itself, but becomes of consequence when the object is from the figures set down to discover by what process they have been obtained.

We come now, however, to that highly interesting part of the work before us which treats of the study of causes, in general ; and in the peculiarly complex form it assumes, in those moral and social inquiries, the data for which are gathered by statistical enumeration. A few remarks on the part which the theory of

probabilities plays in these inquiries will not be out of place here.

This theory is connected with the general philosophy of causation and with inductive inquiry in two distinct ways—the one theoretical, and the other practical. When we see an event happen several times in succession in some particular manner, there arises, in the first place, a *primâ facie* probability that it will happen once more in that manner; which, if the number of repetitions be large, forms of itself a very cogent ground of expectation. But the probability that such repetition has not been merely fortuitous, but has resulted from a determining, or at least a biasing cause, increases with each repetition in a far higher ratio, than the simple probability of the once more happening of the event itself. The distinction is that between a geometrical and an arithmetical progression. Thus, for example, the expectation that the sun will rise to-morrow, grounded on the sole observation of the fact of its having risen a million times in unbroken succession, has a million to one in its favour. But to estimate the probability, drawn from that observation, of the existence of an influential cause for the phenomenon of a daily sunrise, we have to raise the number 2 to the millionth power—thus producing a number inexpressible in words and inconceivable in thought, and the ratio of this enormous number to unity, is that of the probability of the phenomenon having happened *by cause*, to that of its having happened *by chance*. The theorem on which depends this curious application of the doctrine of *probabilities* to the expulsion from philosophy of the idea of *chance*, is known to geometers by the name of its first promulgator, Bayes. It must be observed, that as to the nature of the cause thus insisted on, the calculus says nothing. There may be opposing causes, and a daily struggle between them for the mastery. In this case we are simply forced to admit that the arrangements of Nature are highly favourable to the successful exertion of the one, and highly unfavourable to the other.

It is however as a practical auxiliary of the inductive philosophy that we have chiefly to contemplate this theory. Its use as such depends on that mutual destruction of accidental deviations from the regular results of permanent causes which always takes place when very numerous instances are brought into comparison. Examples of this sort have been already adduced, and might be multiplied indefinitely in every department of practical inquiry. Indeed, every phenomenon which Nature offers on the great scale

may be regarded as such. Nothing can be more irregular and uncertain than the action of the wind on the waters,—yet, in the most violent storms, the *general* surface of the ocean preserves its level. What more fortuitous than the fall of a drop of rain in a shower, or the growth of a blade of grass? Yet the soil is uniformly irrigated, and the unbroken sheet of verdure testifies to the resultant equilibrium of that and a thousand other causes of inequality. These things, it will perhaps be said, are the results of Providential arrangement. No doubt they are so; but it is an arrangement working through a complication of secondary causes and contingencies,—on which man, if he will philosophize at all, is obliged to do it by reference to the laws of probability. Still there is no one who is not astonished, in cases where what we are obliged to call contingency enters largely, to find not only that the mean results of several series of trials agree in a wonderfully exact manner with each other, but that the very errors of individual trials—precisely those portions of the special results which are purely attributable to that which is contingent in the process—group themselves around the mean with a regularity which would appear to be the effect of deliberate intention.

“This singular result” (says M. Quetelet) “always astonishes persons unfamiliar with this kind of research. How, in fact, can it be believed that errors and inaccuracies are committed with the same regularity as a series of events whose order is calculated in advance? There is something mysterious, which however ceases to surprise when we examine things more closely.”

The rationale of this mystery is this. Where the number of accidental causes of deviation is great, and the maximum effect of each separately minute in comparison of the result we seek to determine,—great total deviations can only arise from the conspiring of many of these small causes in one direction,—the more that so conspire the greater the deviation. Now all combinations being equally possible *individually*, and those combinations which can alone give rise to the extremes of error being necessarily very much fewer in number than those which result in moderate amounts of deviation, we easily perceive that the opportunities for the occurrence of great errors are much rarer than for small ones. And this is in fact the reasoning, which, carried out by exact analysis (assimilating the causes of *plus* and *minus* error to black and white balls in an urn), takes the form of that demonstration of the law of probability, which we have above spoken of as devised by Laplace and simplified to the utmost by M. Quetelet.

There still remains behind, however, this inquiry,—which we have known to occur as a difficulty to intellects of the first order,—*Why* do events, on the long run, conform to the laws of probability? What is the *cause* of this phenomenon as a matter of fact? We reply (and the reply is no mere verbal subtlety), that events do not so conform themselves,—the fact to the imagination,—the real to the ideal,—but that the laws of probability, as acknowledged by us, are framed in hypothetical accordance with events. To take the simplest case, that of a single contingency,—the drawing of one of two balls, a black and a white. We suppose the chances equal, in theory; but, in practice, what is to assure us that they are so? The perfect similarity of the balls? But they need not be similar in any one quality but such as may influence their coming to hand. And, on the other hand, the most perfect similarity in all visible, tangible, or other physical qualities cognizable to our tests is not such a similarity as we contemplate in theory, if there remain inherent in them, but undiscernible by us, any such difference as shall tend to bring one more readily to hand than the other. The ultimate test then, of their similarity in that sense is not their general resemblance, but their verification of the rule of coming equally often to hand in an immense number of trials: and the observed fact, that events *do* happen according to their calculated chances, only shows that *apparent* similarities are very often *real* ones.

The application of this calculus to the detection of causes turns essentially upon this view of the conformity in question, and of the nature and delicacy of this *test by indefinite multiplication of trials* which we are enabled, in many cases, to apply to mixed phenomena. All experience tells us, that where efficient *causes* are known, but from the complication of circumstances cannot be followed out into their specific results, we may yet often discern plainly enough their *tendencies*, and that these tendencies *do* result, in the long run, in producing a preponderance of events in their favour. Were it asked, Why do the strong men, in a general scramble, carry off the spoil, and the weak get nothing? the reply would be, that such is not the fact in every instance; that, although we cannot go fully into the dynamics of the matter, we can clearly see the mode of action in some individual struggles, and that in the whole affair there is a visible enough *tendency* to the defeat of the weaker party. Again, when we reverse this process of reasoning, and declare our conviction that success in the long run is a proof of ability, we give this name to some personal

quality or assemblage of qualities which, acting as an efficient cause through a complication of events we do not pretend to penetrate, has a tendency in that direction which issues in success. Here the tendency becomes known by observation, and the nature of the cause is concluded from the nature of the tendency, by appeal to experience, which, in some instances, has shown us the cause in action, and informed us of its direct effect. But it may happen that observation may plainly enough indicate the direction of a tendency which yet experience has not enabled us to connect with any known cause. And it may further happen that this tendency, which we are driven to substitute in our language for its efficient cause, may be so feeble—whether owing to the feebleness of the unknown cause, its counteraction by others, or the few and disadvantageous opportunities afforded for its efficacious action (general words, framed to convey the indistinctness of our view of the matter)—as not to become known to us but by long and careful observation, and by noting a preponderance of results in one direction rather than another.

And thus we are led to perceive the true, and, we may add, the only office of this theory in the research of causes. Properly speaking, it discloses, not causes, but tendencies, working through opportunities,—which it is the business of an ulterior philosophy to connect with efficient or formal causes; and having disclosed them, it enables us to pronounce with decision, on the evidence of the numbers adduced, respecting the reliance to be placed on such indications,—the degree of assurance they afford us that we have come upon the traces of some deeply-seated cause,—and the precision with which the intensity of the tendency itself may be appretiated.

Such tendencies are often apparent enough, without any refined considerations, or reference to any calculus. Thus, on the consideration of thirteen instances of coincidence between the direction of circular polarization in rock crystal, with that of certain oblique faces in its crystalline form,—it was asserted that the phenomena were connected in that invariable manner which is one of the characters of efficient causation. The chances against such a coincidence happening thirteen times in succession by mere accident are more than 8000 to 1; and this, therefore, was the probability that some law of nature, some cause, was concerned. Subsequent observation has brought forward no exception; but, on the contrary, other cases of a similar character have arisen, which go to place the observed tendency in *uncounteracted* con-

nexion with the efficient cause—which, however, still remains concealed.\*

It is, however, the extreme delicacy of the test above spoken of—that property it possesses of bringing out into salience and placing in indisputable evidence, by sufficient multiplication of observations, any preponderance, however small, among the efficient causes in action—that it becomes applicable to those complicated cases in which we find it resorted to. As an instance of this nature, we shall take a phenomenon which has engaged the attention of all who have written on probabilities, from Laplace downwards; one which has been much insisted on by M. Quetelet, and on whose acknowledged obscurity his inquiries have at length thrown a ray of light; viz., the excess of the number of births of male over that of female infants. As a matter of observation, the phenomenon is indisputable; but it requires the assemblage of a great number of instances to bring it out into evidence. In individual experience, or in the birth registers of a parish or small town, the tendency to excess on the male side is quite overlaid and concealed by accidental irregularities. It is otherwise when those of great cities or whole nations are consulted. The irregularities then disappear by mutual destruction, and the result exhibits the tendency in question in its full prominence. If we extract from the population returns of England and Wales the total numbers of registered births in the seven years, from 1839 to 1845 inclusive, we find 1,863,892 males and 1,772,491 females, the excess being 91,401 on the male side, or 105·157 males to 100 females. Suppose it were urged that this may, after all, be a purely accidental excess. It might be said, not without apparent plausibility, that as it would be the height of improbability to expect in so vast a number an exact equality, so, on the other hand, an excess of 91,401, which, though a large number in itself, is yet but  $2\frac{1}{2}$  per cent. on the total number of cases, *does not seem so very improbable*. To this theory replies that, where such high numbers are concerned, it is so:—that the case assumed in the objection is identical with that of drawing 3,636,383 balls out of an urn containing black and white balls in equal proportion and infinite in number, and that the expectation

\* So again, an examination of the elements of all known cometary orbits has disclosed a tendency to direct or eastward motion, increasing in the degree of its prominence with the approach to coincidence of the orbit with the plane of the ecliptic,—and especially marked in the cases where calculation has assigned elliptic elements to the orbit. Here we have a tendency pointing to a cause, still unknown, but with whose effects we are so far familiar that we can trace its action throughout the planetary system, with only two known exceptions among its most remote and insignificant constituents, and these of a very undecided character.

of drawing such an excess of one colour in such a number, so far from a mere moderate unlikelihood, is, in fact, equivalent, supposing the chances equal, to the expectation of throwing an ace 643 times successively, with a single fair die.\* Even on a total of 20,000 births we might bet many thousand millions to one that the same relative preponderance would not be found, were the chances even.

It is abundantly evident, therefore, that we have here arrived at a proof of a tendency which must be taken as a law of human nature under the circumstances in which it exists, at least in this country; and the constancy with which the proportion is maintained in successive years, and even in different nations, is not less striking than the fact itself, and shows it to be a result of deep-seated causes, acting with almost absolute uniformity on great masses of mankind. Thus in the seven years from which the above ratio has been concluded, taking them *seriatim*, we find 104·8, 104·7, 105·3, 105·2, 105·4, 105·4, 105·2, on totals averaging about half a million each; while in France a similar comparison gives 105·9, 105·7, 106·1, 106·2, 105·8, 105·9, 105·9, on nearly double the total numbers. As to the causes of this most striking phænomenon, much speculation has, of course, prevailed; but the inquiries of M. Quetelet into the statistics of marriage have rendered it extremely probable† that the relative ages of the parents very materially influence the sex of the offspring, and that the effect is therefore a resultant one, due to this physiological cause, acting through the medium of all those prudential and moral considerations which in civilized states determine the relative ages of parties contracting marriage. This view of the subject is strongly corroborated by a separate examination of the registers of illegitimate birth, which indicate an excess of only 3 instead of 5 per cent.

The causes, or tendencies indicative of causes, which may be disclosed by the assemblage and comparison of numerous recorded instances, are classed by M. Quetelet under three heads: constant, variable, and accidental. The latter class may be considered as entirely eliminated by their mutual destruction when vast numbers are concerned, and the whole series of collected cases is so treated as to afford a single result. The same process also will in great measure destroy the effect of variable causes, if their variation be periodical in its law, and the observations be made indifferently in all the phases of their period. It is the peculiar property, however, of causes of this latter description, through whatever

\* The chances against throwing an ace only nine times in succession, are ten millions to one.

† *Essai de Phys. Sociale*, i. 57. Citing Hofacker and Sadler in corroboration.

train of circumstances their action is propagated, ultimately to emerge to view in manifestations equally periodical with the causes themselves. In cases of dynamical action this peculiarity is susceptible of demonstration, and has been so demonstrated under the name of the "principle of forced vibrations:"\* and experience abundantly proves its general applicability to every case of indirect action, whether physical or moral. To those, therefore, who assiduously watch the development of phenomena, and register effects as they arise with sufficient exactness, such causes will be detected, and their periods at the same time disclosed by the periodical fluctuations they occasion; or they may be searched for, if suspected to exist overlaid by accidental errors, by dividing the series of observed results into groups, differing in *phase* (i. e., dividing the extent of the period suspected into several equal portions, and grouping the results observed in each together). The influence of the periodical cause suspected will then become apparent in the form of differences in the mean results of the several groups. Of this process every part of science teems with examples. In astronomy we owe to it the grand discoveries of the aberration of light, the nutation of the earth's axis, the separation of the effects of the sun and moon on the tides, and an infinity of others; in meteorology, that of the diurnal and annual fluctuations of the barometer; in magnetism, the daily and annual changes in the direction and intensity of the magnetic forces; and in statistics, the annual oscillations observable in all the great elements of population, which the researches of M. Quetelet have placed in a distinct light.

But among accumulated masses of results, without any attempt at subdivision into *periodic* groups, the influence of periodical causes may start into evidence on a general inspection of the differences from a mean result, after a totally different manner. We have seen that these differences present *inter se* a definite and perfectly cognizable law of arrangement, so long as their causes are purely casual. Any deviation *from this law* among the differences of the observed values from the mean, then, becomes at once an indication of a determining tendency, and will very often, by the character of the deviation, lead to a well-grounded surmise of the nature of its cause. For instance, if a sudden falling off in the number of observed differences, beyond certain limits either way from the mean, *accompanied with some degree of improbable accu-*

\* Encyclop. Metropol. Article Sound, § 323, *et seq.*

*mulation at or about those limits*, should be noticed, it may be taken as a certain indication of a periodical disturbing influence, having those limits for the maximum and minimum of its effect.

Again, if at any particular point in the scale of results arranged in order of magnitude we should notice a sudden and marked irregularity confined to a small extent, we may be sure that it arises from the action of some single, powerful, and exceptional influence. Thus, from the undue accumulation of conscript measurements below the standard height of 5 feet 2 inches, accompanied with a deficiency to the extent of 2275 cases in the two inches just above that standard, M. Quetelet is led to conclude that an influence foreign to the subject—in fact, a fraudulent practice, favouring the escape of the shorter men, has prevailed to that extent in the formation of the official returns he has employed as the basis of his calculations. (*Transl.* p. 98.)

Astronomy affords us a very remarkable example of this nature, which we adduce, by reason of a singular misconception of the true incidence of the argument from probability which has prevailed in a quarter where we should least have expected to meet it. The scattering of the stars over the heavens, does it offer any indication of law? In particular, in the apparent proximity of the stars called “double,” do we recognize the influence of any *tendency to proximity*, pointing to a cause exceptional to the abstract law of probability resulting from equality of chances *as respects the area occupied by each star*? To place this question in a clear light, let us suppose that, neglecting stars below the seventh magnitude, we have measured the distance of each from its nearest neighbour, and calculated the squares of the sines of half these distances, which therefore stand to each other in the relative proportion of the areas occupied exclusively by each star. Suppose we fix upon a circular space of 4" in radius as the unit of superficial area, and that we arrange all the results so obtained in groups, progressively increasing from 0 by the constant difference of one such unit. Now the fact, to which M. Struve originally called attention\*, and on which we believe all astronomers are agreed, is, that the first of these groups is *out of all proportion richer than any of the others*; and that the numbers degrade in the groups adjacent with excessive rapidity; so that, for example, calculating on the numbers given by Struve†, we find the first group to contain 182 cases; the next three 68, or on an average 22 each; the next twelve 70, or 6

\* *Catalogus Novus Stellarum duplicium*, &c. Dorpati, 1827.

† *Ibid.*, p. xxxii., Introduction. Each of M. Struve's classes is doubled, since each constituent of a double star counts as a separate case.

each on an average; and the next forty-eight only 94 in all, averaging 2 to each; while a general average\* would assign only one star to 540,000 such units of area. The case, then, is parallel to that of a target of vast size, marked out into 6700 millions of equi-distant rings, riddled with shot marks in the bull's eye, and with a tolerable sprinkling in the first fifty or sixty rings, beyond which the whole area offers nothing for remark indicative of any particular local tendency, though *dotted all over with marks*, in the sparing manner above described. Any one who should view such a target, bearing in mind what is said above, must feel convinced that a totally different system of aiming had been followed in planting the interior and exterior balls.

Such we conceive to be the nature of the argument for a physical connexion between the individuals of a double star prior to the direct observation of their orbital motion round each other. To us it appears conclusive; and if objected to on the ground that every attempt to assign a numerical value to the antecedent probability of any given arrangement or grouping of fortuitously scattered bodies must be doubtful,† we reply, that if this be admitted as argument, there remains no possibility of applying the theory of probabilities to any registered facts whatever. We set out with a certain hypothesis as to the chances: granting which, we calculate the probability, not of one certain definite arrangement, which is of no importance whatever, but of certain *ratios* being found to subsist between the cases in certain predicaments, on an average of great numbers. Interrogating nature, we find these ratios contradicted by appeal to her facts; and we pronounce accordingly on the hypothesis. It may, perhaps, be urged that the scattering of the stars is *un fait accompli*, and that their actual distribution being just as possible as any other, can have no *à priori* improbability. In reply to this, we point to our target, and ask whether the same reasoning do not apply equally to that case? When we reason on the result of a trial which, in the nature of things, cannot be repeated, we must agree to place ourselves, in idea, at an epoch antecedent to it. On the inspection of a given state of numbers, we are called on to hold up our hands on the affirmative or negative side of the question, Bias or No bias? In this case who can hesitate?

\* Taking 12,400 as the number of stars of the magnitudes and within the region of the heavens contemplated, viz., from the North Pole to 15° south declination, which number, for the reason in the foregoing note, has to be doubled.

† London, Ed. and Dub. Philosoph. Magazine, &c. Aug., 1849.

Accidentally variable causes overlay altogether the evidence of regular action, so that the elimination of their influence is in all cases synonymous with the extension of knowledge. It is not, however, to this or to any other calculus that we can look for special rules of conduct in this part of inductive inquiry beyond the simple precept of collecting facts in great numbers, and employing mean results in lieu and to the exclusion of single observations wherever numerical magnitude is concerned. This precept is, however, of infinite use in all cases where we test the efficacy of a presumed cause by the numerical correspondence between its known energy and the amount of the observed effect.

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Among those branches of knowledge which are most effectually advanced by the consideration of mean or average results concluded from great masses of registered facts, to the exclusion of individual instances, statistics hold beyond all question the most important rank as regards the social well-being of man. To this subject M. Quetelet devotes the fourth and last division of his work; not, indeed, to the delivery of statistical tables or results, nor to the actual discussion of any particular class of documents, but to the points which it so much imports to have generally well understood of the methods and principles which ought to prevail in the collection and subsequent employment of such documents.

Whether statistics be an art or a science (a question to which he devotes a preliminary letter) or a scientific art, we concern ourselves little. Define it as we may, it is the basis of social and political dynamics, and affords the only secure ground on which the truth or falsehood of the theories and hypotheses of that complicated science can be brought to the test. It is not unadvisedly that we use the term Dynamics as applied to the mechanism and movements of the social body; nor is it by any loose metaphor or strained analogy that much of the language of mechanical philosophy finds a parallel meaning in the discussion of such subjects. Both involve the consideration of momentary changes proportional to acting powers,—of corresponding momentary displacements of the incidence of power,—of impulse given and propagated onward,—of resistance overcome,—and of mutual reaction. Both involve the consideration of time as an essential element or independent variable; not simply delaying the final attainment of a state of equilibrium and repose,—the final adjustment of interests and relations,—but, from instant to instant, pending the process of mutual accommodation, altering those relations, and, in effect,

rendering any such final state unattainable. One great source of error and mistake in political economy consists in persisting to regard its problems as statical rather than dynamical in their character; confounding the propagation of impulse with a step towards equilibrium,—a state unattainable where the interests of masses of mankind are concerned. So long, indeed, as society is little developed, its movements fettered, its commercial activity sluggish, and all things go on leisurely, the distinction is one of small importance; a state of *acquiescence*, nearly approaching to that of equilibrium and final adjustment, being taken up from instant to instant, and following at a little distance, yet *pari passu*, the slow changes of the acting causes. It is otherwise under the increased facilities, excessive mobility, and excited energy which prevail under the high temperature and pressure of modern civilization. Friction (which has an equally real existence in both mechanisms) is diminished, the intensity of the active powers increased, the scale on which movements are carried on enlarged,—a state of things which finds its expression in the “over-speculation,” “gluts,” “panics,” “reactions,” *et hoc genus omne* of modern commerce and social change. The same must be the case whenever efficient causes, of whatever nature, act through a train of varying circumstances, and result in effects of which it can only be securely asserted that their momentary and infinitesimal changes stand under given circumstances in given relations. It may be true, for example, that capital tends to a common level of profit in the choice among its possible employments; but endless fallacies would be involved in any reasoning which should proceed on the assumption that it finds that level. Demand may tend to increase supply by stimulating exertion, but a supply proportionate to the demand, and steadily following its variations, is what no sound political economist will ever expect to see. The Rule of Three has ceased to be the sheet anchor of the political arithmetician, nor is a problem resolved by making arbitrary and purely gratuitous assumptions to facilitate its reduction under the domain of that time-honoured canon.

Number, weight, and measure are the foundations of all exact science; neither can any branch of human knowledge be held advanced beyond its infancy which does not, in some way or other, frame its theories or correct its practice by reference to these elements. What astronomical records or meteorological registers are to a rational explanation of the movements of the planets or of the atmosphere, statistical returns are to social and political

philosophy. They assign, at determinate intervals, the numerical values of the variables which form the subject-matter of its reasonings, or at least of such "functions" of them as are accessible to direct observation; which it is the business of sound theory so to analyse or to combine as to educe from them those deeper-seated elements which enter into the expression of general laws. We are far enough at present from the actual attainment of any such knowledge, but there are several encouraging circumstances which forbid us to despair of attaining it.

The first of these is the exceeding regularity which is found to prevail in the annual march of statistical returns and the constancy of the ratios they indicate where great masses of population are concerned, where leading features of human nature are the obviously influential elements on which the observed results depend, and where temporary or periodical causes of disturbance (evidently such) do not visibly interfere. As instances might be cited the relative proportion in the births of the sexes already spoken of; the ratio of illegitimate to legitimate births in the same country and the same section of the population; nay, even the number of the still-born (with a distinct percentage for town and country), which M. Quetelet has ascertained to be so uniform in Belgium that, on a total number of nearly 6000 annual cases, the yearly deviation from the mean falls short of 140; the ratio of marriages to the whole population, of second marriages to the whole number of annual marriages, and, still more minutely, of widowers with widows, widows with bachelors, and widowers with spinsters; the relative ages of parties intermarrying; and innumerable other particulars; all which, free as air in individual cases, seem to be regulated with a precision, where masses are concerned, clearly proving the existence of relations among the acting causes so determinate, that there is evidently nothing but the intricacy of their mode of action to prevent their being subjected to exact calculation, and tested by appeal to fact. *Taken in the mass*, and in reference both to the physical and moral laws of his existence, the boasted freedom of man seems to disappear; and hardly an action of his life can be named which usages, conventions, and the stern necessities of his being, do not appear to enjoin on him as inevitable, rather than to leave to the free determination of his choice: while yet, throughout, he feels himself to be a free agent.

Another encouraging feature in the aspect of statistical docu-

ments, which shows them, when properly collected, to be trustworthy for the purposes to which we desire to apply them, and holds out a rational hope of their available application,—is their evident *sensitiveness* to the influence of real and unmistakeable causes, which we are sure, *à priori*, ought to influence them. Thus we see the uniform march in the number of annual marriages, corresponding to an increasing population, visibly accelerated in years of prosperity and abundance, and visibly retarded in those of scarcity and public distress. Thus, too, we see in Bavaria laws restraining marriage result in an increased number of illegitimate births.\* Wherever monthly returns, of whatever kind, are compared, the influence of season is marked by a more or less conspicuous annual maximum and minimum. Instances of this, of the most striking character, are adduced by our author in his “*Essai de Physique Sociale*.” In these and similar cases, where we clearly perceive the existence of definite tendencies, or of a generally modifying cause pervading the whole field of their action, it is satisfactory and reassuring to find the result in correspondence with our views. For it must never be forgotten that tendencies only, not causes, emerge as the first product of statistical inquiry,—and this consideration, moreover, ought to make us extremely reserved in applying to any of the crude results of such inquiries the axioms or the language of direct unimpeded causation. The proportionality of cause to effect, for instance, is a principle rather emphatically repudiated in the history of the correspondence of increase of imposts with increase of revenue, and of profits as compared with prices.

“Population,” says M. Quetelet, “is the statistical element, *par excellence*: it necessarily rules all others, since it relates, above all, to the people and the appretiation of their welfare and their wants. It would be vain to attempt to form statistics of value without taking as a basis the results of a census executed with all the care and precision which so delicate an operation requires. The other data have no real value, except in so far as they relate to the number of the population. A census carefully made sums, in a measure, the most important problems which can be proposed to a statist. The classification according to age allows of the establishment of tables of population, of forming correct ideas on mortality, on the forces at the disposal of the state in case of necessity, and of fixing the ratio between the useful fraction which contributes to the general well-being, and the fraction which yet requires assistance and support to become in its turn useful. The classification by professions, indicates the means by which the population provides for its subsistence and tends to augment

\* The vast multitude of illegitimate births in France would seem to be traceable in great measure to the difficulties thrown in the way of marriage by requiring the expressed consent of a great number of relatives of both parties to its celebration.

its prosperity. . . . Those by civil condition, by origin, by education, furnish the administration with no less precious information to assure internal good order, and to facilitate the execution of the laws."—(*Transl.* p. 183.)

A well-organized system of civil registration ("*état civil*," ) is therefore one of the first wants of an enlightened people. No man in such a people is above or beneath the obligation of authenticating his existence, his claims on the protection of his country, and his fulfilment of the duties of a citizen,—or of contributing his individual quota of information, in what personally concerns himself or his family, in reply to any system of queries which the Government in its wisdom may see fit to institute respecting them. Such information may be regarded as a poll-tax, which, in this form, a Government is fairly entitled to make, and which indeed is at once the justest and least onerous of taxes; or rather, it may be looked on as a mode of self-representation, by which each individual takes a part in directing the views of the legislature in objects of universal concern. Nothing, therefore, can be more unreasonable than to exclaim against it, or to endeavour to thwart the views of Government in establishing such a system,—nor anything more just than to guarantee its fidelity by penalties imposed on false returns or wilful omissions.

The analysis of the population returns of a great nation, or rather the drawing from that analysis, duly executed according to rational classifications, just and philosophical conclusions, is a task calling for the exercise of much acuteness and discrimination in appretiating the influence which the relative proportions between the classes, as to age, condition, calling, must necessarily have on national character and habits, and in weighing—with reference to future prospects—the probable influence on that character and those habits which is involved in even a very moderate observed change from time to time, in those proportions.

"The numerical tables of a population, when made with care and with all the development which science requires . . . form, in the annals of a people, the most eloquent page that a statesman can read, if he understand them well. In fact it only belongs to the practical observer completely to understand the language of figures, and not to go beyond what they can teach him. Censuses, well made, and which succeed one another on a uniform plan and at intervals sufficiently near, should present most precise notions of the physical and moral condition of a people,—of the degree of its power,—of its prosperity,—and of the tendencies which may compromise its future: they would teach much better than voluminous inquiries, which are often fettered by prejudices and private interests, what

we ought to think of the retrograde state or the immoderate development of certain branches of industry."

Among the first results of such an analysis, are those general ones which our Continental neighbours technically understand by the "movement" of the population—its increase, that is to say, by the excess of births over deaths and emigrations, and the internal change in the proportions of those living at different ages corresponding to changes, if any, in the law of mortality as indicated by the ages of death. On this point M. Quetelet, in an earlier part of this work, has the following pertinent remark:—

"The movement of a stationary population is often compared with that of a population increasing by an excess of births over deaths. However, this is a comparison of heterogeneous elements: all other things being equal, the latter population should have a greater mortality; for there are more children in it."

So far as this remark goes it is just, but it does not include the whole case, or exhibit fully the influence of the consideration in question. To judge of the extent of this influence it is only necessary to consider that, in a given population now existing, the individuals living at any assigned age are not the survivors of that age among a number equal to that born in the current year, but among a number born antecedently, when the population was less than at present, in a proportion easily calculated, the age being given, and the annual rate of increase known. Thus, supposing the population of a country to double in fifty years, a man fifty years old is the survivor of only half the number of cotemporary births, and of one hundred of only one-fourth those which would appear, on a comparison of the number actually born in a given year with those actually living at the age specified, in that year. Not only, therefore, are there more children in comparison with adults in an advancing population, but at the same time fewer old men. Now the ratios of the helpless, the active, and the meditative elements of a population to the entire mass and to each other,—of giddy youth and adult enterprise to mature experience, timid caution, and declining powers, must necessarily give rise to corresponding features of national character. A disproportion in this respect, influencing all the great lines of development of national activity and impressing the whole career of a people, cannot but make itself felt in every feature of their existence. It is only necessary to contrast the energy displayed by a nation whose population doubles in twenty-five years, as in the United States, with the sobriety of movement, not to say torpor, of another,

where, as in Holland, it is nearly stationary, to perceive the connexion in question to be that of effect with cause.

"An exposition of the political condition belongs essentially to the statistics of a country. We do not, however, know how to express it in figures. The same may be said of information relative to the moral and intellectual condition. The simple recital of what has passed in a locality at a particular time sometimes better teaches the moral condition of a people than all the numerical tables possible."

\*            \*            \*            \*            \*

The chief difficulty to be encountered in aiming at correct results in the collection of agricultural, industrial, and commercial statistics is, that it—

"Requires the intervention of persons who are almost always interested, or think they have an interest, in disguising the truth. When the government collects them, it is generally opposed by the manufacturer, who supposes it done with fiscal views. The desire to obtain freedom for his industry, and to obtain what are called protecting laws . . . almost always tends to exaggeration in one direction or another. Governments also publish documents on importations and exportations. These tables, which are useful to consult, nevertheless often contain very vague returns: they are generally confined either to the fixing of prices from faulty valuations or of quantities without considering either price or quality. In the official valuations, moreover, we only know a part of the truth: it is especially here that information not susceptible of reduction to numbers becomes necessary, in order to determine the probable quantity which escapes the legally stated values."

Owing to these causes of jealousy and partial presentation, many important statistical elements, relating to matters of pecuniary concern, can hardly be collected by official intervention. It is here that a Statistical Society may render most valuable service by setting on foot systematically, yet amicably and unobtrusively, local and private inquiries, with the guarantee of personal veracity for their answers, and the purely scientific and truth-loving spirit of such a body of enlightened inquirers for their fair presentment.

"The statistics of the moral and intellectual condition of a people," he goes on to observe, "present still greater difficulties; for the appretiation can only be founded on facts much more contestable than those given by industry and commerce. When we say that a province produces so many quarters of corn or so many gallons of oil, we know that the figures may be more or less in error; but we understand the nature of the unit. It is not the same when we say that a province produces annually so much crime. . . . Infinite precaution and sagacity are necessary to read with success the statistics of tribunals, for the documents they contain are very complex in their nature, and almost always incomplete." . . .

"What a mass of errors have we not accumulated in treating of pauperism! To probe this leprosy of society we have had recourse to lists

of the poor, and very often without inquiring if these lists were complete and comparable in different countries or even within the limits of the same country. Real poverty is nearly always very different from the poverty officially returned. . . . *In Belgium a man will enter his name on the list of paupers to escape serving in the civic guard or to obtain other advantages, without receiving a farthing of public benevolence*" [! !].

With such difficulties in the way of exhibiting fairly, and interpreting truly, statistical facts, arises a necessity for laying down precautionary rules for the guidance of those to whom is confided the important task of their collection and registry—for checking their correctness when collected—and for their legitimate employment in aid of legislative or administrative purposes. On each of these heads M. Quetelet gives us a letter—short, indeed, and somewhat desultory; but abounding in useful and sensible remarks. Each of them would, in fact, require a treatise for its complete illustration.

A fool can ask questions, but only a wise man pertinent ones; and it often takes a wiser man to ask than to answer. After recommending to the statist a due and ample course of preparatory study of the subject in hand, our author goes on to observe, on the collection of statistical information:—

"The principal considerations which should guide an administration as to the questions to be asked are the following:—

"1. Only ask such information as is absolutely necessary, and as you are sure to obtain.

"2. Avoid demands which may excite distrust, and wound local interests or personal susceptibility as well as those whose utility will not be sufficiently felt.

"3. Be precise and clear, in order that the inquiries may be everywhere understood in the same manner, and that the answers may be comparable. Adopt for this, uniform schedules, which may be filled up uniformly.

"4. Collect the documents in such a way that verification may be possible.

"Simplicity and clearness of demand, together with uniformity in the forms to be filled up, are essential conditions to obtain comparable results. Without them, no statistics are possible. When the question relates to ages, professions, or diseases, it is of the greatest importance to employ classifications perfectly identical, in order that the general information may be compared even to the slightest detail. The most perfect unity should reign throughout the whole. It is to establish a unity like this that in certain states, such as Belgium and Piedmont, central commissions have been formed to collect and arrange the different elements which should be included in the national statistics. The necessity of such institutions is particularly shown when we see in very enlightened countries the principal departments sometimes publish very different numbers to express the same

things, or make classifications which render comparison impossible."—  
(*Transl.* pp. 196, 197.)

Not to secure facility for the verification of the documents we collect is to miss one of the principal aims of the science. Statistics are only of value according to their exactness, without which they can serve but to establish error. Every statistical document requires a twofold examination—a moral and a material one, the former being, in all cases, by far the most important, as it involves the inquiry into the influence under which it has been collected—a point on which the whole colouring of the document essentially depends:—

"During the war of independence, the United States carefully misrepresented the true number of their population: they exaggerated considerably the numbers of inhabitants in maritime cities, in order to put the enemy on the wrong scent. Assuredly no good estimate of the American population could be founded on the documents of this period."—  
(*Transl.* p. 202.)

Every statistical document ought to carry on the face of it, the exceptions, exemptions, and limitations, under which its entries are made. In respect of the use which may be made of it, negligence in this respect may amount in effect, if not in culpability, to a falsification.

"Thus, by means of official numbers, M. Sarauw pretended to prove that in the island of St. Croix, in the Danish Antilles, the mortality of the black slaves was less than that of white men even in Europe; and this assertion might appear so much the more imposing, as M. Sarauw resided in the island in question."

This result (which was arrived at in good faith) rested solely on the omission of negro children, dying before attaining their first year, from the register of births, such children being exempt from poll-tax, and therefore their omission being deemed of no importance.

The material examination of statistical documents rests chiefly on the internal evidence they may offer of self-consistency. It is singularly aided by diagrams. A simple line, properly laid down from a consecutive series of numbers, by what is called graphical projection, enables us to apprehend at a glance the continuity and regular progression of their succession; and, what is of still more importance, to apprehend correspondences between two series so projected, which often afford immediate conviction of a relation between them, such as the most subtle mind would find it difficult to perceive without such aid. They give to the study of pheno-

mena the same advantage which algebra has introduced into calculation—they generalize and allow of abstraction; and they enable us at once to detect and often to rectify errors which, if undetected, would affect mean results, and throw everything into confusion. We are glad to find M. Quetelet strong in his advocacy of this mode of dealing with a series of observations which the generality of French *savans* affect, very unwisely, to despise as inconsistent with their notions of mathematical rigour.

There is nothing more indicative of a man's fitness or unfitness for the duties of a legislator and a statesman than his manner of dealing with statistical documents. When appealed to, as they too commonly are, for the purpose of establishing extreme positions, or of lending support to party views, or to particular interests, we are continually reminded of the doctrine of one long accustomed to listen to such arguments. "Nothing can be more fallacious than theories—except facts!" Those who use them in this manner will be found invariably to sin against truth and common sense in one or other of the following ways, viz.:—

- "1. By 'having preconceived ideas of the final result.'
- "2. By 'neglecting the numbers which contradict the result they wish to obtain.'
- "3. By 'incompletely enumerating causes, and only attributing to one cause what belongs to a concourse of many.'
- "4. By 'comparing elements which are not comparable.'"

To which we may add a 5th, the most common of all and the most inexcusable, viz.: singling out the extreme partial results which tell on the side to be defended, and ignoring all the rest.

With such eclecticism we may find in statistics the means of defending almost every position. In politics, especially, they

"Become a formidable arsenal, from which the belligerent parties may alike take their arms. . . . Some figures, thrown with assurance into an argument, have sometimes served as a rampart against the most solid reasoning; but when closely examined, their weakness and nullity have been discovered. Those who allow themselves to be frightened by such phantoms, instead of looking to themselves, prefer rather to accuse the science than to confess their blind credulity, or their inability to combat the pernicious arms opposed to them.

"We see persons profoundly convinced of a truth, seek to establish it directly by the authority of figures, and give, as they think, a mathematical demonstration. However, by means of the statistical documents which they unskilfully employ, they most frequently produce an opposite effect to that which they desired. Thus we cannot reasonably doubt that enlightenment contributes to man's happiness, by illuminating his intellect and fortifying his morals. In the attempt to demonstrate this what has been done? It has been thought necessary to establish that the number of

crimes is inversely as the number of children sent to school—as if the number of crimes, even were it known, had as its only cause the greater or less development of the intellect; and as if the development of intellect were measured by the number of children sent to school. What has been the result of this? It has been found, after well examining statistical documents, that the number of crimes is more generally in a *direct* proportion to the number of children sent to school, than in the *inverse* proportion. The conclusion is exactly the opposite of what was at first desired—a new error, which some have, with the same levity admitted.”—(*Transl.* p. 214.)

The necessary incompleteness of all statistical documents is sometimes urged as a general argument against trusting implicitly to conclusions drawn from them. The argument is valid, in so far as we have reason to believe that the unenumerated cases differ systematically, *i.e.*, in some essential point of classification; from the enumerated; so as to render the proportions in which the several classes are represented in the returns different from what they would be were the enumeration complete. But granting their incompleteness—and granting even that the incompleteness is such as to affect injuriously the proportionate numbers in classified results—this does not preclude the drawing of many sound and valuable conclusions from such documents, if only we are assured that in comparing similar ones for several successive years, or under circumstances otherwise different, the same causes of incompleteness prevailed and continued to affect the several classes in an invariable ratio.

This position M. Quetelet illustrates by a reference to the Criminal Statistics of Belgium.—Prior to 1830 the official returns gave only the number of crimes *known* and *prosecuted*, but for the seven years from 1833 to 1839 they included also the number of crimes known, but which were not prosecuted because the authors were unknown. Now it was found that this latter number proceeded from year to year with even more regularity than that of crimes prosecuted. No doubt, therefore, the number of crimes altogether unknown to justice, could it have been made a matter of registry, would have presented a similar constancy. Of known crimes against person, two thirds were regularly prosecuted, and one third escaped, the authors being undiscovered. In the case of crimes against property the proportions were reversed, and were nearly those of one fourth and three fourths; the graver crimes being those most sure of detection. On the whole it would appear from these records that out of 1154 crimes annually known to justice in Belgium, only 416, or little more than one third, formed

subjects of prosecution. Assuming, then, that the number of unknown crimes is equal to that of known (this would hardly be admissible for crimes against person), the amount of prosecuted crimes in Belgium would not exceed one sixth of those actually committed.

"I am absolutely ignorant and shall never know whether the crimes on which the tribunals have to pass judgment form the sixth or seventh or any other part you will of the total number of crimes. What is important for me to know is that this ratio does not vary from year to year. On this hypothesis I can judge *relatively* whether one year has produced more or less crimes than another."

Admitting that this ratio remains invariable from year to year, and that justice pursues criminals with the same activity, two countries or two provinces of the same country might be compared in respect of morality. But as the latter condition almost certainly does not hold good under different administrations, it becomes impossible, from the official returns of prosecutions, fairly to institute such a comparison between nations. Even should the same legislation, the same repression, and the same activity to bring criminals to justice, subsist, if the result be made to depend on a comparison of the number of *condemnations*, instead of those of *prosecutions*, a difference in the mode of trial would alone suffice to destroy the comparability of the cases.

"We know, in fact, that the establishment of the jury in Belgium has doubled the number of acquittals."—(*Transl. p. 227.*)

The letters on the use of statistics to the administration and on the ulterior prospects of this branch of science, though they can hardly be said to contain anything very new or striking, yet come opportunely at a period like the present, when vast changes, both legislative and economical, are in progress, and when opportunities are lapsing of seizing *in transitu* results which will one day be most valuable for future comparison. Steam, railroads, and free-trade principles are making such inroads into all that used to be considered fixed or slowly alterable, that it will be of the utmost interest to have secured points of departure in the new career which opens on society.

"Statists should be eager to register, from this time forward, all the facts which may assist in the study of this vast transformation in the social body, which is in process of accomplishment.

"A government in modifying its laws, especially its financial laws, should collect with care documents necessary to prove, at a future stage, whether the results obtained have answered their expectation. *Laws are*

*made and repealed with such precipitation that it is most frequently impossible to study their influence."*

These words deserve to be written in letters of gold. They point to an evil whose tendency is to degrade social policy from the list of sciences of observation and experiment to the rank of an empirical art. *Avant nous le Chaos ! Après nous le Déluge !* should be the motto of that statecraft which, under a momentary sense of pressure from those whom even the uneasiness of change makes restless and impatient, urges on the social movement faster than a sound philosophy can count the revolutions of its mechanism or register the work accomplished ; or of that which, by the simultaneous alteration of every condition, makes the separate estimation of any single effect hopelessly impracticable.

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## HOME AND FOREIGN INTELLIGENCE.

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### WESTMINSTER AND GENERAL LIFE ASSURANCE ASSOCIATION.

*Established 1836.*

#### EXTRACT FROM THE REPORT OF THE DIRECTORS.

THE Directors have much pleasure in meeting the Members on this the Sixth Quinquennial General Meeting of the Association, as it is their agreeable duty to report that the Accounts show the result of the business for the last five years to have been the most favourable which has occurred for any similar period since the establishment of the Association.

The Association during that period has progressed steadily, though not with any marked rapidity. The number of New Policies issued since 1862 having been 900, assuring £363,947., the New Premiums on which have been £12,306. 10s. 10d.

The total Income of the Association has increased from £34,769., in 1861, to £42,830., in 1866; £68,794. has been added to the assets, the funds of the Association having increased from £188,786. to £257,580., as shown by the accounts appended.

143 Policies have become Claims, assuring £73,263. 18s. 4d. on 120 lives. On these Policies there has been a further sum of £3722. 2s. 2d. paid for Bonus additions, making the total amount paid £76,986. 0s. 6d. On this occasion, at the last periodical division, the Directors have to observe that the average sum assured by the Policies which have become Claims exceeds the average sum assured by the Policies of the Association generally.

£13,097. 8s. 6d. has been received for the purchase of 33 Immediate Annuities, amounting to £1241. 0s. 2d.; on this branch of business, the Directors have to report, that hitherto the result has been very favourable to the Association.

The careful Investment of the Funds of the Association has constantly occupied the attention of the Directors, and they believe the Securities selected are in all respects sound, and are such as to combine perfect safety with a fair and remunerative rate of Interest. The average rate of Interest on the Productive Funds of the Association is £4. 5s. 11d. per Cent.

The valuation of the assets and liabilities is, on this occasion, made on the same principles as on all previous occasions, viz.,—the valuations of the Policies are based on the Carlisle Table of Mortality, assuming the rate of Interest at which the funds of the Association will accumulate to be 3 per Cent.; none of the future profits are anticipated, and ample reserves are made for all future management expenses. It will be seen, by reference to the accounts, that the amount reserved, in addition to the premiums hereafter to be received, as the fund to provide for the payment of the existing Policies, as they may become Claims, is £211,786.

The amount of profit now divisible, after making all the reserves above referred to, is £27,348. 5s. 10d. The Deed of the Association provides that of this amount £2734 16s. 7d. be carried to the Guarantee Fund, and that a similar sum be divided as a Bonus among the Shareholders. After making these provisions, the balance, £21,878. 12s. 8d., will allow of an average reversionary addition being made to all Policies entitled to participate in profits of about 40 per Cent. on the premiums paid since the last division.

*State of Accounts on the 31st day of December, 1866.*

Dr.	£	s.	d.
* Shareholders Paid-up Capital, with Additions thereto, to be ultimately paid to them according to the provisions of the Deed of Settlement...	10000	0	0
Value of the outstanding and existing Policies, Assuring £815644., entitled to participate in profits, including the Sum of £71682. 12s. 4d. reserved for Expenses and future Profits .....	157955	9	6
Value of £21184. 3s. 4d. the Additions made to the Sums Assured by the above Policies .....	13872	15	0
Value of the outstanding and existing Policies Assuring £173018. not entitled to participate in Profits, including Annuities and the Sum of £9761. 0s. 11d. reserved for Expenses and future Profits .....	39957	18	2
Interest to Shareholders on the Guarantee Fund for the year 1866 ....	420	4	10
Due to Shareholders, Arrears of Interest.....	27	2	5
Unpaid Accounts .....	307	16	11
Bonuses on Policies to be received in Cash not yet paid .....	102	7	3
Unpaid Claims .....	6086	6	0
Annuities due but unpaid .....	99	19	4
One Quarter's Ground Rent to the Duke of Bedford .....	21	10	0
Redemption Account for No. 28, King Street, Covent Garden .....	248	18	8
	229100	8	1
<b>Excess of Assets over Liabilities .....</b>	<b>28479</b>	<b>19</b>	<b>1</b>
	<b>£257580</b>	<b>7</b>	<b>2</b>
From excess of Assets over Liabilities .....	28479	19	1
Deduct amount reserved for future Assurance Guarantee Fund .....	1131	13	3
<b>Clear Surplus now divisible .....</b>	<b>£27348</b>	<b>5</b>	<b>10</b>

\* N.B.—The above Account is debited with £10,000., the full sum to be paid to the Shareholders out of the Guarantee Fund when that Fund shall amount to £110,000.

		Gr.					
		£	s.	d.	£	s.	d.
<b>Guarantee Fund—</b>							
£3980. 3 per Cent. Consols, at the cost price.....		3679	3	3			
6000. Railway Debenture Stock .....		6000	0	0			
800. 3/8 India 5 per Cent. Stock .....		320	16	9			
Reserved for Proprietors .....		10000	0	0			
1058. 16/4 India 5 per Cent. Stock at the cost price—Reserved for future Assurance Guarantee Fund.....		1181	13	3			
					11181	13	3
<b>Assurance Fund—</b>							
Advanced on Mortgage .....		68270	0	0			
£25000. New 3 per Cent. Annuities—at the cost price .....		23358	15	0			
19000. 3 per Cent. Consols .....		17780	1	4			
6000. 3 per Cent. Reduced Stock .....		5707	10	0			
50000. Railway Debenture Stock, at par.....		50000	0	0			
Advanced on Debenture Bonds .....		47000	0	0			
Advanced on Security of Policies .....		13589	4	8			
Cash on Deposit at Bankers .....		3500	0	0			
Cash at Bankers and in the hands of the Actuary ....		2706	12	7			
Balances in hands of Agents .....		2030	4	6			
Premiums due, but unreceived, including unpaid Instalments of Annual Premiums on Policies, renewable half-yearly and quarterly .....		3648	11	1			
Value of Policies for Re-Assurances .....		1600	0	0			
Premises No. 28, King Street, Covent Garden .....		4218	15	5			
Value of Furniture in ditto .....		500	0	0			
Dividends on Consols and India 5 per Cent. Stock, payable January 1867 .....		372	7	5			
Interest on Debenture Stock and Bonds, payable January 1867 .....		1952	0	1			
Interest on Mortgages due 31st December, 1866 .....		189	5	10			
Rent of Chambers due 31st December, 1866 .....		12	12	0			
Interest on Current Account with the Union Bank of London.....		12	14	0			
					246448	13	11
					£257580	7	2

## NORTH BRITISH AND MERCANTILE INSURANCE COMPANY.

### *Sixth Septennial Investigation.*

The Proprietors are aware that, according to the bye-laws of the Company, the Septennial investigation into the affairs of the Company required to be made as at 30th December 1865, in order that the result of the Life Business since the last period of investigation at 31st December 1858 might be ascertained and reported.

Before stating the results of this Investigation, it may be interesting to contrast the amount of business done by the Company between the periods included in the present and the preceding Investigations.

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### *Life Business.*

In the LIFE BUSINESS, the number of Policies issued, the Sums Assured, and the Premiums received during the same two periods are as under:—

For the period 1851 to 1858—

	No. of Policies.	Sums Assured.	Premiums.
1852	259	£256,327	£9,000 18 6
1853	318	237,103	8,098 17 9
1854	528	303,455	10,340 18 2
1855	423	292,978	9,979 19 8
1856	392	274,274	9,061 14 9
1857	325	276,931	8,299 16 6
1858	455	377,425	12,565 18 8
	<u>2700</u>	<u>£2,018,493</u>	<u>£67,348 4 0</u>

Being an average of 386 in the number of Policies issued, and £288,356 per annum of the Sums Assured.

For the period 1858 to 1865—

	No. of Policies.	Sums Assured.	Premiums.
1859	605	£449,913	£14,070 1 6
1860	741	475,649	14,071 17 7
1861	785	527,626	16,553 2 9
1862	870	622,224	18,872 9 8
1863	1071	953,839	33,002 14 0
1864	1242	1,035,906	32,168 19 7
1865	1486	1,018,707	34,041 6 4
	<u>6800</u>	<u>£5,083,864</u>	<u>£162,780 11 5</u>

Being an average of 971 in the number of Policies issued, and £726,266 per annum of the Sums Assured.

#### *Results of Investigation.*

In proceeding to carry out the Septennial Investigation, the Directors have been anxious as formerly, that it should be based on the safest principles. The method adopted in the valuation of the Policies has been precisely the same as was followed in the years 1844, 1851, and 1858, and the surplus brought out arises entirely from the profits on the business for the Septennial period ending 30th December last; and as the whole loading or addition to the Premiums has been deducted, no part of the future profits has been anticipated.

The number of Policies subsisting at 31st December last was 8889, assuring £6,369,482, exclusive of 31 Policies assuring £3318 per annum of Deferred and Survivorship Annuities.

The value of the Company's liability under these was . . . . . £823,778 4 6

To which must be added the value of the Bonuses or

Additions declared up to 31st December 1858 . . . . . 188,930 6 10

Value of Deferred and Survivorship Annuity Policies . . . . . 5,056 8 9

Amount of Claims outstanding as at 31st December last . . . . . 5,614 15 9

£978,379 15 10

From which falls to be deducted the value of Policies and Bonuses, applicable to £832,359 reassured . . . . . 105,011 19 0

£868,367 16 10

Add—Reserve for Contingencies . . . . . 9,598 17 6

Total Liabilities . . . . . £877,966 14 4

The Assets in the Life Department were . . .	£1,028,377	2	1
Deduct the above Liabilities . . . . .	877,966	14	4
	<hr/>		
Leaving a surplus of . . . . .	£150,410	7	9

According to the principle of division adopted at last Septennial Investigation nine-tenths of the above surplus belong to the parties assured on the participating scale, and the Directors have thus been enabled to declare a Bonus of 25s. per cent per annum on all sums assured with profits. This is in many instances equivalent to £1:18s. per cent. per annum on the original sum assured, in consequence of the present Bonus being declared not only on the original sum assured, but also on the previous Bonus additions.

The Directors would further recommend that the usual Prospective Bonus of one per cent per annum should be paid on all Policies existing at 31st December last, which may become claims prior to the next Division of Profits.

The Bonus and Prospective Bonus are not payable on any Policy which has not been five years in existence.

The remaining tenth part of the surplus belongs to the Proprietors, and has been carried to the accumulated fund of undivided profits. To this fund also has to be added the profit arising upon the result of the annuity transactions during the Septennial period, amounting to £9211:14:11.

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Several complaints having been made of the long interval which elapses between the periods of investigation, the Directors have resolved to meet this objection by altering the period at which the investigation and declaration of Bonus takes place from a Septennial to a Quinquennial period.

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## NOTICES OF NEW BOOKS.

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*A Treatise upon the Law of Life Assurance, upon the constitution of Assurance Companies, the construction of their Deeds of Settlement, the sale of Reversionary Interests, and Equitable Lives arising in connection with Life Policies: with an Appendix of Precedents for the Assignment of Policies by way of sale, mortgage, and settlement; Notes of Cases; Statutes; and an Index of Private Acts obtained by Insurance Companies.* By CHARLES JOHN BUNYON, M.A., of the Inner Temple, Esq., Barrister-at-Law. London: Charles and Edwin Layton, 150, Fleet Street.

It is now fifteen years since Mr. Bunyon first published his "Law of Life Assurance"; and the changes which have subsequently taken place in the law, both through the action of the legislature and through the decisions of the Courts of Justice, have been so numerous and so important, that the work had become quite out of date and of comparatively little practical value. It had consequently become absolutely necessary that a second edition of the work should be published, if Mr. Bunyon was to retain the

honour of being the author of the only complete treatise upon its subject. This second edition is now before us ; and we propose to give a short outline of its contents, and of the changes which have been made in it, as the best mode of enabling our readers to judge of its value.

The work commences with a highly interesting introductory chapter "On the Amendment of the Law," in which the Author draws the attention of the reader "not only to the changes which have occurred in this and "collateral subjects, but to amendments which may still be effected"; and his remarks on the latter subject appear to us to be extremely judicious.

First in order of time, and, we may almost say, of importance, comes the overruling of the celebrated case of "*Godsall v. Boldero*" by the decision in the case of "*Dalby v. The India and London Assurance Company*." It will no doubt be remembered by our readers that the first edition contained a long and elaborate argument to prove that the ruling in *Godsall v. Boldero* was bad law and should be overruled. The same view had been taken by other authors, in particular by Professor De Morgan in his *Treatise on Probabilities* (p. 244); but it must be especially gratifying to the Author to find that he has anticipated, if not assisted by his arguments in bringing about, this change in the law.

It is next noted that the position of the referees of a person proposing an assurance has been placed on a juster footing by the decision in the case of "*Wheelton v. Hardisty*," where it was held "that they are not necessarily the agents of the proposer so as to affect him with the responsibility "of a fraud on their part." The Usury Laws have been swept away, and as a necessary consequence, the Annuity Act. Proceeding in the same spirit, the legislature has, still more recently, passed "*The Sales of Reversions Act*," which has put an end to the rule of equity that required the purchaser of a reversion to show that he had paid a full and adequate consideration for it, and in default of such proof allowed the contract of sale to be annulled, even after the reversion had fallen into possession. The most recent Act affecting Life Assurance is "*The Policies of Assurance Act, 1867*"; and to this the author has devoted considerable attention, both in this introductory chapter, and in the body of the work. His view is that although this Act may be esteemed a step towards making policies assignable at law, still it avoids doing so in express terms; for it provides only that any person *possessing a right in equity to receive*, and a right to give an effectual discharge for the insurance money, may sue at law in his own name. It does not simplify titles, and the author with good reason calls attention to the hardship imposed on Insurance Companies "in that in settlement of claims on policies the necessity is thrown upon them of adjudicating, to a great extent at their own peril, upon titles which can often only be fairly settled in Courts of Equity." He "apprehends that a statute on this subject, to satisfy the wants of the community, should render life policies assignable at law by transferring the right of action, and should, in all cases, enable the assignee to give a good discharge in "equity to the Office." Such a statute would be very useful, if only as a declaratory Act; but some of the best lawyers hold that at the present time an assignee can in all cases give a good and effectual discharge to the Company.

Some legislation is urgently required to facilitate and legalize the amalgamation of Life Insurance Offices. On this topic the Author dilates

at some length in his usual perspicuous and argumentative style. He suggests the passing of a general enabling statute, "which should legalize" such transactions, only providing some tribunal which should see that "equal justice is done to all parties in the arrangement." Such a tribunal he considers might be the Board of Trade or Court of Chancery, which should publish some general rules framed with the assistance of the Council of the Institute of Actuaries.

The main point to be borne in mind is, that if the policyholders of an Office are transferred to another without their own consent, as must generally, or indeed always, be the case, they ought not to be placed in a worse position than before, as regards the guarantee afforded them by the shareholders' capital, paid or subscribed. On this ground, probably, the Court of Chancery, if appealed to at the proper time, would have granted injunctions restraining the greater number of the amalgamations and transfers which have taken place. It need not be pointed out here how far the above principle has been lost sight of in many amalgamations, and notably in those which preceded, and doubtless caused or accelerated, the failure of the Albert Life Office.

Another question discussed by the Author is whether and to what effect any special legislative sanction should be given to voluntary settlements of policies of assurance. Ante-nuptial settlements, in the absence of gross fraud, are good against all creditors; but in post-nuptial settlements there are special difficulties where the property settled is a policy of assurance; for although at the time the post-nuptial settlement is made, the settlor may be perfectly solvent, still the payment of premiums when he had become insolvent might have the effect of invalidating such settlement. This is certainly a point on which legislation is needed; and we think it would not be difficult to devise a plan by which the post-nuptial settlement of policies of insurance might be facilitated and encouraged, without unduly restricting the rights of creditors.

The discussion of this subject leads the Author into a dissertation upon the amendment of the law of property of married women, which is rather foreign to the object of the work, and into which we therefore will not follow him.

The body of the work is divided into three parts, which may be briefly described as treating of the subject in its relation to (1) the common and statute law; (2) the equitable interests; and (3) the claim; or to explain it more fully, the first part treats of the contract of Life Assurance, its nature and details, with the rights and disabilities under it, and the constitution, management, and dissolution of Insurance Companies; the second part treats of the transactions connected with Life Assurance, and those in which Life Assurance forms an important element; and the third part, of the questions connected with the payment and enforcement of the claim. The first part is divided into eleven chapters, of which the first is devoted to an explanation of "The nature of the contract at the common law, and as modified by the statute law." In it the Author points out the difference between Life Assurance and Marine or Fire Insurance, and discusses the question as to how far it fulfils the legal conditions of a wager. He also explains the operations of the Act 14 Geo. III., c. 48, commonly known as the Gambling Act, which defines the legal limits of the contract. This Act, which applied only to Great Britain up to the year 1866, was in that

year extended to Ireland. The second and third chapters, treating of the proposal and declaration, warranties and representations, all of which belong to the same branch of the subject, have been judiciously combined in the new edition. All our readers are familiar with the ordinary proposal form and declaration. The declaration is generally, either expressly or by reference, embodied in the policy, and the facts stated therein, when made unconditionally, are in legal language called warranties, and must be strictly and literally true, their correctness being a condition precedent to the validity of the policy. The principle, as the Author here explains, upon which the maxim *caveat emptor* is founded does not apply to the contract of insurance. All material facts known to the assured, whether he himself considers them material or not, must be fully disclosed; for equity requires the parties to contract *pari passu*. The position of the referees, as already pointed out, has been entirely altered, so that instead of being, as formerly considered, the agents of the proposer, they must now be regarded as the agents of the Company. As regards the private and medical referees, this is no doubt the proper light to look upon the question, since the insurers having the means of making full enquiries, by requiring the proposer to mention the names of those persons best qualified to give information, it would be unjust to hold him accountable either for negligence on the part of the office, or fraud on the part of the referees, of which he is wholly innocent. But it is otherwise with regard to the person whose life is proposed for insurance, since in most cases he is interested in obtaining the policy, and will give the most favourable answers which he conscientiously can; and might, therefore, more often than not, be considered to act as the agent of the proposer: so that notwithstanding the unanimous opinion to the contrary of the judges both in the Exchequer Chamber and in the Court below, in the case of *Wheelton v. Hardisty*, we still think that the rule should be considerably relaxed as regards the life insured. How far the omission or misrepresentation of certain facts in the declaration has been considered to invalidate the policy, has been the subject of numerous actions; and it is needless to say that the decisions have been carefully collected and critically examined by the Author.

"The policy and its conditions" form the subject of the fourth chapter, in which the acceptance of the proposal, the form of the policy and its execution, and the ordinary conditions upon which a policy is issued, are fully considered. As might be expected, all these conditions have been severely contested in our Courts of Law, and have been construed very strictly; and it is therefore of the greatest importance to the assured to make himself fully acquainted with the particulars of his policy, and rigorously act up to the conditions of the contract.

The fifth chapter contains a collection of information with regard to the conditions of the policy, which has not been considered in the previous chapters, such as the burden of proof of the truth of the warranties, the admission of certain facts by indorsement, &c. Policies termed "Indisputable and Unchallengable" have been very largely advertised of late, and appear to be somewhat on the increase; it would therefore be advantageous to know what construction is likely to be put upon them by the Courts. Mr. Bunyon has carefully examined the subject in all its details, and we strongly recommend a careful perusal of his remarks. It is undoubtedly open to the parties to a contract to make their own conditions, but never-

theless, where one party has acted *mala fide*, and used fraud to obtain the contract, there is no doubt that an agreement by the other to fulfil the contract, notwithstanding, is against public morals, and therefore void; so that when a legal construction comes to be placed upon these policies, there will be probably found to be very little difference between them and an ordinary policy. Other important matters are also discussed in this chapter, such as the conditions under which equity will relieve the insurers, the return of the premiums, and the reformation of the contract.

Other contracts not being strictly Life Assurances, but partaking more or less of their character—namely, insurances against accident, insurance against the birth of issue, and the guarantee of fidelity—are considered in the sixth chapter. Insurances against accidents are of two kinds, one providing for the payment of a certain sum on death by accident, and the other for a weekly allowance during total disablement from accident; and it appears to have been equally difficult to decide what an accident is, and what total disablement is, without the intervention of a Court of Law. Insurances against the birth of issue are contracts of very recent origin, but are now granted by many Insurance Offices as part of their regular business, the observations of Mr. Day upon the statistics of first and subsequent marriages among the Peerage having made it possible to calculate the premiums with sufficient accuracy for all ordinary purposes.

The grant of Guarantee Policies, or the insurance of fidelity, cannot claim to be connected with Life Insurance in any way, and it might be said ought not therefore to have a place in such a work as this; but we think it fairly admitted, not only because it is sometimes transacted by Life Offices and combined with Life Insurance itself in one policy, but also on account of the similarity of the conditions and obligations, and of the remedies at law and equity. Guarantee Companies are perhaps not resorted to so much as they should be. The fidelity of servants is as properly a subject for insurance as loss by fire or perils of the sea; and the objections which will apply in the one case against personal insurance, equally apply in the other. No landlord thinks of asking a tenant to find responsible sureties against his setting fire to the house, yet every day we find employers requiring their servants to find private sureties for their fidelity. Possibly the cause may be found in the private sureties not stipulating for sufficient caution on the part of the employer, and not being so fully aware of their rights and remedies as the Guarantee Offices.

The following chapter, which treats of "The Insurer and the Constitution of Insurance Offices," has been partly rewritten owing to the change in the law of Joint Stock Companies. Life Insurance Offices are arranged in five classes, according to the nature of their constitution, namely, "(1) Companies incorporated by Royal Charter or statute; (2) Companies empowered by Letters Patent from the Crown, but not incorporated; (3) Companies neither incorporated nor acting under Letters Patent, but governed by their own Deeds of Settlement only, and which do not, except in the number of partners or members, differ from ordinary partnerships; (4) Companies incorporated by registration only, under the Joint Stock Companies Registration Act, 1844, and re-registered under the Companies Act, 1862; (5) Companies formed under the Act, 1862." A careful explanation is given of the legal position of the Companies in each class.

The eighth chapter treats of "Charters of Incorporation, Deeds of Settlement, and of private Acts of Parliament amending the same, and giving special powers to Insurance Companies, and of the partnership rights of the members *inter se*." Such subjects as the power of a majority of the partners at extraordinary general meetings, and the effect of resolutions unauthorized by the Deed of Settlement, the construction of general powers in such deeds, and the interference of Courts of Equity in such acts, are here considered; as well as the construction of private Acts, the privileges which may be obtained by them, and how far they are binding upon the Company and strangers.

"The dissolution, winding up, and amalgamation of Insurance Companies" are subjects which have greatly increased in importance since the passing of "The Companies Act 1862." In the first edition of this work they formed the latter portion of the seventh chapter; but it has now been found necessary to devote an entire chapter to their consideration. The dissolution of a Life Assurance Company is a matter of peculiar difficulty, since the liabilities are different in kind from those of any other Company, and the ordinary rules of liquidation cannot well be applied. "They are liabilities undertaken to be performed at future and perhaps very distant dates, and although susceptible of valuation with some accuracy in the mass, are in detail scarcely capable of valuation at all"; and the only way of fairly meeting the difficulty is by inducing another company to undertake the liabilities. There are three modes by which a Life Assurance Company can be dissolved, namely: (1) "by virtue of a special provision for this purpose inserted in the deed of settlement or articles of association," the provisions of which the Company must strictly follow, or they will be restrained by the Courts of Equity at the instance of a single shareholder or policyholder. (2) "By consent of the whole body, both policyholders and shareholders," which it would practically be quite impossible to obtain. No doubt, says Mr. Bunyon, the majority of successful amalgamations have been carried out by assuming the necessary powers to exist, and obtaining the consent of the majority; but this is an example which cannot be recommended. Under such circumstances the Author has considered the position of shareholders after a transfer, the rights of policyholders, and the effect upon collateral contracts, &c. (3) Or "a Company may be wound up under The Companies Act, 1862"; either compulsorily by the Court, or voluntarily, and in the latter case either with or without the intervention of the Court. Under a voluntary winding up, a transfer or sale of the business may be effected by a special resolution, passed concurrently with the resolution for winding up; but the law upon the subject is very unsatisfactory and will require considerable amendment before the dissolution of a Life Assurance Company can be carried out in a satisfactory manner. The next chapter refers briefly to the law affecting those so-called "Friendly Societies" established before the Act 18 and 19 Vict., c. 63, and still in existence as insurance offices.

In the eleventh chapter we are presented with information, important not only to Life Assurance Companies, but to all Companies of whatever kind. It is "Concerning the powers and duties of Directors, Officers and Agents"; how far the Directors are restricted in their action, the effect of their exceeding their powers, and the result of publishing false or fraudulent statements. In the last case, they will be personally liable to those who

have been deceived. Mr. Bunyon is of opinion that "the declaration of a fictitious bonus payable out of capital instead of realized profits, would be a fraudulent statement, since it would impute a false prosperity to the concern, and thus deceive intending insurers"; and although there appears to have been as yet no prosecution of Directors on these especial grounds, yet this is a point well deserving the attention of the Directors and Managers of Companies. The duties and powers of agents are next considered, and the effects of their exceeding their authority. It is pointed out that the principal officers of a Company, such as the Managing Director, the Actuary, and Secretary, are general agents, and will be held to possess all necessary powers for enabling them to conduct the business.

The second part of the work—devoted to the subject in its equitable relations—commences with a chapter "on the assignment of policies." Policies being *choses in action* are, at Law, with a few exceptions, incapable of assignment; but the right to receive the policy moneys may, in Equity, be transferred for a valuable consideration, the vendor becoming a trustee for the purchaser. The Author has here very carefully considered the question of "considerations," and has also fully investigated and illustrated the subject of the equities of consecutive incumbrancers.

The contract to sell and purchase an annuity being now, in consequence of the repeal of the Annuity Act, a very simple matter, no longer requires a separate chapter for its consideration, as was the case in the first edition. It has therefore been considered at the end of the chapter now under review. It is certainly rather difficult to see its connection with the general subject of the chapter, but the Author has managed to tack it on in so admirable a manner, and it is so difficult to see where else it could have been placed, that we cannot take any strong objection to its position. It is to be regretted however that no mention of the subject is made in the heading of the chapter.

The subject of the following chapter is "concerning the evidences of the contract when a sale is effected or a lien created, and the rights and remedies of the mortgagee as against the mortgagor." Although it is not necessary, as Mr. Bunyon explains, that the contract should be evidenced by deed or in writing, since "there is no legal estate to pass, and the case, except where it is that of a marriage settlement, is not within the operation of the statute of frauds," yet it is of advantage in the case of a subsequent sale to have the contract in writing; and now a very simple form (which may be endorsed on the policy) is given in the "Policies of Assurance Act, 1867." The Author explains the constructions of the ordinary covenants in a mortgage deed, the operation of the Bankruptcy Laws in discharging certain of the covenants, and the rights and remedies of the mortgagee under various circumstances.

The third chapter is "concerning notice." Notice of the assignment of a policy should now be given to the Office in accordance with the provisions of the "Policies of Assurance Act, 1867," which we reprinted in the number of this *Journal* for October, 1867, with a few remarks which occurred to us at the time. We agree with the learned Author in considering that the Act has been most thoughtlessly constructed, and cannot fail to give rise to much litigation, but we think he has gone too far in his criticisms in the following passage (p. 241):—"The Act however goes on to say, 'and the date on which such notice shall be received shall regu-

"late the priority of all claims under any assignment." This is very "obscure and badly expressed. All claims under any assignment must undoubtedly be regulated by the terms of the assignment itself. But the section probably means that the priorities as between successive assignees shall be determined by the dates at which notices of the assignments shall be respectively given." For the Act says that the date of notice shall regulate not "the claims," but "the priority of all claims under any assignment." The sense would certainly have been clearer, if instead of "all claims," the Act had said "the claim." As the Act only applies to assignments made after its date, it is necessary to consider the law relating to the subject previously, so that much of the chapter has not been altered. All the questions concerning notice will be found fully discussed, such as, how far the Company may be affected by constructive notice, the duties of insurers in answering inquiries as to notice, the effect of disregarding notices of assignment, and so on. These are all questions of the most vital importance to the Managers of Life Insurance Companies, and every actuary should therefore carefully study this chapter.

In Chapter IV—"concerning advances by Insurance Offices, by way of mortgage upon their policies, with or without additional security"—the Assurance Offices are considered under their character of capitalists. It is a very common practice for Life Offices to grant loans on life interests and other securities, with a policy effected in the Office itself as collateral security. In this case the Office will occupy the same position as any other mortgagee, and the rights of the assured will be unaffected. Mr. Bunyon has cited three cases at length upon the subject, and we may also mention one that has been tried since, and which will be fresh in the minds of many of our readers, namely, the case of *White v. The British Empire Mutual Life Assurance Company*." Here the Company were mortgagees of a policy in their own Office on the life of a person who committed suicide. The Court decided that, as regarded the insurance, the Office stood in the same position as any other mortgagee; and the Plaintiff, who was the administratrix of the deceased, was entitled to the benefit of the condition in the policy making it good under such circumstances to the extent of any *bond fide* interest of any third party therein, and the amount of the claim was set off against the mortgage debt.

The next chapter, "On the Sale of Reversionary Interests," has been, of necessity, extensively altered in consequence of the passing of the "Sales of Reversions Act, 1867"; but still treats of many points of great importance relating to a married woman's reversionary property, her power of disposition over her separate estate, and her husband's power over her unsettled personality when reversionary, &c., including of necessity the effect of "Malins's Act, 20 & 21 Vict., c. 57."

The sixth chapter is "Concerning the Equities arising from contracts other than contracts of sale, or the creation of liens by the assured"; and embraces such questions as the rights of debtor and creditor to the policy moneys when one or other has paid the premiums, and the disposition of the moneys when the policy is effected by persons as trustees. Here also is considered the question as to the title to the policy upon the life of the *cestui que vie* on the repurchase of an annuity, which is one of great difficulty, and has been often before the Courts. The different cases on the subject are fully set forth and discussed.

The chapter "concerning the renewal of leaseholds for lives or years that "have been the subject of settlement, and the application of policies as "securities for the fines payable on such renewals," will be found useful to the Actuary when he has questions relating to such matters submitted for his opinion; but the subject is only incidentally connected with Life Assurance—inasmuch as that is the only proper mode of providing for the renewal of leaseholds for lives.

Chapter VIII is on a subject of greater importance, as affecting the disposition of the moneys assured, and consequently the receipt for the claim. It is "concerning voluntary settlements." It has been the Author's object to point out in this chapter under what circumstances voluntary settlements have been considered fraudulent within the meaning of the Act known as the Statute of Elizabeth, and in connection with the Bankruptcy Laws—when they have been supported, and when set aside, in equity; and also what conditions are required to make the gift complete; all of which he has most effectually done. His statements are clearly put and amply illustrated with most of the leading cases on the several points.

We now come to the third part of the work, which, although the shortest, is by far the most important to the Offices. It treats of the claim and the subjects connected with its payment. The first chapter is "Upon the rights and interests of persons under disabilities." An infant, for instance, as pointed out by Mr. Bunyon, is capable of having a policy of assurance effected in his own name for his benefit either upon his own life or upon the life of another; but then the ordinary rules affecting the contracts of infants apply equally here as in all other cases; and should the policy become a claim during minority, it becomes important to know to whom the money is to be paid or what is to be done with it. As regards a married woman again, marriage, operating as a gift to the husband of all *choses in action*, belonging to the wife and not settled to her separate use, that he may be able to reduce into possession during her life, places the wife under a disability to give a good discharge to the office for the sum assured under any policy to which she is entitled upon the life of a third person, if the death of that person happen during coverture. But the "questions respecting policies of assurance to which married "women are entitled, more often arise in cases in which the insurance is "effected during the coverture upon the life and in the name of the wife." Various cases arise according as the policy was effected by her with or without the husband's assent, according as the premiums were paid out of his moneys, or out of the income of property settled to her separate use; all of which are fully considered, together with the effect in such cases of the death of either husband or wife, or of the dissolution of the marriage by the Divorce Court. Again, a lunatic or idiot is not capable of contracting; but an originally valid contract is not affected by subsequent lunacy; and it is necessary for the Office to know to whom the sum assured should be paid when the policy is the property of a lunatic. The rights and disabilities of aliens and of persons convicted for treason or felony are also discussed.

The vital importance of the next chapter, which is "concerning the claim "and its payment, the proof of death, the receipt and custody of evidence "of title," to all parties interested in Life Assurance need not be enforced here. In the policy it is usual to stipulate for satisfactory evidence of the

death, which means "sufficient and not such as the mere caprice of the insurers may require." The Author explains what evidence should be furnished under various circumstances, and also what circumstances have been considered to give rise to presumptive proof of death. Proof of age, also, he says, must be furnished, either at the death or sometime during the assurance, and he explains what will be considered as sufficient evidence. "The facts upon which the claim arises having been thus proved, the claimant must deduce a good title to, and be prepared to give a sufficient receipt for the sum assured." Here then is discussed the title of the executor or administrator when the policy has not been the subject of assignment or charge; and the taking out of probate or letters of administration. This has been much simplified since 1858 by "The Court of Probate Act, 1857," but the Author has considered it advisable to state the general rules as to cases both prior and subsequent to the Act. He also points out the course to be pursued when the claimant is abroad; and the discharge to be given where there are several grantees of a policy; and also when the policy is mortgaged, or assigned upon trusts, or is the subject of a voluntary assignment. "Not only are the insurers entitled to a receipt which shall be an equitable discharge, but also to be rendered secure from any exercise of the legal right to their prejudice." When therefore a claim is paid, the questions as to the custody or covenants for the production of deeds are subjects of vital importance.

If the Company does not duly pay the assurance moneys, the claim may be enforced by action at law. The subject of the third chapter is to show how this is to be done; how and against whom proceedings are to be taken, and the remedy of the Company when there are conflicting claims. This chapter has of necessity been considerably altered, for since the first edition, "The Policies of Assurance Act, 1867," has been passed, empowering an assignee under certain circumstances to sue at law in his own name; and the Trustees' Relief Act, as the Author points out, will now relieve Insurance Companies in the case of conflicting claims, where formerly a motion by way of interpleader was the only course open to them. "The Companies Act, 1862," has also caused some alterations to be made.

The last two chapters treat of questions of simple detail, and are more fitted for an appendix than to form a portion of the main body of the work. The first of these is "concerning stamps;" and here we notice a most important correction, namely, the insertion in the Life Assurance policy stamps of the 3*d*. stamp, "where the sum insured shall not exceed twenty-five pounds," which was omitted in the first edition; and a similar error with regard to the stamps for conveyance on sales has also been corrected. The other chapter treats of "The succession duty and income tax Acts"; from which the dissertation on the Report of the Select Committee on Assurance Companies, 1853, has been wisely omitted. There is a copious appendix, containing sixteen precedents for the sale, mortgage, and settlement of policies; and the text of "The Companies Act, 1862," and of "The Companies Act, 1867." There is also an index of Private Statutes obtained by Insurance Offices; an index of cases cited; and a general index.

Mr. Bunyon established for himself a high reputation as a lawyer by the production of the first edition of this work, and this will be increased by the skill he has employed in the revision of the work in preparing the second edition. Some of the arguments which could not be considered

altogether satisfactory have been either suppressed or altered; and information which had become superfluous by reason of change in the law or general opinion, has been carefully eliminated; while the greatest care has been taken to introduce, wherever requisite, all the necessary information consequent upon the changes which have taken place in life assurance itself, the later decisions in the Courts of Justice and the subsequent alterations in the Law. . But in some minor points further revision might with advantage have been bestowed on the work. Thus in the foot note on page 4, the Gambling Act is cited as 14 Geo. III., c. 38, whereas it really is c. 48. This is evidently a printer's error, as is, no doubt, a similar error upon p. 390, where the case of *Fenn v. Edmonds* is cited as *Fenn v. Edwards*. Again, the word "late," as applied to cases cited, has been retained too frequently in this edition; and on p. 219 a case is still termed "very recent," which was so termed in the first edition. Surely 14 or 15 years will reduce a case from the class of "very recent" to a lower grade. Altogether this work forms a most complete treatise on the Law of Life Assurance, and will be found a most valuable addition to both the legal and assurance libraries; and it will no doubt obtain, as it certainly deserves, an extensive sale both among lawyers and actuaries.

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## INSTITUTE OF ACTUARIES.

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SOLUTIONS OF THE SECOND YEAR'S EXAMINATION QUESTIONS,  
1869; WITH REMARKS. BY THE EXAMINERS, PETER GRAY,  
F.R.A.S., AND RALPH P. HARDY.

1. Define the logarithm of a number.

*Ans.* The logarithm of a number is the index of that power of the base of the system in which the logarithm is taken, which is equal to the number. Thus, in the equation

$$b^x = a,$$

$x$  is the logarithm of  $a$  in the system whose base is  $b$ .

The object in view in proposing this question was to test the possession on the part of the candidates of the power of clear conception and correct definition, a power in constant requisition in dealing with questions involving complex contingencies. We find not unfrequently the logarithm of a number stated to be "that power of the base which is equal to the number." A power is here confounded with its index. In the above equation  $b^x$  is *the power*,—the  $x$ th power—of  $b$ ; and it is *equal* to  $a$ . It is the index,  $x$ , which is the logarithm of  $a$ .

This very loose mode of expression, if it did not originate with the late D. Jones, at least received abundant countenance from him. In his work on *Annuities* we are frequently directed to raise a certain quantity to "the same power as the number of years," his meaning being, to "the power whose index is the number of years." See his vol. i., pp. 11, 20, 27, &c.

2. The system of logarithms known as Briggs's, or the common system, is more convenient for numerical computations than any other. State the reason.

*Ans.* We have two advantages in the use of Briggs's system of logarithms:—First, that of two or more numbers consisting of the same succession of digits, but differing in the position of the decimal point, the mantissæ of the logarithms are the same; and secondly, as incidental to the first, that by inspection of a number we are able at once to assign the characteristic, or integer portion, of its logarithm. And these advantages arise in consequence of the radix of our numeral system and the base of Briggs's system of logarithms being identical.

Let  $a$  be any number. Then, if  $r$  be the radix of the numeral system, and  $n$  any integer,  $a$  and  $ar^n$  will consist of the same succession of digits, and the point of separation between the integer and the fractional part of  $ar^n$  will be  $n$  places to the right or the left of its position in  $a$ , according as  $n$  is positive or negative.

Taking the logarithms in any system we should have

$$\log ar^n = \log a + n \log r.$$

But in every system the logarithm of its base is unity. Hence if we use the system whose base is  $r$ , the equation will become

$$\log ar^n = \log a + n.$$

In this system then, which is Briggs's, the logarithms of all numbers consisting of the same succession of digits, differ only in the value of the integer  $n$ , which thus indicates the number of places by which the position of the separating point differs in the numbers.

Again. In a numeral system whose radix is  $r$  the  $n$ th power of  $r$  (or  $r^n$ ),  $n$  being an integer, is an integer consisting of unity followed by  $n$  ciphers; and in like manner  $r^{n+1}$  is an integer consisting of unity followed by  $n+1$  ciphers. Now by definition  $n$  and  $n+1$  are the logarithms of  $r^n$  and  $r^{n+1}$  to base  $r$ , respectively. Hence the logarithms of all numbers comprised between  $r^n$  and  $r^{n+1}$ , that is of all numbers whose integer part consists of  $n+1$  digits, will be comprised between  $n$  and  $n+1$ . Their characteristic or integer portion will therefore be  $n$ .

In virtue of the possession of this property it suffices to tabulate only the mantissæ of Briggs's logarithms.

3. An homogeneous coin is tossed three times. Assign the probability of the result giving, (a) one head; (b) two heads; (c) three heads. The sum of these probabilities differs from unity. Why?

*Ans.* Such a very simple question as this admits of solution on the most purely elementary principles. Write down all the results that can arise from three tossings, thus:—

H H H	T T T
H H T	T T H
H T H	T H T
H T T	T H H

They are eight in all; and they are all equally probable, since in a single trial the probability of H is the same as that of T. Of these eight results, three (Nos. 4, 6, 7) favour (a); three (Nos. 2, 3, 8) favour (b); and one (No. 1) favours (c). Hence the probabilities of these several events are  $\frac{3}{8}$ ,  $\frac{3}{8}$ , and  $\frac{1}{8}$ , respectively.

But the solution of all such problems is best effected by the aid of the expansion of the binomial  $p + q$ ,  $p$  and  $q$  denoting respectively the probabilities of the two contrary events (here the occurrence of head and the occurrence of tail) in a single trial. Raising to the *third* power, there being here *three* trials,

$$p^3 + 3p^2q + 3pq^2 + q^3,$$

we have in each term the probability of the occurrence indicated by the argument of that term. Thus,  $p$  and  $q$  being each  $\frac{1}{2}$ , we have from the expansion the probabilities of the various results, as follows:—

1st term,  $\frac{1}{8}$ , 3 heads.

2nd „  $\frac{3}{8}$ , 2 heads and 1 tail, in any order.

3rd „  $\frac{3}{8}$ , 1 head and 2 tails, „ „

4th „  $\frac{1}{8}$ , 3 tails.

The first three terms, in reverse order, give the probabilities required.

The events whose probabilities are  $p$  and  $q$  being contrary events,  $p + q$  is equal to unity. Hence also  $(p + q)^n = 1$ , for all values of  $n$ ; in other words, the sum of the terms of the expansion of  $(p + q)^n$ , is always unity. The meaning of this is, that, the terms of the expansion being the probabilities of *all* the compound events which can arise in  $n$  trials, their sum is the probability that *one or other of them* will arise; and this is an event of which, as it is certain to happen on the  $n$  trials being made, the probability is unity.

From the above it follows, that if the sum of the probabilities, correctly determined, of a number of the events that can arise out of  $n$  trials, does not amount to unity, one or more of the possible events must have been omitted. In the present case the omission is, the occurrence of three tails.

Galloway *On Probability* may be advantageously studied.

4. A coin being tossed as before, assign the probabilities of the following results: (a) one head at least; (b) two heads at least; (c) three heads at least.

*Ans.* The probabilities here required are the sums of the probabilities of the possible cases in which (a) one head or more occur, (b) two heads or more, (c) three heads. Thus we have,

$$(a) \quad p^3 + 3p^2q + 3pq^2 = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8};$$

$$(b) \quad p^3 + 3p^2q = \frac{1}{8} + \frac{3}{8} = \frac{4}{8};$$

$$(c) \quad p^3 = \frac{1}{8}.$$

The sum here differs from unity because the events in question are not *contrary* events.

5. Give the formulæ which represent the accumulated amount of an annuity, when the first payment is made in advance:—

(1) When the annuity is receivable annually;

(2) When the annuity is receivable half-yearly.

*Ans.* (1). The amount of an annuity for  $n$  years, at the period of the last payment being made is

$$\frac{(1+i)^n - 1}{i}.$$

If paid in advance, each payment will have been improved for one year

more than in the former case, and the amount at the same period will consequently be

$$\frac{\{(1+i)^n-1\}(1+i)}{i} = \frac{(1+i)^n-1}{1-v}.$$

(2). The forms of the foregoing expressions are in no degree due to the circumstance that the interval between successive payments is a year. They will hold in reference to any interval when  $i$  denotes the interest accruing on one pound during one such interval, and  $n$  denotes the number of payments. If then we retain  $n$  to denote the number of years in the duration of an annuity, and  $i$  the yearly interest on one pound, to adapt the above formulæ to half-yearly payments, we must write in them  $2n$  for  $n$ ,  $\frac{1}{2}i$  for  $i$ , let  $v$  denote  $1 \div (1 + \frac{1}{2}i)$ , (writing it in this case  $v_1$ ), and finally multiply by  $\frac{1}{2}$ , each payment now being half one of the yearly payments. The formulæ then become

$$\frac{1}{2} \cdot \frac{(1 + \frac{1}{2}i)^{2n} - 1}{\frac{1}{2}i}, \text{ and } \frac{1}{2} \cdot \frac{(1 + \frac{1}{2}i)^{2n} - 1}{1 - v_1}, \text{ respectively.}$$

The usual tables of the amounts of annuities, readily give the amounts when the annuities are payable in advance. Thus the amount of an annuity for  $n$  years, payable in advance, is the same as the tabular amount for  $n+1$  years, diminished by £1, its last payment.

So also the common tables can be adapted to the case of annuities payable half-yearly, &c. Thus the amount of an annuity for  $n$  years, payable half-yearly at a specified rate, is half the tabular amount of an annuity for  $2n$  years, at half the rate.

These properties are so obvious as not to stand in need of proof.

6. A and B (and their representatives) are put in possession, in equal shares, of an annuity certain of £ $a$  for  $2n$  years. They arrange to take the payments alternately, A taking the first. How much ought he now to pay to B for the advantage he thus receives?

*Ans.* The method of proceeding which would here probably first suggest itself would be to take the difference between the sums of the following series, which express the values of the payments to be received by A and B, respectively, viz:—

$$a(v + v^3 + v^5 + \dots + v^{2n-1}),$$

and

$$a(v^2 + v^4 + v^6 + \dots + v^{2n});$$

but the trouble of summing the series may be avoided thus:—

We know the value of the entire annuity, which is  $\frac{a(1-v^{2n})}{i} = M$ , say;

and we know also that the values of the payments receivable by A and B are in the ratio of  $1:v$ . Hence, if  $x$  denote the value of A's payments, that of B's will be denoted by  $vx$ . We shall consequently have

$$(1+v)x = M; \text{ whence } x = \frac{M}{1+v}.$$

$$\text{and } vx = \frac{vM}{1+v}.$$

The difference of these is  $\frac{(1-v)M}{1+v}$ , which is the excess of the value of A's payments over that of B's; and the shares of the two parties will be equalised by A making payment to B of half this amount, namely  $\frac{(1-v)M}{2(1+v)}$ .

If we restore the value of  $M$ , the value just deduced becomes

$$\frac{av(1-v^{2n})}{2(1+v)}.$$

7. State the leading characteristics of the Carlisle Table of Mortality. State also for what class of risks you consider it to be best adapted.

*Ans.* The Carlisle Table of Mortality was constructed by the late Mr. Milne, his data being two enumerations of the inhabitants of Carlisle, having between them an interval of nearly eight years,\* with the registered number of births and deaths during the same interval; all as contained in a tract published by Dr. Heysham. In both enumerations the ages are given for the same quinquennial periods. But it has been recently pointed out by Mr. Makeham (*Journal*, vol. xii., pp. 319-21) that in both enumerations the distribution of the numbers living amongst the several periods is absolutely identical. This seems to render it all but certain that in only one of them—the first—were the ages distinguished. If so, and as the numbers were but few, we have most likely in this circumstance an explanation of those irregularities in the indications of the Carlisle Table of Mortality, which must be held to impair to some extent its authority as an exponent of healthy life in England. As an illustration of what is here referred to it may be mentioned that this table gives the mean duration for ages 92 to 97 greater than that for age 91.

8. Explain De Moivre's hypothesis as to the probabilities of life; and state whether it indicates a stationary or a variable rate of mortality, with the reasons for your opinion.

*Ans.* De Moivre's hypothesis is that of 86 persons born one dies each year till all are extinct. It was not proposed by its author as giving a correct representation of the probabilities of life, but as affording the means of readily deriving the usual results of the mortality table, while the results so formed should not, at least for the middle ages of life, deviate much from the truth.

Take any age,  $x$ . Then  $86-x=c$ , is called the complement of life; and if the numbers living were arranged as in the mortality table, we should have  $l_x=c$ . But it is usual to express the tabular functions in terms of  $c$ . Thus, we have,

$$p_{x:n} = \frac{c-n}{c}$$

$$e_x = \frac{c}{2}$$

And if  $a_{(c)}$  denote the present value of an annuity certain for  $c$  years, we further have:

\* This interval is usually stated to have been nine years. But January, 1780, to December, 1787, gives only seven years and eleven months.

$$a_x = \frac{1}{c^x} \{c - (1+v)a_{(c)}\}$$

$$A_x = \frac{a_{(c)}}{c}$$

$$w_x = \frac{(1-v)a_{(c)}}{c - a_{(c)}}$$

This hypothesis indicates an *increasing* rate of mortality. The probability of dying in a year is a fraction whose numerator is constant, being unit, and whose denominator,  $c$  or  $l_x$ , decreases as  $x$ , the age, increases.

9. Give expressions for the following functions; and state clearly the meaning attached to them, and the practical uses to which they can be put:—(1) Expectation of life; (2) *Vie probable*; (3) Mean age at death.

*Ans.* (1) Expectation of life, *vide* Gray's "Tables and Formulæ," § 135, *et seq.* See also *Journal*, vol. xiii., p. 381.

(2) *Vie probable*, *vide* Gray's "Tables and Formulæ," § 170.

(3) Mean age at death, *vide* Farr's Introduction to English Life Table No. 3, p. xxxv.

No essentially practical use has yet been made of these functions. The values of some benefits might be computed from the "expectation of life"; but the process would be laborious. For an application, *vide Journal*, vol. iii., 325.

10. How do you express symbolically the probability that an assigned life will survive a year? Explain what is meant by such a function, and state under what conditions and limits it is a fair representation.

*Ans.* By  $p_x = \frac{l_{x+1}}{l_x}$ ; by the ratio between the number attaining age  $x$ ,

and the number attaining age  $x+1$ . Assuming that the life selected be a fair specimen of its class, and that the mortality of the class be adequately represented by the table of mortality employed, this ratio is the probability, or a numerical expression of the force of the impression upon our minds before the determination of the event, certainty being represented by unity.

11. State the constitution and characteristic properties of the Single Life Commutation Table, and show the relations which subsist between corresponding values in the several columns.

*Ans.* See *Journal of the Institute*, vols. xii., p. 328, and xiii., p. 129.

12. If  $w_x$  and  $w_{x+1}$  represent the annual premiums for the ordinary whole life assurance at ages  $x$  and  $x+1$  respectively, show symbolically the probability that ( $x$ ) will die in a year:—

(1) If the premiums be taken as net;

(2) If the premiums be taken as loaded to the extent of 25 per cent.

*Ans.* We have  $w_x = \frac{1}{1+a_x} - (1-v)$ ;

$$\therefore \frac{1}{1+a_x} = w_x + (1-v),$$

$$\text{and} \quad 1+a_x = \frac{1}{w_x + 1-v} \quad \dots \quad (1)$$

Similarly, 
$$1 + a_{x+1} = \frac{1}{\omega_{x+1} + 1 - v} \dots \dots \dots (2)$$

Now  $a_x = v p_x (1 + a_{x+1}) = \frac{v p_x}{\omega_{x+1} + 1 - v}$ , from (2) by substitution.

But from (1), 
$$a_x = \frac{v - \omega_x}{\omega_x + 1 - v}.$$

Hence, equating, 
$$\frac{v p_x}{\omega_{x+1} + 1 - v} = \frac{v - \omega_x}{\omega_x + 1 - v};$$

whence, 
$$p_x = \frac{v - \omega_x}{v} \cdot \frac{\omega_{x+1} + 1 - v}{\omega_x + 1 - v} \dots \dots \dots (3)$$

And 
$$1 - p_x = \frac{\omega_x - (v - \omega_x)\omega_{x+1}}{v(\omega_x + 1 - v)},$$

which is the probability of dying in a year.

The use of this problem is to enable us, when a table of premiums is given, to deduce thence the mortality on which it is founded. To do this with certainty it is obviously necessary to know the rate of interest involved, and also the loading, if any, that the premiums bear. If these be known or assumed,  $\log p_x$  for each age can easily be determined by (3).

If the loading be  $p$  per £, the premiums must be diminished in the ratio of  $1 + p$  to 1 before being used in the formula.

It appears from (1) and (2) that (3) may be written as follows :

$$p_x = \frac{v - \omega_x}{v} \cdot \frac{1 + a_x}{1 + a_{x+1}}.$$

Papers on this subject will be found in the *Assurance Magazine*, vol. ii., p. 391, by Mr. William Wylie; vol. vi., p. 231, by Mr. George Scott; and vol. vi., p. 297, by H. A. S.

13. If the claims expected by an Office in a given year were more than those actually experienced, can the amount stated to have been so saved be termed profit? Give full reasons for your opinion.

*Ans.* No: the whole of this difference would not appear in the surplus upon a valuation of the assets and liabilities of the Office. The liability is not extinguished by the non-occurrence of the exact number of expected deaths within the specified period. The mortality in respect of these suspended claims may emerge gradually or at once;—may be distributed over future years according to the normal rate, or follow closely upon the period of rest.

In the first case, the profit would be the difference in question less the reserve necessary to be held against such suspended cases. In the second, merely the interest upon the sum assured so long as it is not called for, and the accumulation of premiums received in the meanwhile.

Let the reserve fund of the Office be  ${}_4V_{30/10}$ , and let the deaths of the ensuing year be 60, instead of 66 as provided by the Carlisle Table.

Then, the difference between the expected and the accrued claims will be 6. How much of this may be appropriated as profit?

*Case 1.* If the mortality of the six unexpected survivors (*i.e.* 5009 + 6) be assumed to follow the normal rate, the profit

$$P = 6(1 - V_{30|11}) \\ = 5.221 \text{ at Car. 3 per cent.}$$

This is the greatest amount of profit resulting from the year's favourable experience, and would be properly appropriated if selection has a calculable value.

*Case 2.* If these six deaths be only postponed, and if the time of postponement be assignable—say  $n$  years—so that the mortality of the  $n$ th year would be excessive in respect of these deaths, then the sums suspended might be carried to a special account, and the present value of both the interest on the same and the accumulation of the premiums meantime receivable, might be appropriated as profit. Hence

$$P = v^n \{ 6((1+i)^n - 1 + w_x(A_{n+1} - 1)) \}$$

Taking  $n=1$ , we get  $P = 6\{w_x + (1-v)\}$ .

$$P = .9708 \times 6(.03 + .01952 \times 1.03) \\ = 2919 \text{ at Car. 3 per cent.}$$

14. Find the formula for the present value of the remainder of an annuity certain for  $n$  years, to be entered upon at the death of  $(x)$ , the present possessor.

*Ans.* The entire value of the annuity is  $a_{(n)}$ , and the value of  $(x)$ 's share of it is  ${}_n\bar{a}_x$ . Hence the value of what remains is

$$a_{(n)} - {}_n\bar{a}_x.$$

This expression cannot become negative, since  ${}_n\bar{a}_x$  is always less than  $a_{(n)}$ . For  $a_{(n)}$  is  $v + v^2 + \dots + v^n$  ( $n$  terms), while  ${}_n\bar{a}_x$  is  $vp_x + v^2p_{x+1} + \dots + v^n p_{x+n-1}$  (also  $n$  terms), and the co-efficients,  $p_x$ , &c., are all less than unity.

15. Investigate a formula for the annual premium for an assurance on the life of  $(x)$ , which is to be 1, 2, 3, . . .  $n$  pounds according as the death of  $(x)$  shall take place in the 1st, 2nd, 3rd . . .  $n$ th year, and to remain constant at the last-named amount during the remainder of life:—

- (1) For a premium payable  $n$  times, provided  $(x)$  so long lives;
- (2) For a premium payable throughout life.

*Ans.* Here the benefit side of the equation is,

$$R_x - R_{x+n};$$

and the payment side, according as the premium is payable for  $n$  years or for life, is

$$w(N_{x-1} - N_{x+n-1}), \text{ or } w_1 N_{x-1}.$$

Equating, we get

$$w = \frac{R_x - R_{x+n}}{N_{x-1} - N_{x+n-1}}, \text{ and } w_1 = \frac{R_x - R_{x+n}}{N_{x-1}}.$$

16. Deduce the expression for the single premium for an assurance on  $(x)$  against  $(y)$ .

*Ans.* The sum assured, say one pound, will be receivable in the  $n$ th year if  $(x)$  die therein, and if he be survived by  $(y)$ . There is no restriction as

to the duration of the term of  $(y)$ 's survivorship: the second condition will be fulfilled if he be alive at the instant of  $(x)$ 's death.

It is at once obvious that for the solution of this problem it is necessary to take account of the manner in which the deaths of the  $n$ th year, in each of the classes to which  $(x)$  and  $(y)$  respectively belong, are distributed over the year. It is usual to assume that the deaths in each class are uniformly distributed over the year in which they take place; so that the deaths which occur amongst that class in any specified portion of the year are the same in number as those that occur in any other equal portion. This assumption, which would be rigorously true if the annual decrements of the mortality table were equal for the whole duration of life, is, as matters stand, sufficiently near the truth for practical purposes. The decrements, except for a few years at the close of life, vary but slowly; and the error committed by assuming them to be uniformly distributed in each year, is found to be immaterial. In what follows the usual assumption is made.

Denote, in the meantime, for brevity, the number of  $(x)$ 's class now alive by  $A$ , of whom  $a$  enter on the  $n$ th year, and  $d$  die therein; and denote the like for  $(y)$ 's class by  $B$ ,  $b$ , and  $d_1$  respectively. Let the  $n$ th year be divided into  $m$  equal parts,  $m$  being any positive integer; then, confining our attention to one of the parts, say the  $t$ th in order, the deaths in this part will be, amongst  $(x)$ 's class  $d \div m$ , and amongst  $(y)$ 's  $d_1 \div m$ .

Hence the probability that  $(x)$  will die in the  $t$ th part is

$$\frac{1}{A} \cdot \frac{d}{m}; \dots \dots \dots (1)$$

and that  $(y)$  will be alive at the end of it, the probability is,

$$\frac{1}{B} \left( b - \frac{td_1}{m} \right), \text{ or } \frac{1}{mB} (mb - td_1) \dots \dots \dots (2).$$

Hence also, the probability of the concurrence of two independent events being the product of their individual probabilities, by multiplying together (1) and (2), we get

$$\frac{d}{m^2 AB} (mb - td_1) \dots \dots \dots (3)$$

for the probability that  $(x)$  will die in the  $t$ th part of the  $n$ th year and  $(y)$  be alive at the end of the same part.

Now this concurrence may happen in any one of the  $m$  parts into which the  $n$ th year is divided. Hence if in (3) we make  $t=1, 2, 3, \dots m$ , successively, and add the results, the sum will be the probability that  $(x)$  will die in, and  $(y)$  be alive at the end of, some one or other of the  $m$  parts; that is, that  $(x)$  will die in the  $n$ th year, and  $(y)$  survive him by a portion of time not exceeding one  $m$ th part of a year.

Replacing in the meantime  $d \div m^2 AB$  by  $Q$ , we have,

$$\begin{array}{ll} \text{for } t=1, & Q(mb-d_1); \\ t=2, & Q(mb-2d_1); \\ t=3, & Q(mb-3d_1); \\ \vdots & \vdots \\ t=m, & Q(mb-md_1); \end{array}$$

and summing, we get, on restoring the value of  $Q$ , since  $1+2+3+\dots+m=\frac{1}{2}m(m+1)$ ,

$$\frac{d}{m^2AB} \left( m^2b - \frac{m(m+1)}{2} d_1 \right), \text{ or } \frac{d}{AB} \left( b - \frac{m+1}{2m} d_1 \right), \dots (4)$$

for the probability in respect of the  $n$ th year, as above.

We can now, by giving to  $m$  in (4) the proper value, assign the probability that ( $x$ ) will die in the  $n$ th year and be survived by ( $y$ ) by a portion of time not exceeding any that we choose to name. Thus, for a period of survivorship not exceeding a month, making  $m=12$ , and observing that the coefficient of  $d_1$  may be written  $\frac{1}{2} + \frac{1}{2m}$ , we have

$$\frac{d}{AB} \left[ b - \left( \frac{1}{2} + \frac{1}{24} \right) d_1 \right];$$

and in like manner, for a term not exceeding a day, we have, making  $m=365$ ,

$$\frac{d}{AB} \left[ b - \left( \frac{1}{2} + \frac{1}{730} \right) d_1 \right].$$

From these instances it appears that by increasing  $m$  we diminish the term of ( $y$ )'s survivorship, and so assimilate the event, whose probability is thus assigned, more and more nearly to that whose probability is sought. Carrying this process to its limit,  $1+2m$  vanishes, and we get finally, for the probability that ( $x$ ) will die in the  $n$ th year, and ( $y$ ) be alive at the instant of his death,

$$\frac{d}{AB} (b - \frac{1}{2} d_1).$$

And this becomes, on replacing the symbols by their equivalents,

$$\frac{(l_{x+n-1} - l_{x+n}) \{ l_{y+n-1} - \frac{1}{2} (l_{y+n-1} - l_{y+n}) \}}{l_{x,y}} \\ = \frac{(l_{x+n-1} - l_{x+n}) (l_{y+n-1} + l_{y+n})}{2l_{x,y}};$$

or,

$$= \frac{1}{2} (p_{x,n-1} - p_{x,n}) (p_{y,n-1} + p_{y,n}).$$

Multiplying these by  $v^n$  and summing with respect to  $n$ , we obtain the usual expressions for the present value of the assurance, which it is unnecessary to reproduce.

17. Do you consider the formula usually given for the foregoing benefit to be absolutely or only approximately true? If the latter, state the objection to it.

*Ans.* It has been already intimated that the expression here referred to, (which is the one just deduced,) is only approximately true. The number of survivors, at the expiry of the fraction  $t+m$  of the  $n$ th year ( $t$  being less than  $m$ ) of  $l_{x+n-1}$  who enter on that year, is truly denoted by

$$l_{x+n-1} + \frac{t}{m} \Delta l_{x+n-1} + \frac{t(t-m)}{1.2m^2} \Delta^2 l_{x+n-1} + \dots$$

the series being continued till the terms become insignificant.

In the foregoing investigation, in assuming that the deaths of the year are uniformly distributed over it, we in effect make use of the first two terms of the above development to denote the number living at the end of the time  $t \div m$ , neglecting all that follow. And hence the error in the resulting formula. It is not on account of any analytical difficulty in dealing with the complete development that we content ourselves with using the first two terms only: the reasons are, first, that the employment of any terms beyond the first two lands us in an expression far too complex to admit of being commodiously used; and, secondly, that the error committed by neglecting terms beyond the second is too small to be of any practical importance.

See *Assurance Magazine*, vol. i., pp. 187 to 151.

There still exists in the minds of a good many a lingering belief, notwithstanding the various demonstrations that have been given, that the expression under consideration is a comparatively rude approximation to the truth, and subject to at least one other source of error than that above referred to. We trust it will not be considered presumptuous in us to say, after the very explicit demonstration we have given, that there exists no reason for attributing to the expression deduced any other inaccuracy than that which arises from the assumption made as to the distribution of the deaths.

18. Find the annual premium for an assurance on the life of ( $x$ ), deferred  $n$  years, and contingent upon ( $y$ ) surviving that term. The premium to be payable for the term, provided that both ( $x$ ) and ( $y$ ) live so long.

*Ans.* Here the benefit term is

$$\frac{M_{x+n}}{D_x} \cdot \frac{l_{y+n}}{l_y} = \frac{M_{x+n}l_{y+n}}{D_{x,y}}, \text{ if } x > y;$$

and the payment term is

$$\frac{wN_{x-1,y-1|n}}{D_{x,y}}.$$

Therefore, equating,

$$wN_{x-1,y-1|n} = M_{x+n}l_{y+n},$$

whence,

$$w = \frac{M_{x+n}l_{y+n}}{N_{x-1,y-1|n}}.$$

If  $x < y$ , a not very likely case to occur in practice, we should find,

$$w = \frac{M_{x+n}l_{y+n}v^{y-x}}{N_{x-1,y-1|n}}.$$

19. Investigate a formula for the annual premium for an assurance of £1 payable in the event of a life aged  $x$  attaining the age  $x+n$ , with the provision that the premiums paid are to be returned (without interest) in the event of death happening before the attainment of the specified age. Allow for a loading of 10 per cent upon the premium.

*Ans.* The specialty in this question is as to the manner of applying the loading. This is a point which is not touched upon in any of the text-books, the reason for the omission no doubt being that in nearly all the cases that

arise it suffices to apply the loading to the net premium. This is however obviously not sufficient when, as in the present case, there is to be a return of premium, since every increase of the premium involves also an increase of the return. In such cases therefore a different mode of treatment becomes necessary.

For generality, let the sum assured be  $A$ , let  $p$  be the loading per £, and let the required premium be  $\omega$ .

The benefit consists of first, an endowment of  $A$ , receivable in  $n$  years; secondly, an assurance of  $A$  for  $n$  years; and thirdly, an increasing assurance of  $\omega$ ,  $2\omega$ ,  $3\omega$ , &c. also for  $n$  years. These give for the value of the benefit (omitting the denominator,  $D_x$ )

$$A(D_{x+n} + M_{x|n}) + \omega R_{x|n};$$

and the value of the premium (subject to the like omission) is,

$$\omega N_{x-1|n}.$$

If we were to form with these the usual equation, it is obvious that the resulting value of  $\omega$  would be the *net* premium. But, by condition, the present value of the premium is to be  $1+p$  times that of the benefit. Hence our equation is,

$$\omega N_{x-1|n} = (1+p)\{A(D_{x+n} + M_{x|n}) + \omega R_{x|n}\};$$

and solution gives

$$\omega = \frac{(1+p)A(D_{x+n} + M_{x|n})}{N_{x-1|n} - (1+p)R_{x|n}}.$$

This expression differs from that which we should obtain by applying the loading to the net premium, in having the *negative* term in the denominator multiplied by  $1+p$ . The value of  $\omega$  is obviously thereby increased.

20. A person aged  $x$  is assured for  $m$  pounds, at an annual premium  $\omega$ , payable until death. He desires to convert his assurance into one of the same amount, payable in  $n$  years or on previous death. Required the future premium  $\omega'$ , payable until the risk is determined.

#### Benefit Terms.

*Ans.* (x) receives, first, an assurance of  $m$ , payable at  $x+n$  or death. . . . .  $m(D_{x+n} + M_{x|n})$ ;  
and, secondly, remission of the premium  $\omega$ , now payable . . . . .  $\omega N_{x-1}$ .

#### Payment Terms.

He gives up his present assurance . . . . .  $mM_x$ ;  
and he is to pay a premium  $\omega'$ , for  $n$  years . . . . .  $\omega' N_{x-1|n}$ .

Hence, equating the sums of the Benefit and the Payment Terms,

$$\omega' N_{x-1|n} + mM_x = m(D_{x+n} + M_{x|n}) + \omega N_{x-1}$$

$$\omega' = \frac{m(D_{x+n} - M_{x|n}) + \omega N_{x-1}}{N_{x-1|n}}.$$

If the Office charge a commission on the transaction it will be sufficient in this case to apply the loading to the net premium  $\omega'$ , determined as above.

See *Assurance Magazine*, vol. i., p. 98.

25. Will a loading of 25 per cent upon a net premium provide for an additional mortality also of 25 per cent? Give your reasons.

*Ans.* It is not easy to assign analytically the relation that subsists between a variation in the annual rate of mortality and the corresponding variation in either the single or the annual premium for assurance; but it is easy enough to show that there is not between them the simple relation that seems implied in the question.

First. The subsisting relation, whatever it is, must involve the rate of interest. The increase in the value of the liability on subsisting policies, consequent on an increase in the mortality, arises not from there being more claims to pay, but from the claims provided for by the premiums arising sooner than was anticipated. It will vary therefore with the rate of interest used in the calculation, and hence the implied relation, which ignores the rate of interest, cannot be that which actually subsists.

Secondly. Put the matter to the test by trying an extreme case. The mortality at age 94 (Carlisle) is .25. If this be increased by 300 per cent, the value of an assurance on (94) will be  $A_{94}=v=.9709$ , at 3 per cent. Now subject to the normal mortality, and at 3 per cent interest, we have  $A_{94}=.8912$  and  $\omega_{94}=.2386$ , which become, when increased in the same ratio as the mortality, 3.5648 and .9544, respectively. Hence as by the implied relation these ought to be each equal to .9709, it inevitably follows that this relation is not that which actually subsists.

An addition of  $p$  per cent to a net premium will (leaving expenses, &c. out of view) exactly provide for a bonus of  $p$  per cent on the sum assured. But this is by no means the same thing as providing for an increase of  $p$  per cent in the annual mortality. In the one case the claims arise as anticipated, and their increased amounts will be duly met by the increased premiums; while in the other the claims arise sooner than was anticipated, and fewer payments of premiums are received.

#### ERRATA.

Mr. W. Sutton points out the following *errata* in Mr. Sprague's paper *On the Value of Apportionable Annuities*.

Vol. xiv., p. 88, line 4 from bottom;

$$\text{For } \frac{1}{4} + \frac{1}{2n} + \frac{1}{6n^2} - \frac{1}{30n^3} \text{ read } \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}.$$

Vol. xiv., p. 42, line 8,

$$\text{Add the term } + \frac{1}{12} \frac{D''_k}{D_k} \frac{\delta r^2}{2} \left( \frac{1}{m} - r \right)^2.$$

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*On some problems in the theory of Probabilities. By WILHELM LAZARUS, of Hamburg, Corresponding Member of the Institute of Actuaries. Read at the second meeting of the Berlin Life Assurance Institute, on 11th March, 1868.*

*Translated from the German by THOMAS B. SPRAGUE, M.A., and J. HILL WILLIAMS, Vice-Presidents of the Institute of Actuaries.*

THE Author commences by stating that he has in the course of his investigations found the theory of probabilities in a far from perfect state. In many cases, when he has wished to make use of the theory of probabilities, he has searched in vain for demonstrated formulæ capable of immediate application, and he has generally been obliged to begin by working out the formulæ independently for himself.

This appears strange when we remember that the names of the greatest mathematicians of all ages are intimately connected with the theory of probabilities; but may perhaps be accounted for in the following manner.

The theory of probabilities deals with questions in which such an immense number of cases are possible, that the formulæ derived directly from it turn out to be, except in the simplest cases, too complicated for calculation; and we have in practice to seek for

simpler formulæ which shall give results approximately true. The sufficiency of these approximations will however mostly depend on the values of the probabilities with which we have to do; and a formula which gives a close approximation for probabilities equal to  $\frac{1}{4}$ , and  $\frac{1}{8}$ ,—the values prevailing in questions arising out of games of chance, will prove perhaps quite incorrect for a probability equal to  $\frac{1}{36}$ , and altogether useless for a probability equal to  $\frac{1}{1000}$ ,—the values occurring in life contingency computations.

Now the great mathematicians above mentioned have only occasionally turned their attention to the application of the theory of probabilities to life insurance, having been principally occupied with another problem, viz., the adjustment of errors of observation.

He then proceeds:—

Let us denote by  $p$  the probability of the happening of an event (or we may say, for brevity, the probability of E); and by  $q(=1-p)$  the probability of its not happening (or briefly, of F). Let us assume that  $\mu$  trials take place, and that the most probable combination is that the event will happen  $m$  times, and fail  $n$  times ( $m+n=\mu$ ), the numbers being so chosen that there is only one most probable combination. Lastly, let us understand that when  $p$  and  $q$  are not equal,  $p$  is the smaller.

The problems I propose to consider are the following:

(1) What is the value of the probability,  $\Omega_0$ , of the occurrence of the most probable combination,  $m$  E's and  $n$  F's?

(2) What is the value of the probability,  $\Omega_1$ , that E will happen oftener than in the most probable combination, at least  $(m+1)$  times, and at most  $(m+z)$  times, F therefore occurring between  $(n-1)$  and  $(n-z)$  times?

(3) What is the value of the probability,  $\Omega_2$ , that E will happen less often than in the most probable combination, at most  $(m-1)$  times, and at least  $(m-u)$  times, F occurring between  $(n+1)$  and  $(n+u)$  times?

Before proceeding further, we will take an example. Let the event observed be the throwing of a given number, say ace, with an ordinary six-sided die. Here  $p=\frac{1}{6}$  and  $q=1-p=\frac{5}{6}$ . Let there be 12,000 throws, or  $\mu=12,000$ . To determine the values of  $m$  and  $n$  for the most probable combination we have the following inequalities:

$$m < p(\mu + 1) \qquad m > p(\mu + 1) - 1 \quad . \quad . \quad . \quad (1)$$

whence we find  $m=2,000$ , and consequently  $n=10,000$ .

We now ask :

(1) What is the probability that in 12,000 throws with a die, there will be thrown exactly 2,000 aces?

(2) What is the probability that in 12,000 throws, there will be thrown more than 2,000 aces—say at least 2,001, and at most 2,030—putting  $x=30$ ?

(3) What is the probability that in 12,000 throws, there will be thrown less than 2,000 aces—say at most 1,999, and at least 1,960—putting  $u=40$ ?

The solution of (1) is well known to be

$$\Omega_0 = \frac{1.2 \dots \mu}{1.2 \dots m.1.2 \dots n} p^m (1-p)^n \dots \dots (2)$$

and provided that  $m, n, \mu$  are sufficiently large numbers, Stirling's formula of approximation

$$1.2 \dots v = e^{-v} v^{v+\frac{1}{2}} \sqrt{2\pi}$$

gives approximately:

$$\Omega_0 = \sqrt{\frac{\mu}{2mn\pi}} \dots \dots \dots (3)$$

The numerical calculation in this example is very simple. Putting  $\mu=12,000, m=2,000, n=10,000$ , we have

$$\Omega_0 = \sqrt{\frac{3}{10000\pi}} = \frac{1}{100} \sqrt{\frac{3}{\pi}} = .0098.$$

It certainly would appear surprising, to a man not acquainted with mathematics, that the probability of throwing exactly 2,000 aces in 12,000 throws is nearly .01; indeed, I believe that, considering there are 12,001 possible cases, viz.: that there might either be thrown no ace at all or any number of aces, 1, 2, 3 up to 12,000, he would estimate this probability at very much less. How much more then will he be surprised to find, as the result of our investigation will show, that the probability of throwing at least 1,970 and at most 2,030 aces, is more than  $\frac{1}{2}$ , so that it is more probable that one of these 61 events will happen, than that one of the remaining 11,940 events will happen.

Notwithstanding careful researches through all the books that treat of the Theory of Probabilities, I have not found that anyone has directly attempted the solution of questions (2) and (3); and yet it certainly is a matter of great importance to the Theory of Risk to know how to ascertain the separate probabilities,  $\Omega_1$ , and  $\Omega_2$ , of deviation in each direction from the most probable event. Instead of this, the same value has been assigned to the

deviations in both directions, and a formula then found for the sum of  $\Omega_0$ ,  $\Omega_1$  and  $\Omega_2$ , from which conclusions have been drawn as to the values of  $\Omega_1$  and  $\Omega_2$ .

Laplace proceeds in the following manner:—Supposing the deviation from the most probable event to be  $v$  in each direction, then we have to find the probability—say  $\Omega$ ,—that in  $\mu$  trials  $E$  will not happen oftener than  $(m+v)$  nor less often than  $(m-v)$  times.

The probability that there will happen  $(m-v)$   $E$ 's and  $(n+v)$   $F$ 's, where  $v$  is arbitrary, is well known to be

$$\frac{1.2 \dots \mu}{1.2 \dots (m-v) 1.2 \dots (n+v)} p^{m-v} (1-p)^{n+v}$$

and supposing that the numbers  $\mu$ ,  $m-v$ ,  $n+v$ , are sufficiently large to warrant the employment of Stirling's formula for approximation, we find this value equal to

$$\sqrt{\frac{\mu}{2mn}} \cdot \frac{e^{-\frac{\mu v^2}{2mn}}}{\sqrt{\pi}} \left\{ 1 - \frac{(m-n)v}{2mn} + \frac{(m^2-n^2)v^3}{6m^2n^2} + \dots \right\}.$$

If for  $v$  we now substitute  $-v$ , and expand in the same manner, we find the probability that there will be  $(m+v)$   $E$ 's and  $(n-v)$   $F$ 's, equal to

$$\sqrt{\frac{\mu}{2mn}} \cdot \frac{e^{-\frac{\mu v^2}{2mn}}}{\sqrt{\pi}} \left\{ 1 + \frac{(m-n)v}{2mn} - \frac{(m^2-n^2)v^3}{6m^2n^2} - \dots \right\}.$$

Adding these two probabilities, putting for brevity

$$\sqrt{\frac{\mu}{2mn}} = h \quad \dots \quad (4)$$

and calling the sum,  $y$ , we find that

$$y = \frac{2h}{\sqrt{\pi}} e^{-h^2 v^2} \quad \dots \quad (5)$$

This is therefore the probability in favour of one or other of two combinations occurring, which are at equal distances on the two sides from the most probable event. Giving to  $v$  the values 1, 2, 3, ...,  $v$  successively, expanding  $y$ , and adding the several values of  $y$ , and finally adding the probability of the most probable combination, we shall have the required probability,  $\Omega$ . Now observing that if we make  $v=0$ ,  $y$  becomes equal to twice the probability of the most probable event, or to  $2\Omega_0$ , we readily find

$$\Omega = \sum_0^v y - \Omega_0 \quad \dots \quad (6)$$

Now applying to  $\Sigma y$  Euler's formula of summation, and bearing in mind that under the previous supposition of large numbers, the differential coefficients arising may be neglected, we shall find

$$\sum_0^v y = c + \frac{2h}{\sqrt{\pi}} \int_0^v e^{-h^2 v^2} dv + \frac{h}{\sqrt{\pi}} e^{-h^2 v^2} \dots \quad (7)$$

where  $c$  is the constant of integration. For the determination of this we know that  $y = 2\Omega_0$ , when  $v = 0$ , consequently  $c = \Omega_0$ . Thus finally we have

$$\Omega = \frac{2h}{\sqrt{\pi}} \int_0^v e^{-h^2 v^2} dv + \frac{h}{\sqrt{\pi}} e^{-h^2 v^2} \dots \quad (8)$$

I must confess that, apart from the fact that this solution by no means exhausts our problem, I have never been quite satisfied with it. Starting with the hypothesis of numbers sufficiently large to admit of the application of Stirling's formula, it leaves us completely at fault when we have to do with smaller numbers. Above all, I miss the consciousness of certainty in the results, when, after having begun with Stirling's formula of approximation, I have by and by to content myself with a further approximation in applying Euler's formula of summation. Lastly, the final result has a tendency to mislead, inasmuch as that term of the equation which is affected with the sign of integration may give rise to the erroneous assumption that the probabilities on the two sides of the most probable event are equal. This is certainly not the case, although most recent writers have fallen into this mistake.

Although I have above substantially followed the analysis of Laplace, I must not forget to mention that this problem has been considered last year (1867) by Professor Wittstein in his work ("Die Mathematische Statistik") and treated by him with the same general conception of the various probabilities which I have considered separately. This treatise is extremely valuable and suggestive, and its principal aim being the application of the theory of probabilities to the question of mortality, I would strongly recommend it to those who have not yet seen it. But Wittstein's demonstrations of formulæ are deficient in clearness and exactitude. On this point I refer you to the book itself, not doubting that you will agree with me; and I will only observe that Wittstein neglects that term in our formula (8) which is free from the sign of integration, and arrives at the result,  $\Omega = \frac{2h}{\sqrt{\pi}} \int_0^v e^{-h^2 v^2} dv$ , which is quite inaccurate for small values of  $v$ .

I must now ask you to follow my method of demonstration, which is certainly somewhat more laborious. Both for  $\Omega_1$  and  $\Omega_2$  we have to sum a series of terms of an expanded binomial.

It is well known that

$$\begin{aligned}\Omega_1 = & \frac{1.2 \dots \mu}{1.2 \dots m+z \cdot 1.2 \dots n-z} p^{m+z} q^{n-z} \\ & + \frac{1.2 \dots \mu}{1.2 \dots m+z-1 \cdot 1.2 \dots n-z+1} p^{m+z-1} q^{n-z+1} + \dots \\ & + \frac{1.2 \dots \mu}{1.2 \dots m+1 \cdot 1.2 \dots n-1} p^{m+1} q^{n-1} \dots \dots (9)\end{aligned}$$

$$\begin{aligned}\Omega_2 = & \frac{1.2 \dots \mu}{1.2 \dots m-1 \cdot 1.2 \dots n+1} p^{m-1} q^{n+1} \\ & + \frac{1.2 \dots \mu}{1.2 \dots m-2 \cdot 1.2 \dots n+2} p^{m-2} q^{n+2} + \dots \\ & + \frac{1.2 \dots \mu}{1.2 \dots m-u \cdot 1.2 \dots n+u} p^{m-u} q^{n+u} \dots \dots (10)\end{aligned}$$

These sums we will denote by the sign  $\Sigma[p^\mu]$  putting the highest power of  $p$  at the bottom, and the lowest power at the top, so that we have

$$\Omega_1 = \sum_{m+z}^{m+1} [p^\mu] \quad \Omega_2 = \sum_{m-1}^{m-u} [p^\mu] \dots \dots (11)$$

$$\text{Hence,} \quad \Omega_1 = \sum_{\mu}^{m+1} [p^\mu] - \sum_{\mu}^{m+z+1} [p^\mu] \dots \dots (12)$$

$$\Omega_2 = \sum_{\mu}^{m-u} [p^\mu] - \sum_{\mu}^m [p^\mu] \dots \dots (13)$$

in which all the  $\Sigma$ 's have the same lower limit  $\mu$ , and only differ in the upper limit.

In the next place we have to find a general expression for  $\sum_{\mu}^{\rho} [p^\mu]$ , i.e., in other words, to find the probability that, in  $\mu$  trials,  $E$  will happen at least  $\rho$  times, or what is the same thing, that  $F$  will happen at most  $\mu - \rho = \tau$  times.

The author then demonstrates by means of a process used by Poisson in connection with a different subject, that

$$\sum_{\mu}^{\rho} [p^\mu] = \frac{\int_{\frac{1-p}{p}}^{\infty} \frac{x^{\mu-\rho}}{(1+x)^{\mu+1}} \cdot dx}{\int_0^{\infty} \frac{x^{\mu-\rho}}{(1+x)^{\mu+1}} \cdot dx} \dots \dots (20)$$

He proceeds :

This result, you observe, has been arrived at without the assistance of any *approximations*, and is quite general, whether  $\rho$ ,  $\tau$  and  $\mu$  are great or small numbers.

I recently communicated these operations to my friend Dr. Laudi, of Trieste, and I am indebted to him for a second method of investigation shorter than the foregoing, but not less strict. If we make

$$\begin{aligned} \sum_{\mu}^{\rho}[p^{\mu}] = y = & p^{\mu} + \mu p^{\mu-1}(1-p) + \frac{\mu \cdot \mu - 1}{1 \cdot 2} p^{\mu-2}(1-p)^2 + \dots \\ & + \frac{\mu \cdot \mu - 1 \dots \mu - \tau + 1}{1 \cdot 2 \dots \tau} p^{\rho}(1-p)^{\tau} \dots \quad (21) \end{aligned}$$

and differentiate with respect to  $p$ , we shall find, since all the terms but one cancel each other,

$$dy = \frac{\mu \cdot \mu - 1 \dots \mu - \tau + 1}{1 \cdot 2 \dots \tau} \rho p^{\rho-1}(1-p)^{\tau} dp \quad (22)$$

But since  $y=0$ , if  $p$  is put equal to 0 in (21), it follows by integration that

$$y = \frac{\mu \cdot \mu - 1 \dots \mu - \tau + 1}{1 \cdot 2 \dots \tau} \rho \int_0^p p^{\rho-1}(1-p)^{\tau} dp$$

or, since the  $p$  under the integral sign has only the force of an independent variable

$$y = \frac{\mu \cdot \mu - 1 \dots \rho + 1}{1 \cdot 2 \dots \tau} \rho \int_0^p x^{\rho-1}(1-x)^{\tau} dx \quad (23)$$

and since  $y=1$ , when  $p$  is put  $=1$  in (21), it follows that

$$1 = \frac{\mu \cdot \mu - 1 \dots \rho + 1}{1 \cdot 2 \dots \tau} \rho \int_0^1 x^{\rho-1}(1+x)^{\tau} dx$$

and

$$\frac{1 \cdot 2 \dots \tau}{\mu \cdot \mu - 1 \dots \rho + 1 \cdot \rho} = \int_0^1 x^{\rho-1}(1-x)^{\tau} dx \quad (24)^*$$

Thus we finally find

$$\sum_{\mu}^{\rho}[p^{\mu}] = \frac{\int_0^p x^{\rho-1}(1-x)^{\tau} dx}{\int_0^1 x^{\rho-1}(1-x)^{\tau} dx} \quad (25)$$

That this result is the same as we obtained in (20), differing from it only in form, will be readily seen if in (25) we make

\* The truth of the equation (24) may be also readily proved by means of successive integration by parts.

$x = \frac{1}{1+\phi}$ , whence  $1-x = \frac{\phi}{1+\phi}$ , and  $dx = \frac{-d\phi}{(1+\phi)^2}$ . Also, since  $\phi = \frac{1-x}{x}$ , we shall find for the limits, if  $x=0$ ,  $\phi=\infty$ ; if  $x=1$ ,  $\phi=0$ ; if  $x=p$ ,  $\phi = \frac{1-p}{p}$ . Making these substitutions, (25) is transformed into (20).

Applying now our results to the equations (12) and (13), we find by means of equation (20)

$$\Omega_1 = \frac{\int_{\frac{1-p}{p}}^{\infty} \frac{x^{n-1} dx}{(1+x)^{\mu+1}} - \int_{\frac{1-p}{p}}^{\infty} \frac{x^{n-s-1} dx}{(1+x)^{\mu+1}}}{\int_0^{\infty} \frac{x^{n-1} dx}{(1+x)^{\mu+1}} - \int_0^{\infty} \frac{x^{n-s-1} dx}{(1+x)^{\mu+1}}} \dots \dots (26)$$

$$\Omega_2 = \frac{\int_{\frac{1-p}{p}}^{\infty} \frac{x^{n+u} dx}{(1+x)^{\mu+1}} - \int_{\frac{1-p}{p}}^{\infty} \frac{x^n dx}{(1+x)^{\mu+1}}}{\int_0^{\infty} \frac{x^{n+u} dx}{(1+x)^{\mu+1}} - \int_0^{\infty} \frac{x^n dx}{(1+x)^{\mu+1}}} \dots \dots (27)$$

and by means of equation (25)

$$\Omega_1 = \frac{\int_0^p \frac{x^m (1-x)^{n-1} dx}{\int_0^1 \frac{x^m (1-x)^{n-1} dx}} - \frac{\int_0^p \frac{x^{m+s} (1-x)^{n-s-1} dx}{\int_0^1 \frac{x^{m+s} (1-x)^{n-s-1} dx}}}{\int_0^1 \frac{x^m (1-x)^{n-1} dx}} \dots \dots (28)$$

$$\Omega_2 = \frac{\int_0^p \frac{x^{m-u-1} (1-x)^{n+u} dx}{\int_0^1 \frac{x^{m-u-1} (1-x)^{n+u} dx}} - \frac{\int_0^p \frac{x^{m-1} (1-x)^n dx}{\int_0^1 \frac{x^{m-1} (1-x)^n dx}}}{\int_0^1 \frac{x^{m-u-1} (1-x)^{n+u} dx}} \dots \dots (29)$$

Thus we have expressed the probabilities required by definite integrals; and that, without the help of any formulæ of approximation, by means of a strict and perfectly general process. Our expressions possess in consequence an absolute and general value, whether the number,  $\mu$ , of the trials to be made be large or small, whatever be the relation of  $m$  and  $n$ , or  $p$  and  $q$ , to one another, and whatever whole number we may choose for  $z$  between the limits 1 and  $n$ , and for  $u$  between the limits 1 and  $m$ . While in the old manner of proceeding it must be presupposed that  $z$  and  $u$  respectively are small relatively to  $m$  and  $n$ , our formulæ allow of the calculation of the probability in all the possible cases on both sides of the most probable event.

In applying our formulæ for the numerical determination of

probabilities, we have still to contend with the difficulties which so often occur in numerical integrations, and we shall scarcely be able here to avoid the employment of formulæ of approximation. We are now touching upon another branch of our subject; having left the theory; and entered upon the practice. According to the special circumstances of the case and of the numbers involved in it, we will now choose that mode of proceeding which appears to us most suitable; but we always have the power of carrying our approximations to any degree of accuracy required.

It would occupy too much space to enter upon the different cases of numerical integration in connexion with our formulæ; and I will merely observe that in my opinion much remains to be done in this department, and that the formulæ of approximation with which I am acquainted appear to require both to be simplified and made more precise. I may, however, show how far our formulæ and their results agree with that which I have called the old method of solution (8), and how far they differ from it.

I select for this purpose the formulæ (28) and (29), specially because in these the form of the function under the sign of integration is the more interesting to us, inasmuch as it is the very same function we meet with in deducing the probability of hypotheses from observed events; but the treatment of the formulæ (26) and (27) differs very little from the mode to be adopted here. I will follow the ingenious guidance of Laplace and Poisson.

Required the values of  $\int x^a(1-x)^\beta dx$ , between the limits 0 and  $p$ , and 0 and 1.

The function  $x^a(1-x)^\beta$ , under the integral sign, vanishes, both when  $x$  is made  $=0$ , and when it is made  $=1$ . It is a maximum, in accordance with the well-known rule, when

$$x = \frac{a}{a+\beta} = M; \text{ and in that case is } = \frac{a^\alpha \beta^\beta}{(a+\beta)^{a+\beta}} = N.$$

By introducing a new independent variable  $t$ , we may write

$$x^a(1-x)^\beta = Ne^{-t^2} \dots \dots \dots (30)$$

$$x^a(1-x)^\beta dx = Ne^{-t^2} \frac{dx}{dt} dt \dots \dots \dots (31)$$

and consequently express  $\frac{dx}{dt}$  as a function of  $t$ . If we write for brevity  $\log \{x^a(1-x)^\beta\} = f(x)$ , it follows that  $\log N = f(M)$ ; and if we likewise write  $x = M + y$ , whence  $dx = dy$ , it follows from (30) that

$$t^2 + f(M+y) - f(M) = 0 \dots \dots \dots (32)$$

Expanding  $f(M+y)$  by Taylor's Theorem, and observing that  $f(M)$  is a maximum, and consequently  $f'(M)=0$ , we get the equation

$$t^2 + \frac{y^2}{1.2} f''(M) + \frac{y^3}{1.2.3} f'''(M) + \dots = 0 \quad (33)$$

Here  $f''(M)$ ,  $f'''(M)$  &c.; are easily determined, and are of course independent of  $y$ . We find, for example,  $f''(M) = \frac{-(a+\beta)^3}{a\beta}$ , and

$$f'''(M) = \frac{-2(a+\beta)^2(a-\beta)}{a^2\beta^2}, \text{ and so on.}$$

Making  $y = At + Bt^2 + Ct^3 + \dots \quad (34)$

we have  $\frac{dy}{dt} = \frac{dx}{dt} = A + 2Bt + 3Ct^2 + \dots \quad (35)$

where we may determine  $A$ ,  $B$ ,  $C$ , &c., by substituting in (33) the values of  $y^2$ ,  $y^3$  . . . derived from (34), and then putting the coefficients of the several powers of  $t$  separately  $= 0$ .

We thus find  $A = \sqrt{\frac{2a\beta}{(a+\beta)^3}}$ ,  $B = \frac{2}{3} \frac{\beta-a}{(a+\beta)^2}$ , &c.; and the convergency of the coefficients  $A$ ,  $B$ ,  $C$  . . . will be more rapid as  $a$  and  $\beta$  are greater. For the proof of this I would refer you to Laplace, merely remarking that, if we may neglect the terms of the order  $\frac{1}{a}$  and  $\frac{1}{\beta}$  (and this was the supposition of the old method in applying Stirling's formula) then the coefficients  $C$ ,  $D$ , &c., may also be neglected. We will here make the same supposition, and we thus get

$$x^a(1-x)^\beta dx = N e^{-t^2} (A + 2Bt) dt \quad (36)$$

where  $A$  and  $B$  have the values found above.

If we now integrate with respect to  $x$  from 0 to 1, we must evidently take  $t$  from  $-\infty$  to  $+\infty$ .

$$\int_0^1 x^a(1-x)^\beta dx = N \int_{-\infty}^{+\infty} e^{-t^2} (A + 2Bt) dt \quad (37)$$

and since  $\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}$ ; and  $\int_{-\infty}^{+\infty} e^{-t^2} t dt = 0$ , it follows that

$$\int_0^1 x^a(1-x)^\beta dx = NA\sqrt{\pi} \quad (38)$$

If, however, we integrate from  $x=0$  to  $x=p$ , we must

distinguish whether  $p$  is  $>$  or  $<$   $M$ . If we have for any one value of  $p$

$$x^a(1-x)^\beta = p^a(1-p)^\beta = N e^{-\theta^2} = \frac{a^\alpha \beta^\beta}{(a+\beta)^{a+\beta}} e^{-\theta^2} \quad (39)$$

it follows that

$$e^{-\theta^2} = \frac{(a+\beta)^a p^a}{a^a} \cdot \frac{(a+\beta)^\beta (1-p)^\beta}{\beta^\beta} \quad (40)$$

and putting for brevity

$$k^2 = a \log \frac{a}{p(a+\beta)} + \beta \log \frac{\beta}{(1-p)(a+\beta)} \quad (41)$$

it follows that  $\theta = \pm k$ , i.e.

$$\left. \begin{array}{l} \theta = -k \text{ when } M > p \\ \theta = +k \text{ „ } M < p \\ \theta = 0 \text{ „ } M = p \end{array} \right\} \quad (42)$$

To  $x=0$  corresponds  $t=-\infty$ , consequently when  $M > p$

$$\int_0^p x^a(1-x)^\beta dx = N \int_{-\infty}^{-k} e^{-t^2} (A + 2Bt) dt \quad (43)$$

and after a similar reduction

$$\int_0^p x^a(1-x)^\beta dx = \frac{NA\sqrt{\pi}}{2} - NA \int_0^k e^{-t^2} dt - NB e^{-k^2} \quad (44)$$

On the other hand, for the case where  $M < p$

$$\int_0^p x^a(1-x)^\beta dx = N \int_{-\infty}^{+k} e^{-t^2} (A + 2Bt) dt \quad (45)$$

and after similar reduction

$$\int_0^p x^a(1-x)^\beta dx = \frac{NA\sqrt{\pi}}{2} + NA \int_0^k e^{-t^2} dt - NB e^{-k^2} \quad (46)$$

Thus we find at last

$$\frac{\int_0^p x^a(1-x)^\beta dx}{\int_0^1 x^a(1-x)^\beta dx} = \frac{1}{2} \mp \frac{1}{\sqrt{\pi}} \int_0^k e^{-t^2} dt - \frac{B}{A\sqrt{\pi}} e^{-k^2} \quad (47)$$

where we use the upper or *minus* sign of the second term, when

$\frac{a}{a+\beta} > p$ , and the lower or *plus* sign, when  $\frac{a}{a+\beta} < p$ . For determining the limit  $k$ , if we write  $p(a+\beta) = a + \varepsilon$ , consequently  $(1-p)(a+\beta) = \beta - \varepsilon$ , then

$$k^2 = - \left\{ a \log \frac{p(a+\beta)}{a} + \beta \log \frac{(1-p)(a+\beta)}{\beta} \right\} \\ = - \left\{ a \log \left( 1 + \frac{\varepsilon}{a} \right) + \beta \log \left( 1 - \frac{\varepsilon}{\beta} \right) \right\}.$$

Expanding and reducing, we find

$$k^2 = \varepsilon^2 \frac{a+\beta}{2a\beta} + \varepsilon^4 \frac{a^2-\beta^2}{3a^2\beta^2} + \varepsilon^6 \frac{a^3+\beta^3}{4a^3\beta^3} + \dots$$

$$k = \varepsilon \sqrt{\frac{a+\beta}{2a\beta}} \sqrt{\left\{ 1 + \varepsilon \frac{2(a-\beta)}{3a\beta} + \varepsilon^2 \frac{a^2+\beta^2-a\beta}{2a^2\beta^2} + \dots \right\}}$$

in which we must take  $k$  as always positive,  $\varepsilon$  being evidently positive when  $p > \frac{a}{a+\beta}$ , and negative when  $p < \frac{a}{a+\beta}$ .

By help of (47) we get from (28)

$$\Omega_1 = \frac{1}{\sqrt{\pi}} \int_0^{k_2} e^{-t^2} dt \mp \frac{1}{\sqrt{\pi}} \int_0^{k_1} e^{-t^2} dt + \frac{B_2}{A_2\sqrt{\pi}} e^{-k_2^2} - \frac{B_1}{A_1\sqrt{\pi}} e^{-k_1^2} \quad (48)$$

where in the second term the sign  $-$  is to be taken when  $\frac{m}{\mu-1} > p$ , and the sign  $+$  when  $\frac{m}{\mu-1} < p$ , and where  $k_1, A_1, B_1$  correspond to the relative values of the first part, and  $k_2, A_2, B_2$  correspond to the analogous values of the second part of equation (28).

From (29) we get

$$\Omega_2 = \frac{1}{\sqrt{\pi}} \int_0^{k_3} e^{-t^2} dt \mp \frac{1}{\sqrt{\pi}} \int_0^{k_4} e^{-t^2} dt + \frac{B_4}{A_3\sqrt{\pi}} e^{-k_3^2} - \frac{B_3}{A_4\sqrt{\pi}} e^{-k_4^2} \quad (49)$$

where in the second term the sign  $+$  is to be taken, when  $\frac{m-1}{\mu-1} > p$ , and the sign  $-$ , when  $\frac{m-1}{\mu-1} < p$ ; and where  $k_3, A_3, B_3$  are deduced from the first part, and  $k_4, A_4, B_4$  from the second part of equation (29).

Making  $z=u$ , expanding  $\Omega_1$  and  $\Omega_2$  under this supposition, and adding  $\Omega_0$ , which may be done in the simplest manner as we expand equation (28) or (29) while calculating the most probable case—making also  $\Omega_0 + \Omega_1 + \Omega_2 = \Omega$ , we find in place of equation (8),

$$\Omega = \frac{1}{\sqrt{\pi}} \int_0^{k_2} e^{-t^2} dt + \frac{1}{\sqrt{\pi}} \int_0^{k_3} e^{-t^2} dt + \frac{B_2}{A_2\sqrt{\pi}} e^{-k_2^2} - \frac{B_3}{A_3\sqrt{\pi}} e^{-k_3^2} \quad (50)$$

This equation gives, on the supposition of large numbers, nearly the same numerical result as (8) notwithstanding its very different

form. The form now exhibited, however, shows more certainly the composition of  $\Omega$ , which (8) leaves out.

In conclusion, I subjoin a few numerical results, obtained by means of the formulæ under consideration.

I find the probability that in 12,000 throws with a die, there will be thrown

at least 2,001 aces and at most 2,030 aces, equal to	0.267
„ 1,970 „ „ 1,999 „ „	0.269
exactly 2,000 „ „	0.010

∴ at least 1,970 aces and at most 2,030 aces, equal to	0.546
at most 1,999 aces equal to	0.4962
at least 2,000 „ „	0.4941
exactly 2,000 „ „	0.0098

The probability that the most probable number of aces, 2,000, will be fallen short of at most 29, is nearly as great as the probability that it will be exceeded by at most 28.

We thus see, that the differences of probability, if we keep before our eyes the deviations from the most probable case either on one side or on the other, are not so insignificant as to be safely neglected, even when we deal with large numbers.

*Address to the Actuarial Society of Edinburgh.* By EDWARD SANG, F.R.S.E., *President.*

[Read 5th November 1868.]

[The length of this address precludes our reproducing it in its entirety; but we have so high an opinion of its value and suggestiveness that we cannot refrain from giving the following rather copious extracts. We regret that pressure of other matter has prevented their earlier insertion.—ED. J. I. A.]

IN estimating the value of an engagement depending upon the life of a nominee, we proceed on the principle that, although the issue of a single transaction be uncertain, that of the aggregate of many transactions may be relied on. Hence, in estimating the value of a life-contingency, we imagine a multitude of cases, as many, to wit, as our life table gives, and consider in how many of those cases the specified contingency will occur; we then compute the present value of the expected payments, and divide it among the number of assumed cases. Thus the only algebraic process essentially involved is that for finding the present value of a known prospective payment. All the formidable-looking formulæ which

you see in our books on life calculations notwithstanding,—this is the sum and substance of the matter.

The intricacy of the calculations arises, not from the profundity of the principles involved, but from the multitude of steps of which the operations in general consist.

It is only when the contingencies relate to several lives, and when we have to take into account the deaths within the year, that any beyond the most elementary operations of algebra are called for.

Seeing that, in the valuation of an annuity, we have to make a great many separate discounts and to collect the results, one very naturally asks whether the sum could not be obtained by some concise process, such as discounting the whole payments as if due at the average date. We know that the sums of the terms of many progressions can be got without computing the separate items, and that it is sometimes easier to obtain the sum of the whole, than to compute even one of the terms. It would be a great boon if some such method could be contrived for life-contingency calculations. In cases involving three, four, or five lives the calculations are so enormously laborious that the actuary hardly ever attempts to go through them; he is obliged to content himself with a rough kind of guess\* by substituting for two of the lives a single life, on which the annuity is equivalent to the two-life annuity. The discovery of a concise process would enable us to overcome these at present insuperable difficulties.

Now, the only way in which such a process can arise is from the discovery of an algebraic or analytical law among the numbers entered in our life-table. Then, and then only, will it be possible to sum the progression at once. Hence the next step for which we are to look in the improvement of actuarial calculations is the invention of a formula for the number alive at a given age; this invention would complicate the principles, and by that complication facilitate the process.

In order to speculate a little upon this prospective discovery, let us represent the law of life geometrically by the familiar method of co-ordinates. Taking the abscissæ for the ages, we draw perpendiculars proportional to the numbers alive at the several ages, and trace a line through the ends of those perpendiculars. The line so obtained is not straight, it is not uniformly curved. There are multitudes of algebraic, analytical, geometrical, and mechanical curves: Is there among these any one which will passably well agree with our life-line?

\* This appears to us to be much too strong a phrase.—ED. J. I. A.

The answer to this question is not to be sought for in the actuary's office; the mere student of Life Assurance would not be able to comprehend it were it found. In order barely to qualify yourselves for your profession you have been forced to study the doctrine of logarithms, a doctrine so far foreign to the subject that you could have done without it. If the life-law were discovered you would, in the same way, be compelled to extend your studies to new and higher departments of the science of quantity.

The problem "To represent the law of human life by an analytic function" presents itself to the biologist under an aspect much more comprehensive than that under which the actuary would regard it. To the latter the problem is restricted, because no formula can be of use to him unless it be susceptible of integration; he therefore confines his search to integrable functions, and so has, first of all, to inquire as to what classes of functions admit of being summed after having been multiplied by the powers of the rate of improvement of money. Without going at all deeply into the matter, I may indicate the general nature and the results of this inquiry.

If we represent the number alive at the age  $n$  by some symbol such as  $\phi n$ , and multiply that number by the power of  $v$ , the inverse of the rate of improvement of money, we obtain  $\phi n.v^n$  as the general representative of those quantities which we have to sum; and we have to inquire what must be the nature of this unknown function  $\phi n$ , in order that the composite expression may be integrable.

The expression to be integrated is the product of two functions, one known, the other unknown. Now the integral or primitive of a product is a series composed of the products of the derivatives of the one by the primitives of the other factor. The primitives and derivatives of  $v^n$  are known; they form an interminate geometric progression proceeding both ways; wherefore, we shall be able to integrate the product  $\phi n.v^n$  in two cases—the one, when the derivatives of  $\phi n$  terminate,—the other, when these derivatives recur. In the present state of the science of logistics no life-function which does not belong to one or other of these two classes would be of the least use to us. It may, then, be interesting to study their nature.

The only functions which have a finite number of derivatives are algebraic formulæ, into which the positive integer powers of the primary alone enter; such as

$$P + Qn + Rn^2 + Sn^3 \dots$$

the highest power determining the order of the last derivative.

If a formula of this kind could be found approximating even passably well to the life-table, the computation of annuities, assurances, joint-annuities, and reversions would be effected at once by the operations of ordinary algebra. In order to explain to you the mode of conducting such a search, let me remind you that we can obtain an algebraic curve passing through any number of given points. If, then, we assume several points in the life line and draw an algebraic curve through them, we shall have a line coinciding, at least in those points, with the life line. The difficulty is, so to select the points of agreement as that the two lines elsewhere shall not separate widely from each other. For this many trials are needed. In general, the order of the equation is given by the number of the assumed points or intersections, but it may happen that the co-efficients of the higher terms may be zero, and that so the number of intersections may be more than the order of the equation.

As a matter of curiosity to those who are disposed to study this branch of the subject, I may exhibit the following formula, which, though containing only six terms, yet gives a line crossing that of the Carlisle Bills at nine places. The abscissa of it is reckoned from the supposed end of life, viz., 96 years, backwards, an arrangement which is convenient for the summations.

. Putting  $u = 96 - \text{age}$ , the formula is—

$$\begin{aligned} & u^3 \times \{ +.294 \ 93 \} \\ & + u^3 \times \{ +.483 \ 016 \} \\ & + u^4 \times \{ -.022 \ 718 \} \\ & + u^5 \times \{ +.000 \ 430 \ 174 \} \\ & + u^6 \times \{ -.000 \ 003 \ 755 \ 57 \} \\ & + u^7 \times \{ +.000 \ 000 \ 012 \ 515 \}. \end{aligned}$$

The results of this formula differ from the table very seriously at the beginning, and they agree passably well from age 8 to age 80. On trial you will find it exceedingly difficult to obtain anything like the characters of the two ends of the curve.

The lines obtained from algebraic functions can hardly give approximations sufficiently close for business purposes without using a great many terms; and then the formulæ for two and three life cases become excessively long. We may, therefore, turn to those functions which have recurring derivatives. The only functions of this class which occur in ordinary practice, and with which you are likely to be familiar, are the exponential function  $e^x$  and the trigonometrical one  $\sin x$ ; to these we may add the catenarian functions  $\frac{1}{2}(e^x + e^{-x})$  and  $\frac{1}{2}(e^x - e^{-x})$ .

The first of these,  $e^x$ , is its own derivative, and the curve resulting from it is the logarithmic curve which bears a general resemblance to the life line, only that it is asymptotical to its line of abscissæ. Wherefore an expression of the general form

$$\phi n = Pe^{-kn} - Q$$

may be made, in a very rough way, to represent the law of life. On comparing this line with any of the life lines, you will observe that the protuberance toward the middle ages is wanting.

\* \* \* \* \*

Here we are, thus far advanced in the history and practice of Life Assurance, without having a single unobjectionable series of observations on the law of mortality.

The simplest kind of observation, that which naturally presents itself first to the mind, is to note the births of a great many children, and to watch the deaths as they occur: the numbers alive at each successive year form, without further trouble, our life table. This process, however, is altogether unworkable, none of those who begin the observations can live to see them ended; and the result, even if it were obtained, would involve all the influences of advancing civilization; it could not show the expectation of life due to the present habits and customs of society.

A second process is, on a fixed day, to make a nominal roll of a great multitude of individuals of all ages, a roll, say, of the inhabitants of a large city, and to ascertain who of them are alive on the same day of the next year. In this way we obtain the probability of living from each year of age to the next, and are able, by the combination of these probabilities, to fill up our required table. In such a result there would be accumulated all the peculiarities of that year, its exceptional healthiness or unhealthiness, its specific influences on old people or on children; and, therefore, it is necessary to repeat the observations for many years. It is also expedient to extend them to various localities, for the obvious reason that the peculiarities of the situation must necessarily influence the data.

Such observations can only be carried out under the auspices of some central authority which shall insure the trustworthiness of the returns; and even with all the help which can be obtained, the difficulties are very great, for we have to contend with the movements of the population from place to place; and above all, with the almost impossibility of ascertaining, to a year, the ages of the greater number of the people.

Those who have pursued the investigation of this most interesting subject, have, in the absence of exact data, been compelled to accept of such imperfect information as the mortuary registers have furnished; in no single particular is this information trustworthy.

To begin at the beginning, we have no security that the records have been well kept. Each death, or rather each burial, is understood to have been registered; but then the recorder had not in view any other object than the convenience of his cemetery arrangements, or the accounting of his fees. To him the entry of the burial of a labourer or of a vagrant could hardly have appeared of any importance, and thus it is by no means unlikely that many omissions have been made.

Next there is the uncertainty as to age. No one is particularly interested, except as a matter of mere curiosity, in recording the age truly. In taking the census of the living population, we may expect that each man knows something about his age, yet even there, there is great carelessness. But in the case of the dead it is often a mere guess; now we never attempt to guess within a year, and hence we arrive at round numbers, mostly the tens. Up to 25 or 28 the age is often talked of, and hence you will find that the entries thus far run smoothly; but even at 30 the influence of the guessing is distinct. The number of deaths recorded at 50 greatly exceeds those at 49 or 51. However, when people get very old, their ages come to be talked about and discussed, so that the relatives and neighbours come to remember that old John is just 96, and that Bridget is two years older. So the result is that the ages above 70 are more truly recorded than those from 30 upwards to that age.

Thirdly, we have the migration of the inhabitants. The populations of many towns are kept up by influx from the country, and that influx is determined by the industries which are carried on, or by the amenities of the situation. In some places the attraction is employment for female workers, in others it is for strong men, sometimes only young persons are employed, who, as they grow up, leave the town for other employments, and altogether the conditions become so complex that it is impossible to unravel them.

In addition to all this, we have the gradual internal growth of the population. If we had had a yearly census of the town all the way back to the birth of the oldest person whose death is recorded, and if the migration had been insignificant, we might have been able to approximate passably well to the general law of mortality

during the century. In the absence of such enumerations, we are driven back upon some estimate of, or guess at, the average annual rate of increase.

Thus in every essential particular, the information obtained from obituary town-records is defective, the life-table deduced from such information becomes a mere rough estimate, and all that can be said in its favour is, that we have no better.

The publication of the Registrar-General's Report for the year 1838, in which report the deaths at each year of age for the whole population of England and Wales were given, promised to inaugurate a new era in the history of mortality bills. The want of similar returns for Ireland and Scotland was indeed to be regretted, on account of the large influx from both of those countries, but the supply of that want was in contemplation. Even with this very serious drawback the return was of great value. It shows distinctly the uncertainty as to age, by the marked preference for the *tens*, and its annual continuation would have enabled us to judge of the growing intelligence of the population by watching the gradual diminution of these irregularities.

You will observe in that return a marked unevenness at the ages 8 and 9. Now, at such early ages there can have been little or no uncertainty; how then does this occur? why are there fewer deaths at 8 and more at 9 than we should expect from the comparison of those at 7 and 10? It may be that there is something in the development of the human body which renders it less liable to the influence of disease from the seventh to the eighth year, than from the eighth to the ninth, or that there is something in our national customs which leads to this anomaly; or else there may have been something exceptional in the numbers of births for 1829 and 1830. If the cause be constitutional or national, the same unevenness may be expected in next year's report; if it be in the peculiarities of previous years, we shall see it transposed one year in each subsequent return, while, if it be merely an accident of this year's epidemics, it may not be repeated.

A similar irregularity, but much more strongly marked, occurs at the ages 14, 15, concerning which analogous questions may be propounded.

All expectations of answering such questions were destroyed by the appearance of the Report for 1839.

The returns are now given from five years to five years; that is to say, all deaths recorded as having occurred between the ages, say of 25 and 30, are summed up, and the amount only given; so

that we are left in ignorance, I do not say of how many died, but of how many were said to have died in each single year. Thus the extent of the uncertainty and its character are concealed from us; and so we are unable to take any step towards its rectification. Granting that the five year aggregates are true, we cannot thence interpolate the single years, because we know of no law to guide the interpolation; and thus the quinquennial return is almost useless to the actuary. But more,—the very argument which has been set forth in favour of the quinquennial period, convincingly shows that the inaccuracy which was sought to be avoided is in reality preserved. Thus, if you glance at the numbers opposite ages 29, 30, 31, you cannot fail to be convinced that the superior attraction of age 30 has drawn many from the adjoining ages, and that the projection which you there see in the death-line ought to be smoothed down, part of the excess being given to 29 and part to 31, so as to fill up the hollows there. The same thing is obvious at ages 40 and 50; it is seen in a much smaller degree at 35, 45, 55. Now if we take the sum of the numbers from 26 to 30, we shall certainly include therein part of those deaths which have been drawn from 31, while the sum from 31 to 35 will contain only the much smaller number drawn from 36. Thus the quinquennial summation must, at this part of the table, show too many deaths in the latter half of the decade and too few in the first half.

These considerations show conclusively that the introduction of the quinquennial system into the Registrar's returns has deprived us of much useful information, tends to error, and has only done for us what we could easily have done for ourselves. It is therefore to be hoped that the system of annual returns may be resumed.

However carefully the national mortuary records may be kept, we can only obtain from them a loosely approximate result; the value of which is greatly augmented when the whole community is included, because then the effects of migration from one part of the country to another are eliminated. There still remain the influences of immigration and emigration, the extent of which we have scarcely the means of even guessing at. Besides, to the practical actuary, there appears this inconvenience, that the result does not give the probability of life among that class with which he is mostly concerned, viz., the provident part of the community.

The only satisfactory data which can be obtained for the guidance of those engaged in the business of Life Assurance are to be found in the records of the offices. The dates of birth have been ascertained with very considerable precision, the influence of

migration is reduced almost to zero, while the deaths, as well as the ages at death, are all authenticated; we only need a considerable number of cases at each age to make these records all that can be desired.

It is now more than the quarter of a century since the desirableness of combining their experience was urged upon the Scotch offices. The results of the experience of seventeen offices have been published, and now again we are on the eve of receiving a statement of the results obtained from the records of all the old establishments in the country.

It does not look to be a difficult affair, that of doing away with these spasmodic efforts, and of substituting for them a continuous system according to which each year's experience of all the institutions may be added to the past experience, and the results of the whole at once placed before us. Such a system would put us in possession at all times of the very latest information on the subject.

When we have to deal with confessedly imperfect data, we are forced to adopt some plan for smoothing down their irregularities. Now, if we knew the cause of the imperfection and the law according to which that cause operates, we could compute and allow for its effects; the data would then have ceased to be imperfect. For example, if we could estimate the relative attractions of the different ages, we could apply this law of preference to the tabulated numbers, and thus have made a true table. Hence, when we come to consider the matter deeply, the uncertainty is seen to be not in the numbers returned, but in our ignorance of the many causes which have combined to produce the return.

In the case, however, of the office records, which must be regarded as almost entirely freed from disturbing influences, it becomes a question whether it be or be not allowable to smooth the results. This question belongs to the general subject of observations made to determine physical laws, and we can hardly obtain a satisfactory answer to it if we confine our attention to one limited inquiry. Let us step out of our own department and take a look at other matters. Astronomers have set themselves to discover the manner of the earth's motion round the sun; they have observed the sun's apparent angular distance from a fixed line, and obtained his relative linear distance by micrometric measurements of his diameter at various times during the year. By combining these two sets of co-ordinate measurements they obtain a series of points representing the successive positions of the earth, and the line

passing through these points is the earth's orbit. The operation is identical in principle with that which we follow in seeking the law of life. The line drawn through the points thus obtained will be uneven on account of the unavoidable errors of observation, and the astronomer will naturally set about to smooth it. Let us suppose him to be yet in ignorance of any law, and inquire on what principle he may proceed. Casting a glance over the general form of the observed path, he notices with Kepler its strong resemblance to an ellipse; he assumes the ellipse as representative of the true law of the earth's motion, seeks out from among all ellipses that one which passes most closely to his observed points, adopts that as the true orbit, and attributes the various deviations from this line to errors of observation.

Improved telescopes, larger and better divided circles, more accurate time-keepers, enable him greatly to reduce the errors of observation, and he finds that the deviations from the elliptic path are too great to be accounted for by these; the law is more complex than he at first thought;—the earth does not move round the sun in an ellipse, and the astronomer has to investigate the law of the deviation, or inequality as he calls it. Now it must be clear to you that if, instead of recording the actual observations, astronomers had been in the habit of previously smoothing them down to suit the supposed proper curve, we should never have been able to expiscate the true law of the variations, because the data from which alone it is to be found would have been concealed.

It is the same in our business. There is, we must presume, a fundamental law of life accompanying the organization of the human being; but this law is traversed by many artificial and accidental conditions; just as the disturbing attraction of the planets are combined with that of the sun. The disturbances caused by these conditions take the appearance of unevennesses in the life-line, and to smooth these down for the sake of accommodating them to any pre-conceived idea, is virtually to prevent our learning anything about them.

We can hardly over-estimate the influence of circumstances upon the duration of life. Think on the ravages of helminths and other parasites before the discovery of fire, and when all food was eaten raw; consider the catarrhs and bronchitis, and the nauseous skin-diseases when men wrapped themselves in untanned hides. Or, not to go so far back, remark the change which has been caused by the discovery of vaccination by Jenner, or the immunity from ague obtained by the use of quinine, from scurvy by that of

lime-juice. Every improvement in life, every fluctuating fashion of dress, each new article of food, each new material for clothing, each change in our customs, must produce its effect, and leave its impress on the life-line; so that the sinuosities which we attribute to accidents in our observations, may in truth indicate the existence of active influences. Those of the irregularities which are due to minute errors in observation will gradually be compensated, while such as depend on changes in the state of society will continue to be represented in the returns for succeeding years; and hence, on all accounts, we must declare against the smoothing down of results obtained from trustworthy data.

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Notation is a contrivance for facilitating inquiry; and though only of secondary importance, it exercises a perceptible influence on the progress of the student, who is apt to direct his attention more to the symbol used than to the idea conveyed by it. The mental operations, the reasoning, and the results, are the same in whichever way they are expressed, but the mode of expression may be cumbrous or may be clear.

In order to make my meaning clear I shall take an example. Let us suppose that a mechanician is investigating the theory of the steam-engine. He says, let  $z$  be the radius of the cylinder,  $y$  the length of the stroke,  $x$  the number of strokes per minute,  $w$  the pressure of the steam,  $v$  the horse-power of the engine; and, in virtue of the laws of mechanics, he establishes a set of equations among these five quantities,  $z, y, x, w, v$ : and after a variety of algebraic manipulations he obtains, for some particular case, a set of results,  $z$ =so much,  $y$ =so much, and so on. You see at once that he must now recur to his original positions in order to know the meaning of these results, for most probably during the manipulations he has lost sight of the import of each letter. Whereas, if he had said let  $r$  be the radius of the cylinder,  $l$  the length of the stroke,  $n$  the number of strokes per minute,  $p$  the pressure of the steam, and  $HP$  the horse-power, he would have carried with him easily their imports all through his manipulations. I think that you will all admit this: but let us look at the principle involved. The words *radius*, *length*, *number*, and so on, are, in essence and originally, arbitrary sounds used to recal ideas: in other languages, other words are used for the same ideas. In employing the initial letters of these words, we avoid the multiplication of arbitrary marks for the same thing, we save unnecessary loading of the memory, and this is the principle which regulates all sound systems

of notation:—The symbol should recal, by some simple analogy or resemblance, the thing symbolized.

In order to indicate the relation between connected magnitudes we use what are called *functional* symbols, which generally take the form of abbreviated words; as in the expressions  $\log a$ ,  $\sin a$ ,  $\tan a$ , and so on. This scheme leaves us at liberty to add to our stock of symbols as our researches are extended into new fields; the new relations which are therein discovered are necessarily represented by words, and the abbreviation of these words furnishes us with the requisite signs, without disturbing in any way our previous notation.

Again, the letters or symbols which we are using may refer to various cases, and it may be necessary to distinguish between these. Thus the mechanician may have to do with two or three steam-engines at once; he then writes  $r_1$ ,  $r_2$ ,  $r_3$ , for the radii of the cylinders,  $l_1$ ,  $l_2$ ,  $l_3$ , for the lengths of the strokes, and so of the rest, because it has become usual, and is also most convenient, to assign the place occupied by those subscribed numerals to what may be called the definite article or demonstrative pronoun of algebra. Similarly the symbol  $\log a$  denotes generally the logarithm of the number  $a$ ; but then there are various systems of logarithms, and so when we wish to define the very logarithm of  $a$  of which we are treating, we write  $\log_{10} a$  when the denary system is meant, or  $\log_e a$  when we are to use that system of which the number  $e$  is the basis.

Such being the general principles of notation, the actuary has to arrange his symbols in accordance with them. His subject forms too minute a portion of the science of logistics for him to set up an independent system, even if he could devise one clearer or more comprehensive than that which is in common use; he only needs to devise functional symbols appropriate to the peculiar relations which arise in the course of his investigations.

The plan of using arbitrary letters is, perhaps, the worst that can be devised. In making a set of computations, we title the columns A, B, C, D, as they arise in the work, and, instead of designating the quantities therein entered by names or symbols descriptive of them, we denote them by the letters which happen to be written at the heads of the columns. Thus we come to have the D column, the N column, and so on.\*

\*       \*       \*       \*       \*

\* The author has overlooked the circumstance that D and N are the initial letters of *Denominator* and *Numerator*. This we had supposed was generally known, until we read Professor De Morgan's remarks in the note on p. 363 of our number for Oct. 1868.—Ed. J. I. A.

In its notation, as well as in the principles of its operations, the calculus of life contingencies is intimately related to the other departments of the science of numbers. So also the business of the actuary is closely connected with all the other pursuits of life.

I would, therefore, impress upon you this truth—that, in order to become thoroughly masters of your own profession, you must give time and attention to the study of many things beyond it. The arithmetical part, though an important, is but a small fraction of your studies. You have to examine the statistical evidence on which your computations are founded, to consider the influences of trade and climate, to weigh the arguments for and against a gradual improvement in the general health, and to study the influences of new medical discoveries. Then also the average rate of interest, the likelihood of its increase or decrease, and the general course of business over the world, have to be looked into; while, for the guidance of the affairs of an office, legal knowledge and an acquaintance with securities and investments are indispensable. All these subjects bear directly upon your business, they may almost be said to belong to it; yet with even these you are not to rest satisfied.

The business of the intelligent student is not to acquire knowledge—this is only the means to an end—his business properly is to acquire the power of knowing, to cultivate his perceptive and discriminative faculties, to learn to analyse the subjects before him, to make himself ready in weighing arguments, and in general well to perform that complex function of the human mind which we call thinking.

You enter the gymnasium to learn to climb poles, to overleap hurdles, to throw weights, to run races! No. You go to strengthen your bodies, to gain command over your muscles, to become ready in action, able to do; and you carry that ability along with you everywhere. How often, when you thought not of it, has skill in balancing saved you from a fall? how often has the readiness of eye and quickness of hand which you acquired in your games, prevented accidents and mistakes?

So it is in the intellectual gymnasium. The habits of thought which we have acquired during our studies accompany us in our daily life. The care with which we have been accustomed to scrutinize the logic of a geometrical theorem, has its influence long after the theorem has been forgotten. So the relations which we have observed among chemical or mechanical phenomena, the graces of arrangement and of style which we have enjoyed while

perusing the work of some able writer, the beauties which we have admired in a fine painting, every perception of appropriateness of beauty, of goodness, continues to aid us in our further progress, because it has assisted in developing and strengthening our mental powers.

Let us then, in striving to become actuaries, not forget that we have first to become men. Nor let us fear to engage, even keenly, in pursuits which may seem alien to our professional duties, satisfied that every piece of work which is thoroughly done, every study which is faithfully prosecuted, will aid us in that peculiar vocation to which we have been called by circumstances or by choice.

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*Suggestions for Legislation to regulate the calculation and investment of the reserve in Life Assurance Companies.* By HERR FINANZ-RATH HOPF, *Manager of the Gotha Life Office.*

*Translated by D. A. BUMSTED, F.I.A., of the General Reversionary and Investment Company.*

[Read before the Institute, 29th November, 1869.]

AT the present time the subject of legislation in reference to Insurance Companies is attracting much attention in Germany as well as England. Until quite recently the regulation of insurance matters in Germany was under the sole control of the several States, as is still the case in the United States of America: the former German Confederation possessing neither the organization nor the authority to interfere in the matter. As a consequence of this state of affairs, we are informed that a company organized in one of the German States was regarded in the others as a foreign company, and all companies, whether German or foreign, which desired to extend their business over the whole of Germany, were compelled to put themselves in communication with more than thirty different Governments, each of which had its own special regulations. Some of the larger States had passed laws on the subject, but the provisions of them differed very widely from each other. In the remaining States, each Government imposed on the companies such conditions as it thought proper. This state of things is evidently very unfavourable for Insurance Companies, which require—more than any other undertaking—a wide field for their successful development. Under the old political organization of Germany, a uniform system of legislation, though long and

ardently desired, was quite unattainable. But after the events of the year 1866 had led to the formation of the present North German Confederation, which gives a common legislation to at least three-fourths of Germany, the hopes of reformers revived. The 4th article of the Constitution of the Confederation expressly provides that legislation upon insurance shall belong to the province of the Confederation. More than a year ago, the Confederation having taken no steps in this direction, the Prussian Government attempted to pass a new law on the subject for Prussia proper. The bill however was strongly opposed, on the ground that the Confederation is pledged to legislate on the subject; and it was accordingly withdrawn without its provisions being discussed. In the mean time the German Life Insurance Institute, considering the Prussian Government's bill too restrictive, prepared and presented to the House of Representatives an opposition bill founded on liberal principles. On the other hand, several managers of German Insurance Companies met together last summer, and drew up and submitted to the council of the North German Confederation two bills, one relating to all kinds of Insurance Companies, and the other relating specially to Fire Insurance Companies. These bills are founded upon the principle that all the present burdensome restrictions upon the carrying on of insurance business and the establishment of new companies should be entirely removed; but that the law should take steps to insure such publication of accounts and other particulars as will enable persons to acquire a knowledge of the position of each company.

Herr Finanzrath Hopf took part in the discussion and preparation of these Bills; but as finally settled by the majority of the managers, they do not fully meet his approval. He proposes therefore that the following sections should be added to the bill which relates to Insurance Companies in general.—*ED. J. I. A.*

§ *a.*

1. Every Company granting assurances or annuities upon the contingencies of human life, and required by the law to be registered, is required to file with the Board of Trade\* a statement of the principles which it has decided to employ in the calculation of its reserve; giving the tables of mortality or sickness (*Invalidität*)

\* The author suggests "Court of Commerce" as a translation of the original "*Handelsgericht*," but we think it more convenient for the English reader to write "Board of Trade," although this is not strictly accurate.—*ED. J. I. A.*

upon which the valuation is to be based, and stating the rate of interest assumed.

2. These principles and data shall be entered in the Register of Life Assurance Companies.

3. Similarly, all future alterations of the same shall be filed and registered before their adoption.

4. The registration of the aforesaid statement, but not the contents of the same, shall be published by the Board of Trade at the expense of the Company, in the newspapers mentioned in § x. of the Bill.

5. These regulations, when registered, shall be binding upon the Company, to the extent of prohibiting it from making a smaller reserve than the regulations require, in respect of assurances proposed before an alteration of them has been registered.

6. Alterations of the same shall apply to the assurances proposed after the day on which the regulations were published by the Board of Trade, unless the Company has itself appointed a later time. They may only be applied to assurances previously proposed, when their adoption will increase the amount of the reserve.

7. The aforesaid Companies are bound to require that the Actuaries to whom they entrust the calculation or revision of the reserve, shall make a declaration upon oath to the Board of Trade, that they will conform to the foregoing conditions.

#### § b.

As to the publication of accounts by the Companies.

In Companies granting life assurances, annuities, &c., the amount of the reserve stated in the balance sheet, in respect of the assurances in force at the end of the year, must be at least equal to that which is required, according to the principles and data for the calculation of the reserve, filed with the Board of Trade (§ a); deducting the reserve in respect of the reassurances which may have been effected with other Companies, the amount of which is to be stated in the Balance Sheet.

The Actuary must, in the Balance Sheet, state on his oath (as mentioned in § a) that the reserve is estimated according to the above instructions.

#### § c.

As to the investment of the reserve.

In Companies granting life assurances, annuities, &c., the

reserve, so far as not advanced upon security of their own policies, may only be invested on mortgage of real estates, to the extent of one half of their value, estimated according to their permanent income; or on mortgage deeds (Pfandbriefen), annuity deeds (Rentenbriefen), or similar bonds, granted by corporations and secured by direct or indirect mortgage of real property, or annuities payable out of it; or on bonds which the North German Confederation, or any one of the States belonging to it, have issued, or guaranteed. Those Companies which, at the passing of this law, shall not have invested their reserve in the above manner, must do so within the next five years.

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In support of these propositions Herr Finanzrath Hopf adds the following remarks.

### 1. *General Remarks.*

Life Assurance is distinguished from other branches of assurance by two characteristics. The first is that human life, which is the object of life assurance, is necessarily subject to destruction before a very long period has elapsed; and secondly, that the risk of destruction does not remain constant during this period, but is continuously varying with a certain degree of regularity. The chance of dying is greatest immediately after birth, then decreases until about the age of 10, remains nearly constant for some years, and then increases continually, at first slowly, but after the age of 55 rapidly. This, at least, is its general course, if both sexes are combined.

In the first of these periods, (from birth to the age of 10), during which the chance of dying continually diminishes, and the expectation of life consequently increases, scarcely any assurances proper are effected, but only endowments, in which the payment of the sum depends upon the assured living to a given age. Assurances proper are effected almost exclusively at ages above 15; but constitute the great majority of life assurances. It will be observed that, in both classes, the liability of the Company increases, every year, with the age of the assured.

If the Company should agree to take, each moment, from the assured, in respect of assurances against death, the exact amount necessary to cover the risk for that particular moment, this payment—the premium—would vary at every moment, increasing with the lapse of time. The system would obviously be quite

impracticable; and although possible if applied to yearly periods, it could never be adopted in practice, as the premium required under it from persons attaining the higher ages, would become oppressively great.

There are two principal ways in which these difficulties may be avoided. The premium may either be paid, for the whole term of the assurance, in one or more capital sums; but this, for obvious reasons, is seldom done. Or, what is by far the most convenient and usual course, it may be paid by a constant average premium, depending on the age at the time of effecting the assurance. This average premium is, of course, at first much higher than the premium for the current risk of the year; but the disproportion gradually diminishes, until at one certain moment the premium exactly covers the risk; after which the reverse is the case, and the premium in each successive year falling more and more short of the risk. On the whole, however, complete equalization is produced. It is necessary in both these methods, for the stability of the Company, and the ultimate fulfilment of all its engagements, that the amount received in excess during the earlier years, should not, in any way, be consumed, either entirely, or in part, but be laid aside and invested at interest, in order to provide for the loss certain to be incurred at a later period. The sums thus accumulated form what is called the reserve. It comprehends the equivalent received, either in one sum, as in the case of the single premium, or in the yearly average premiums, to provide for the claim which will become due at a future time; and is in fact only anticipated premium.

The *premium-reserve* is therefore very nearly related to the *premium-carried-forward*,\* but by no means identical with it. Life Assurance Companies are similar to Fire and other Assurance Institutions in this respect, that, whenever the period for which the premium was contracted to be paid by the assured, extends beyond the current year of valuation, the corresponding part of the premium paid must be carried forward to the following year or years of valuation, and laid aside as *premium-carried-forward*—or unearned premium. While, however, the *premium-carried-forward* is released at the expiration of the year of valuation to which it extends, the same is not the case with the *premium-reserve*. This should increase uninterruptedly (in the ordinary modes of life

\* *Prämien-übertrag*. The author here makes use of this expression to distinguish it from the *premium-reserve* (*prämien-reserve*), which we have generally translated simply by "reserve."

assurance), from a cause which it is not necessary here further to investigate; and should become equal to the sum assured, when the assured life has attained the highest age according to the table of mortality. It, therefore, very soon becomes much greater than the premium-carried-forward.

For this reason, the importance of the reserve, in relation to the stability and future solvency of an Office, is much greater. Now it has already been felt to be necessary, with respect to the premium-carried-forward, to impose upon the Companies certain rules, contained in Sec. 17 of our Bill. Much more pressing therefore must be the necessity for similar rules, in regard to the reserve.

The only principle which, according to the views now prevalent in politics, can properly form the basis of the much-to-be-desired regulation, by the State, of assurance matters, may be thus defined:—Abolition of the system of dependence upon the State (concession and supervision); and by way of compensation, legislative guarantees that a full statement shall be published of the condition of every Company.

The Bill above referred to has been drawn up on this principle, which was adopted after mature deliberation. But it ought not to stop half way with life assurance, and deny the most important consequences of the principle. The publicity which the law is to enforce, must *à fortiori* extend to the principles and data for the calculation of the reserve.

It would be a glaring inconsistency, and a serious hindrance to the efforts which are now directed towards the legal regulation of assurance matters, if, upon this vital point, concealment should be allowed, and even sanctioned, by the law. Nay, publicity alone will not be sufficient, considering the immense importance of the subject. The interest of the assured, who make their payments in advance for so long a term, and place such implicit confidence in the Offices, requires that a further step should be taken. A legal guarantee must be obtained that these payments shall not be lightly and improvidently squandered, as might still happen under a system affording the utmost publicity, but that they shall be, so far as necessary, faithfully reserved until the period shall arrive when they will be required for payment of assurances, annuities, &c.

The following figures will place the extraordinary importance of the subject in a clearer light.

There were, for example,

	Sums Assured.	Premium-carried-forward.	Reserve.
	£	£	£
In the Gotha Life Assurance Bank, established in 1829 (Mutual Office), at the end of 1868	9,068,295	152,218	1,686,565
In the Leipzig Life Assurance Company, established 1830 (Mutual), at the end of 1867 . .	1,941,870	'37,097	282,908
In the Berlin Life Assurance Company, established in 1836 (Proprietary), at the end of 1868	2,214,114	£496,906	
In the Concordia, at Cologne, established in 1853 (Proprietary), at the end of 1868 . . .	3,737,529	476,065	
In the 35 German Life Assurance Companies, at the end of 1867 . . . . .	50,448,333	About 6,750,000	

In all these Companies, the reserve may be expected to increase for a great number of years, according to the extent of their business, before it reaches its maximum. In the Gotha Office alone, the increase amounts yearly to more than £75,000. It is obvious therefore that, in this question, very large sums are involved, entrusted to the Companies by the assured, to provide for claims which will arise at periods more or less remote. The Companies would not receive these sums, if the assured paid the exact premium for the risk incurred in each year. As, however, they pay so much in advance, in the single or annual premiums, these payments in advance are in a certain sense the property of the assured, and must certainly be treated as such by those Institutions which wish to continue in a position to fulfil their engagements, and to be able to meet all the claims as they fall due.

At present, the Companies possess the unrestricted right of dealing with the fund; and, this being so, there is manifestly great danger of injury being done to private interests. There is a strong temptation to neglect to provide a sufficient reserve, in order to declare large profits, or to cover excessive expenses of management; and again, when a reserve fund has been established upon correct principles, the temptation arises to reduce it, and to represent the amount thus obtained as surplus, distributing it among the proprietors in the form of dividends. In the absence of legislative provisions, a protest against this course could either not be sustained at all, or only with the greatest difficulty to the individuals making it. For the time, manœuvres of the above kind would be very profitable to the proprietors, and would perhaps bring extraordinary profits to the officials, if they are remunerated, as is sometimes

the case, by a percentage on the profits. The fatal consequences would be sure to arrive at last; but on account of the peculiar source from which the reserve is derived, and the long period which elapses before it is required, they would not be felt until after a long term of years, at a period when those who had enjoyed the unfair advantage had long ago retired, or perhaps were no longer in existence. The full weight of the calamity would fall upon a later generation, who would thus be deprived of their due rights.

It cannot be replied that the sound principles upon which the Offices are constituted and managed, render such fears chimerical. Although these principles may be respected at present, they will possibly be entirely disregarded by other persons, and other management, if the law do not restrain their conduct. Companies which are now in a most flourishing condition, which are most prudently managed, and enjoy the fullest confidence, may be destroyed through the self-interest, the carelessness, or the ignorance, of a later administration; and the thousands who have deposited with them their hard-earned savings, may have their dearest hopes destroyed.

Against such possibilities, in the absence of state supervision, the law alone can afford effectual protection. For this, there is now an irresistible demand, and unless it be conceded, the law relating to life assurance will completely fail in its purpose. The statements of accounts, which concern only the thousands, will have been regulated with the utmost anxiety; while the millions are left to take care of themselves!

The question is, By what arrangement can the desired object be attained in the simplest and safest manner, while allowing the greatest possible protection to freedom of management?

In considering what has been done by other legislatures in regard to this subject, we are struck with the highly characteristic manner in which some of the States of the North American Union have acted. The numerous instances of fraud which have occurred, in connexion with life assurance, seem to have impelled them to acts of an extreme kind. They have not only taken a course directly opposed to that which we have urged, by establishing a system of State supervision over the business of every Company, but have even a system of interference of the most minute and personal character. In some of the States, the Government appoints actuaries, through whom it maintains strict control over the calculation of the reserve of the Companies situated within its boundaries; and, where it thinks necessary, prescribes, without further notice,

how much reserve must be laid aside; or prohibits Companies of other States and countries from carrying on business if their reserve appears insufficient. With us, of course, the introduction of such a violent measure cannot be thought of; but it is a proof of the immense importance of the subject, that in a land of the most absolute freedom, such extreme measures should have been adopted.

In England, the cradle of life assurance, its practice has hitherto been quite unrestrained; but the resulting evils caused Parliament, some years ago, to take into consideration the necessity for legislative restrictions; and an investigation into the subject took place, the results of which were published in 1853, in a large blue-book. At that time, however, no law was passed. The subject has recently been again discussed in Parliament, and we are looking forward with anxiety for the result. It is reported that the debates have shown that the condition of many a Company, now in good repute, is very unsatisfactory in regard to funds; and that legislative regulations, if they provide the public with the requisite protection, will render the continued existence of many Companies impossible, and thus precipitate a great calamity—a calamity which certainly cannot be avoided, but which it is sought to defer as long as possible.\* Meanwhile, in order to provide for the poorer classes, at least, a protection from unrighteous robbery, a large Government Savings Bank, a Government Annuity Office, and a Government Life Assurance Office for the assurance of small sums, have been established with the approval of Parliament. These institutions are entirely conducted by the Government, through the Post Office officials, and are managed on account of the State. In a country like England, this also must be looked upon as an extreme act, and shows again the deep importance of the subject.

In France, a law relating to Assurance Companies was passed on the 22nd January, 1868. This is united by its preamble to Art. 66 of the law of 24th June, 1867, in which it is enacted that "The Tontine and Life Assurance Companies, whether mutual or charging fixed premiums, remain subject to inspection and control by the Government. Other Companies may be formed without permission." Life Assurance Companies are therefore here subject to exceptional treatment. It is true, there is no condition in the law, relative to the calculation of the reserve; but as the Government exercises inspection and control, it can

\* It is difficult to say how far these remarks may be true; but there is no doubt a great deal of exaggeration in them, the great majority of the Offices being unquestionably in a sound state.—ED. J. I. A.

impose upon every Company such regulations as it pleases on this important point.

In Austria, a decree was issued in the year 1860, as an instruction to the Imperial Commissioners, "concerning the formation of funds for life assurances and the conduct of the same." This prescribes exactly how the reserve of Life Assurance Companies is to be calculated; and how a special "assurance fund" is to be formed, which must at least equal this reserve. The Government Commissioner must see that attention is paid to the decree; and the balance sheet is to be submitted to him for examination. Unfortunately, however, this instruction is not correct, as far as the "assurance fund" is concerned; for, in the event of the true mortality being less than the mortality to be expected from the table, it requires too great a fund to be laid aside.

In other countries, life assurance has not yet acquired such extension as to become (so far as the author is aware) the subject of legislative regulations.

It remains for us to consider whether it would not answer our objects to introduce into the law of the North German Confederation a definite rule for the calculation of the reserve; establishing, by law, the general principles upon which that calculation should be founded—prescribing, for instance, the rule which science recognizes as the most sound one, and which many actuaries consider to be the only correct one, viz.:—that the reserve should be calculated solely from the assumed table of mortality and the assumed rate of interest—this rate not to exceed 4 per cent—pure premium and pure annuity values being employed, calculated according to the nature of the assurance and the mode of paying the premiums, and no deduction being made on account of ~~commuted commissions~~, &c., nor any account taken of the loading of the office premiums. Even if it should be found that the most substantial Companies do now act, and will be certain to act, in future, upon this principle, I should yet be unwilling to recommend that the same (laid down of course with more precision and detail) should be made by law compulsory upon all Companies. I start from the principle that freedom in commercial matters should not be restrained more than is absolutely necessary; and with this view, the object of the proposed law appears to me to be attainable with a considerably less amount of compulsion. I find this in the following points, which, I am convinced, exhibit the minimum of what is to be required.

1. Life Assurance Companies must be compelled by law to act upon principle, and not capriciously, in respect to the vital question

of the calculation of their reserve. They must be required not to make a random guess at the amount of it, but to deduce it by a scientific calculation, from a table of mortality (or sickness), combined with a definite rate of interest. This requirement involves, in fact, nothing more than the essential one of the solvency of the Offices; for without such a foundation, it would be impossible to imagine an orderly business, but only a chaos, which sooner or later must lead to ruin. On the other hand, no restraint should be imposed upon the choice of the elements for the calculation, but they should be left to the free selection of each Office.

2. The law must compel the Companies to publish in an authentic form accessible to everyone, both the data for the calculation mentioned above, and the principles upon which it is founded. This publicity provides the necessary check upon the absolute freedom reserved in the choice of data. It is the only way, at least to expose fraud, which on account of its inherent boldness will never be totally intimidated. When the State ceases to examine into the stability of undertakings—to attest their solvency by granting concessions, and exercising the necessary supervision, it is relieved from a moral responsibility. But the public, being required to judge for themselves, may rightly demand to be furnished with every means of obtaining information. It is true that many persons are incapable of forming an opinion upon the subject, but science will apply its criticism for the enlightenment of those who, though deeply interested in the matter, are unable to understand it. Mathematical text books are even now published in which the rules for the calculation of annuities and assurances are explained. The more this branch of assurance extends, the more generally will it be studied; and in future there will be many more accountants than at present exist, who, without being learned actuaries, will be able to form a correct opinion upon the subject.

We may also expect that publicity on this point will exert its usual beneficial influence; that it will beget mutual emulation; so that no Company will desire to be inferior to others in the soundness of the principles upon which its reserve is calculated. It is therefore to be hoped that, omitting isolated exceptions, sound principles alone will be registered, and even the few exceptions in which this is not the case, will disappear when the managers of those Companies perceive how much importance the public attach to this point, in the choice of an Office. It is certainly much better that the end should thus be indirectly attained through

mutual emulation, rather than directly through compulsion by the State.

3. A guarantee must be afforded to the assuring public that the Companies will really and permanently act upon their declared principles, with respect to the calculation of the reserve; for an arbitrary change of these principles (at least to the disadvantage of the assurances already effected) ought not to be permitted. They must therefore not be allowed, at a later period, to make less reserves for their assurances than those fixed by the registered conditions in force at the time the assurances were completed. It is otherwise in the opposite case of an increase of the reserve, as the interests of the future cannot be endangered by this. On the contrary, it simply raises the value of the assurances. An injustice might be done to the proprietors of the Company, as the periodical gains or benefits, which come to them in the form of dividends, would be diminished. This, however, is an interest which may be left to be decided by the Deed of Settlement or the articles of agreement of the proprietors. In this direction, therefore, the law need not fix any limit. Indeed it must expressly legalize the increase, for in certain cases, such increase may appear of urgent necessity, in order to ensure the fulfilment of the future obligations of the Company, and in that case individual proprietors should be prevented from objecting to the increase; which they could do, if the law prescribed that the reserve is to be calculated exactly according to the registered principles and data.

The practical necessity, moreover, of preventing the diminution of the reserve, must be more considered than the theoretical object of making the reserve, for old and new assurances, homogeneous, by the introduction of a new and possibly better table of mortality, so far as this homogeneity would have (for particular ages of the assured or particular periods of their assurances) the result of diminishing the reserve. If it were desired to secure this power for the Companies, a higher authority would have to be created, to decide whether the interests of the assured were not illegally prejudiced, through the exclusive employment of the new table of mortality upon the assurances already in existence; and the interference of any higher authority is just what the law is to abolish.

Such are the general considerations which have suggested the foregoing provisions, which cannot, in my opinion, be properly omitted from the Bill to regulate life assurance; containing, as they do, the smallest possible amount of restriction.

The following remarks are made in elucidation of details:—

*2. Special reasons for the above propositions.*

As to § a.

Paragraphs 1 to 4 lay down:—the duty of Life Assurance Companies to publish the principles upon which their reserve is calculated, the extent of this duty, and the mode of publication. The last can only be done by Returns to the Board of Trade—registration in the special Insurance Register—and notice through the Official press. It is, therefore, based upon the general provisions of the law, which again are copied from the forms drawn up in the recent Trade and Partnership Law. That the obligation must apply not merely to the original conditions, but also to subsequent changes in the points concerned, is self-evident, as soon as the principle is once acknowledged. The public announcement is to be limited, as in other cases, to the fact of the registration. The contents of the latter, which may possibly be voluminous, need not be advertised. It is sufficient, that, according to the other conditions of the law, everyone is entitled to examine the register and to have extracts made at his expense.

Paragraph 5 enacts that the Companies must always adhere to the registered principles for all the assurances, which, so to speak, have been effected under their rule; that is so long as no alteration of the same has been made by advertisement and registration, and the public were consequently entitled to consider the former as still in existence.

Paragraph 6 defines the extent of the operation of the aforesaid alterations. The first sentence defines—in opposition to the negative rule in paragraph 5, which protects the assured from alterations to the disadvantage of former contracts—the earliest moment at which a change to a new system can be made; and this takes effect, unless the Company itself should fix a later term of commencement. This must of course rest with the Society, and, so far, the condition is only a subsidiary one. The second sentence expressly authorizes alterations which are to the advantage of the assured, without, however, making them compulsory: this point being no farther dealt with by the law.

Paragraph 7 provides the necessary guarantee for the actual performance of the foregoing conditions. The calculation of the reserve is a technical operation. Many Companies, for greater security, have the work revised by a second actuary. It is obvious that the function of these persons is of the highest technical

importance. They need not necessarily be officers of the institution; for the calculation or the revision of the reserve, may be entrusted to an actuary who is not on the official staff; but it is all the more necessary, from the great importance of the reserve to the assured, that the actuary should be sworn to conform to the law. For the administration of this oath, the Board of Trade naturally suggests itself, as it is already charged with the registration. On the other hand, one oath taken, either by the calculator, or the revisor (where one is employed), appears sufficient.

As to § b.

This rule corresponds with the previous conditions respecting the reserve, and is in fact only a logical consequence of the same, namely its application to the Balance Sheet. The amount which the calculation of the reserve, according to the registered data, gives for each of the assurances, is the minimum to be laid aside as reserve. It is, therefore, also the least which must be brought into the account. It is true it only appears in the balance sheet as a single total. The premium reserve is, in the first place, to be reckoned upon all the assurances in force at the end of the year. With respect to the portion reassured, the Company has transferred to the reassuring Company the corresponding premium, and is thus relieved from the risk. It must, therefore, deduct from the general reserve, a reserve in respect of the reassurances, according to the nature of the same, and bring only the difference into the balance sheet. The case is thus provided for in which the original assurance and the reinsurance do not completely cover one another, whether as regards the nature or the term of the assurance. It is however necessary that the reserve in respect of the reassurances should be separately calculated; and it is, on that account, generally desirable that the amount should be stated in the accounts. An attestation is required to be added to the balance sheet by the calculator or revisor of the reserve, as part of his duty. In fixing penalties, the case in which this certificate has been found to be incorrect should receive particular attention; and for such a case, it should be provided that the actuaries shall not only make compensation, but also undergo severe punishment; and the like compensation and punishment should visit those of the directors, with whose knowledge or consent, an amount of reserve, not corresponding with the conditions of the law, was placed in the balance sheet.

## As to § c.

The accumulated reserves are profitably invested until their application; and this growth of the funds, by the operation of compound interest, forms an essential element in the whole system and economy of life assurance. From the nature of the reserve, we see the importance of investing it in the most substantial securities, and the avoidance of all that bears the character of speculation. If there is any negligence or imprudence in this respect, there is danger of serious losses occurring, which would be equivalent to a reduction of the reserve, such as is forbidden by the foregoing conditions. On this account, it seems just that the law should sanction certain limitations, within which the Companies may select their investments of capital. In the first place, there can be no objection to allow the Companies to advance to the assured upon the deposit of their unincumbered policies, the amount of reserve existing at the time upon them; for the possibility of immediate compensation completely neutralizes the risk of loss. The great bulk of the reserve must, however, always be invested in other ways. With regard to this, the idea suggests itself to enjoin upon the Companies the observance of the State rules about investment of trust funds, and to content ourselves with a simple reference to them. But, on the other hand, we must bear in mind that not all the States have rules of that nature, and even where they do exist, the legislation respecting them varies remarkably. It is therefore better, for the sake of completeness, and to establish uniformity in this important part of the business, that definite rules should be inserted in the law itself. Mortgages form the most suitable kind of investment; for in them, at any rate, the capital suffers no fluctuation. Moreover, if due caution be observed, and the amount advanced be limited to one half of the value of the property, estimated according to the permanent income derived from it, a loss of capital and interest, even in the case of a compulsory realization, can be rendered almost an impossibility. Although mortgages offer such great advantages, it does not seem practicable to limit the investments of the Companies to those securities, and to loans upon their policies, on account of the increasing importance of those funds which are more liable to fluctuation. The permission to invest in interest-bearing bonds is indeed an unavoidable concession to the condition of the money market of the present day; but it is impossible to allow the Companies free choice over the whole catalogue of home and foreign, public and industrial, bonds. Those kinds only must be allowed, in which the defects of

fluctuation, and possible loss on realization, are at least confined within moderate limits. Among these we would include—first, mortgage debentures—then the State bonds of the North German Confederation and of the several States belonging to it; or bonds guaranteed by these States, as specified in the above propositions. Finally, it must be remembered that the existing Companies could not, without great hardship, be required immediately to conform to these rules, on account of the probable loss which would be occasioned thereby. Some time should therefore be allowed them for converting their securities; and a period of five years would appear to be sufficient for the purpose.

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The following is a translation of the former of the two bills prepared by the Managers, as mentioned on p. 271 :

ACT TO REGULATE INSURANCE COMPANIES.

§ 1. After the passing of this Act the permission of the State shall not be required for the establishment of Insurance Companies or the transaction of Insurance business.

§ 2. Every Company of whatever kind, whether Proprietary or Mutual, which is desirous of transacting the business of insurance, is required to file a statement (of the nature hereinafter described) with the Board of Trade in whose district the chief office of the Company is situated.

Until this statement has been filed, the Company may neither grant nor seek to obtain any insurances.

§ 3. The statement (§ 2) to be filed with the Board of Trade by a Proprietary Company,\* in addition to giving the branches of insurance which are to form the objects of the undertaking, must be in conformity with the provisions of the general Trade Code which refer to Companies of the particular class to which it belongs.

The Company must be based on written articles of association; and these also must be sent to the Board of Trade.

§ 4. The statement mentioned in § 3 shall be entered by the Board of Trade in a Register entitled "Register of Insurance Companies" (which forms a part of the Trade Register), and then published without delay by the Board of Trade, at the cost of the Company, in conformity with the provisions of the general Trade Code respecting the registration of Trading Companies, and in the newspapers appointed for that purpose in accordance with Art. 14 of the Code.

This publication must also specify, in the case of Proprietary Companies, how much of the capital is paid up, and whether there is any, and if so, what security, given for the amount unpaid.

Further, the greatest number of shares which a single Proprietor may hold.

\* Under this term, are included the two classes of "Share Companies," and "Commanditis Companies," which are both mentioned in the original, here and elsewhere. These two classes are sometimes spoken of in the bill as "Trading Companies."

Also, whether the formation of a Capital-Reserve is provided for; and if so, in what manner and to what amount.

Finally, the principles on which the Assets of the Company are to be invested, as appearing from the documents filed.

The entry in the Trade Register and the publication of this registration, prescribed by the general Trade Code, are no longer required for carrying on insurance business; but those regulations are to be considered as fulfilled by the entry in the Register of Insurance Companies and the publication of its contents.

§ 5. In the formation of a Mutual Insurance Company, the following regulations must be observed:—

1. The Company's Articles of Association must be settled by a Notary or a Court of Law.

2. A corporate name must be adopted.

The name of the Company must express the object of the undertaking, and may not contain the names of members or other persons. Every new name must plainly differ from the names of all other Insurance Companies registered by the same Board of Trade.

A written declaration, or the acceptance of a Policy, shall be sufficient to constitute membership.

§ 6. The Articles of Association must contain—

1. The name of the Company and the address of its chief office.

2. The particular branches of insurance business carried on by the Company.

3. The district within which its business is to be carried on.

4. The conditions respecting the admission and withdrawal of members.

5. The amount of insurances necessary for the commencement and continuance of the Company.

6. A condition as to the manner in which the costs of establishing the Company and of commencing business, shall be defrayed.

7. The manner in which the payments for claims and expenses are to be met; whether contributions are to be paid in advance, or only as required; and whether the liability of the members is limited or unlimited; particularly whether, and to what extent, the members are bound to make extraordinary contributions in addition to the regular contributions.

8. The principles according to which, in the case of the contributions being paid in advance, the yearly surplus, if any, is to be calculated and distributed.

9. The internal constitution of the Company, particularly as to the representation of the members; whether it is by general meetings, or by committees, or by public bodies belonging to the state or district, or in what other way.

In addition there must be given,

- a.* The mode of convening these representative bodies.

- b.* The regulations as to the right of voting at the meetings, and the manner in which the voting is to take place.

- c.* The questions which must not be decided by a simple majority of votes of those present at the meeting, but require a greater majority or some other condition.

10. The official representation of the Company, and particularly,
  - a. The constitution of the Board of Directors who manage the business and represent the Company judicially and extra-judicially; the manner of their appointment, and of their signature; and the manner of proving that the members of the Board and their deputies have been duly appointed.
  - b. The manner of appointment and the powers given to all the other officials of the Company of every kind.
11. The principles on which the Company's assets are to be invested.
12. The mode in which the Accounts and Balance Sheet are audited.
13. The conditions under which the Company shall be dissolved.
14. The form in which the notifications of the Company are to be issued, and the newspapers in which they are to be inserted.

§ 7. The Articles of Association (§ 6) shall be sent to the Board of Trade (§ 2) by the Directors, and the Board of Trade shall enter the same in the Register of Insurance Companies and publish an abstract thereof, according to the provisions of § 4, at the expense of the Company.

The abstract must contain,

1. The date of the Articles of Association.
2. The name of the Company and the address of its chief office.
3. The objects of the Company and the district within which its business is to be carried on, and the period of its duration (if limited to a certain time).
4. The mode in which the Board of Directors makes known its resolutions and signs for the Company.
5. The form in which the notifications of the Company are issued, and the newspapers in which they are to be inserted.
6. The regulations of the Articles of Association on the points 5-10 in § 6.
7. With the abstract shall be published,
  - a. The names and residences of the members of the Board at the time, and their deputies (if any).
  - b. Whether, and if so in what manner, the appointment of a Committee of supervision to watch over the management of the business, is provided for.
  - c. Also whether, and to what amount, and from what resources, the formation of a Capital-Reserve is provided for.

§ 8. Every resolution of the Company, altering, continuing, or cancelling, the Articles of Association, must be declared before a Notary or a Court of Law; and a certified copy of the resolution (§ 6<sup>9</sup>) must be filed with the Board of Trade.

Any resolution altering the regulations of the Company must be entered in the Register of Insurance Companies, and an abstract of it published (§§ 4 and 7). Before this is done, it will have no legal value.

Every alteration in the Board of Directors of a Mutual Company must be registered either by a Notary or a Court of Law, and be filed with the Board of Trade by the new Board of Directors. This alteration shall then be entered in the Register of Insurance Companies, and published (§§ 4 and 7).

The same shall be done in the case of temporary deputies being chosen for one or more members of the Board.

§ 9. Every registered mutual Company may sue and be sued in its own name, acquire rights, and incur liabilities, and obtain proprietary and other legal rights in land. The Court in which it shall ordinarily sue and be sued shall be that in whose district its chief office is situated.

§ 10. Such Mutual Companies as neither directly, nor through third parties, grant insurances beyond the limits of the hundred (Bezirk der unteren Verwaltungsbehörde) in which they are situated (Landrathsamt, Amtshauptmannschaft, &c.), may elect whether they will be subject to the provisions of §§ 5 *et seq.*, or not. If they are once entered in the Register of Insurance Companies, they shall be subject to all the regulations of this Act affecting Mutual Companies.

§ 11. The aforesaid unregistered Mutual Companies shall obtain the corporate rights mentioned in § 9 on their rules being approved by a superior magistrate (oberen Verwaltungsbehörde).

§ 12. The Register of Insurance Companies and the documents filed shall be open to the inspection of everyone; and plain or attested copies thereof shall be given on payment of the cost to everyone who requires them.

§ 13. The following regulations shall be observed with reference to the book-keeping and statements of accounts of the Insurance Companies coming under §§ 2-9.

§ 14. The books must be balanced every year, and an account prepared therefrom showing—

1. The income and outgo, or the profit and loss, and
2. A true balance sheet.

If the Company has only commenced business during the calendar year, the first accounts may be prepared so as to include this and the following year.

If the business of the Company includes different branches of insurance, the income and outgo in each branch must be separately stated, and the liabilities in each branch must be separately stated in the Balance Sheet.

§ 15. The first part of the accounts, showing the profit and loss, must, in addition to other items required by the nature of the account, contain the following items stated separately.

In the Receipts,

- a. The amount received for premiums during the year (without deducting commissions).

In the case of fire insurance, the premiums shall be separately stated, according as they are

aa, for the current year or for future periods;

bb, for insurances granted by the Company direct or for reinsurances.

In life assurance, they shall be classified according to the principal kinds of assurance (sums payable at death, endowments, annuities, tontines, savings—insurance, &c.),

- b. The interest on investments.

c. The other receipts classified according to nature and amount.

In the Outgo,

- a. The claims, including not only the payments in respect of ordinary assurances, but also for endowments, annuities, &c., after deducting the amounts received under reinsurance policies.

- b.* Reassurance premiums.
- c.* Commission, less the commission upon reassurances and the drawback on freight insurances.
- d.* The other expenses of management.
- e.* The amounts written off in respect of bonds, real estates, office furniture, and preliminary expenses (§ 16).
- f.* The loss on outstanding loans.
- g.* Other payments, classified according to nature and amount.

§ 16. The Balance Sheet shall specify separately (so far as the following items occur in the branches of the business carried on by the Company),

In the Assets,

- a.* In Trading Companies, the unpaid amount of the share capital, stating the manner in which the same is secured.
- b.* The value of the real estate and of the office furniture, in separate sums.
- c.* The investments, classified under the heads—mortgages, bonds, pledges, bills of exchange, loans on policies.
- d.* The agents' balances, money at bankers, and cash in hand, all in separate amounts.
- e.* The proportion of interest accrued to the end of the year, but payable in the following year.
- f.* Other outstanding amounts.
- g.* Preliminary expenses not yet written off.
- h.* The loss, if any, on the current year and on former years.

In the Liabilities,

- a.* In Trading Companies, the amount of shares issued or the amount of capital paid up. In Mutual Companies, the amount of the guarantee fund, if any.
- b.* The claims announced but not yet settled, including sums assured under ordinary policies, annuities, endowments, &c., and deducting the portion secured by reassurances.
- c.* In Life Assurance Companies, the reserves for life assurances proper, for endowments, tontines, and savings bank deposits.
- d.* The amount of unearned premiums to be carried forward; to be distinguished, in the case of insurances for several years, according to the amount for the next year and the following years.
- e.* The capital (or profit) reserve, if any.
- f.* The other debts, classified according to nature and amount; the borrowed capital, without reference to the time of its falling due, and the interest thereon to the date of the account.
- g.* The profits to be divided.

§ 17. In preparing the Balance Sheet, the following regulations shall be observed:—

The property and debts belonging to the Company shall not be taken at more than their selling value. Doubtful debts shall be set down at their probable value, and bad debts written off.

Investments in bonds may not be taken at higher prices than those officially quoted on the last day of the year of account.

The preliminary expenses to the end of the first year only may be entered as Assets in the Balance Sheet; and must, in all Companies

established after the passing of this Act, be completely written off within at most 10 years.

For the claims announced, but not settled (§ 16, Liabilities, *b*), shall be reserved at least the sum that will be probably required for settling them.

The amount to be carried forward of the premiums paid in advance (§ 16, Liabilities, *d*) shall, in the case of term insurances, be calculated at least at the same rate as the portion of the premiums for the past duration of the insurance. The utmost to be deducted is what has been paid for reassurances and commissions; but the costs of management incurred thereon are not to be taken into account.

§ 18. A statement of the insurances current at the close of the year shall be added to the accounts, distinguishing the direct insurances, the reassurances effected with, and those effected by the office, and distinguishing also the different classes of insurance.

§ 19. The accounts (§§ 15–18) must be published by the Board of Directors of the Company, within six months at the latest from the end of the business year, in a newspaper to be appointed once for all by the Government of the State in which the Company's chief office is situated; and this publication must be proved within four weeks to the superior magistrate of the district in which the Company's chief office is situated, by the production of a copy of the document.

Everyone shall be entitled to obtain from the said magistrate a copy of the account at his own expense.

§ 20. The above-mentioned Government is authorized to ascertain whether the Company has complied with the provisions of § 19; and for this purpose to require the production of the books and documents of the Company at the office of the Company.

Violations of the Act, proved in a Court of Law, are to be published by the Court.

§ 21. Foreign Insurance Companies, of whatsoever kind, may carry on the business of insurance within the boundaries of the Confederation, by Agents or other authorized representatives, under the following regulations:—

1. They must establish at least one Branch at a stated place within the boundaries of the Confederation, with an Office and a resident Agent, who must be legally authorized to accept service of all writs and proceedings on behalf of the Company.
2. They must legally engage to abide by the decision of the Courts of Law of the district in which the Agent resides, in respect of insurance contracts entered into with subjects of the Confederation.
3. This shall be expressly stated in every Policy.

It will thus be legally decided in what Court of Law all claims by subjects of the Confederation against a foreign Insurance Company are to be prosecuted.

§ 22. The provisions of §§ 2–8, as well as those of §§ 13–19, shall apply also to these foreign Insurance Companies, but under the following conditions:—

1. The statement to be filed with the proper Board of Trade, when the Branch Office is established, must show that the requirements 1 and 2 of § 21 have been complied with.

2. It must also be stated whether the judicial decrees, including decrees of arbitrators, issued against the Company within the territory of the Confederation (specifying in what particular States of the same), are executed in the country in which its chief office is situated, in the same manner as similar decrees made in that country, or whether this is not the case. In the former case, if the execution of the decrees is not consequent upon laws or published treaties, it shall be proved to the Board of Trade by a certificate issued from the Court of the Chancellor of the Confederacy.

The Board of Trade shall complete the Register of Insurance Companies and the notifications accordingly.

§ 23. Every alteration in the contents of the statement to be filed according to §§ 3, 7 and 22, also the discontinuance of the business, must be reported immediately to the Board of Trade, and shall by it be entered in the Register and published in the same manner as the original statement.

When the discontinuance of the business has been filed, no new insurances may be negotiated or granted.

Trading Companies must conform in all respects to the regulations of the general Trade Code upon the subject of these notices.

§ 24. Companies which only grant reassurances are exempted from the provisions of § 2, *et seq.*

§ 25. The appointment of representatives for the negotiation of insurances (agents, &c.) rests entirely with the Companies themselves. They are in this matter placed under no restrictions, and a concession from the State is not required for agents.

Agents are not limited to a particular district, but may travel about for the purpose of canvassing, &c.

§ 26. Promoters, managers, and officials who do not comply with the provisions of this Act incur penalties of 50 to 1,000 thalers or corresponding imprisonment. Foreign Companies may in addition to this be prohibited from transacting business, either entirely or for a time. Such total or partial prohibition shall be published by the Board of Trade.

Agents who do not comply with the provisions of this Act incur penalties not exceeding 200 thalers or corresponding imprisonment.

§ 27. Native Insurance Companies already in existence, and foreign Companies which before the passing of this Act were permitted to carry on business, must, if they wish to continue the same, file with the proper Board of Trade, within six months after the publication of this Act, the statement (§§ 3, 7, 22) required for the purpose of the notification to be published. In default of this, the directors, officials, and agents are liable to the penalties of § 26.

If the Articles of Association, or the Contract upon which the Company is founded or permitted to carry on business, is at variance with provisions of this Act, the former need not be altered.

But the regulations as to books and accounts (§§ 13-19) are absolutely binding upon these Companies.

§ 28. All laws and regulations which are not in harmony with this Act are hereby repealed.

Friendly Societies (Miners' Funds, Funds for Trade Charities, &c.), depending upon special legal provisions will not be affected by this Act.

The Public Insurance Companies now or hereafter to be established with Government approval (by the State, the provinces, or parishes, &c.), are subject to the provisions of this Act, except that they are not subject to the obligation of filing statements with the Board of Trade, nor to the regulations as to the form and contents of the Articles of Association of the Company (§§ 6 and 7).

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## NOTICES OF NEW BOOKS.

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*Sterblichkeit und Versicherungswesen, neu behandelt* (Mortality and Insurance, treated in a new manner), von Dr. HERMANN SCHEFFLER, Braunschweig, 1868.

At no time has the subject of social economy attracted so much attention as at the present day. Statistics is the language in which social economy finds expression, and it may fairly be said to owe its origin in a great measure to Germany. The very term Statistics appears first to have been introduced in 1749 by Professor Achenwall of Göttingen. But even before that time there were German writers, who, by means of induction from authenticated facts, ascertained the condition of man as existing in society. At the end of the seventeenth century Dr. Carl Neuman was the first to give, with any pretension to exactness, the mortality of the City of Breslau, and afforded materials which Dr. Halley subsequently worked upon. In 1741, J. P. Süssmilch published a remarkable work on Divine Order, as exhibited in the changes to which the human race is subject. It must be admitted that for nearly a century subsequently hardly any work of importance appeared in Germany on human mortality, unless we except perhaps J. H. Lambert's writings, who about the year 1770 investigated the London and Brandenburg Bills of Mortality, and made the first attempt at applying the Differential and Integral Calculus to mortality, the law of which he endeavoured to represent by an algebraical equation.

Of late years, however, a host of profound men in Germany have given the subjects of population and mortality their attention. The statistical returns of Prussia, Saxony, Bavaria, &c., have been thoroughly investigated by such men as Dietrich, Engel, Wappaeus, Meyer and others, and the theory has been gone into by Moser, Fischer, Zillmer, and quite recently by Dr. Knapp,\* whose method is a novel one, which should not be lost sight of by English writers.

The work we propose to review, embraces this branch of enquiry, although the principal portion of the book is devoted to a new mode of working the different problems of Life Assurance. Dr. Hermann Scheffler is an author whose talents are of a versatile character. By profession an architect, and author of no less than ten works on architecture and engineering, he has written not without success on philosophical, optical, and purely mathematical subjects. He also acts as a Government functionary in connection with the local Insurance Office at Brunswick. We regret that in the work in question, we are unable to find much merit in the portion which discusses mortality. The author endeavours to steer clear

\* *Ermittlung der Sterblichkeit*, by Dr. A. F. Knapp, Leipzig, 1868.

of the doctrine of Probabilities,—but his treatment of the subject does not thereby gain in perspicuity. He prefers a table of mortality, exhibiting the number of survivors at each year of life, to one showing the rate of mortality—the more so as the former table is more easily corrected. He touches, tho' rather vaguely, on what is now so well known as the actual intensity or force of mortality. He then investigates how best to obtain the mortality of a stationary population, and discusses Halley's and Euler's methods. He examines how to ascertain the mortality of a population subject to immigration and emigration, as well as to changes in the fecundity, and shows how Mortality Tables may be practically framed from actual periodical enumerations of the people and annual Bills of Mortality. The results of the censuses in the Duchy of Brunswick were applied by the author to the determination of the mortality existing in the capital, in other towns, and in the whole of the principality, and he finds his results correspond to a remarkable degree with those of certain Societies whose returns he had examined. He finds little divergence between the mortality of the two sexes; but on the whole it appeared that the mortality of females was greater than that of males up to the twenty-fifth year, and that above that age the contrary was the rule. The annual increase of the population in the principality during the decennium ending 1861 was 83 percent.

Dr. Scheffler does not however use the Brunswick Tables as the basis of his operations. He proceeds to find an Average Table of Mortality, based on no less than 25 known tables, in addition to one derived from a formula of Dr. Moser.\* The tables he uses are—the English Males and Females, Duvillard's, Quetelet's "Whole of Belgium," Ditto Towns only—Males and Females, Ditto Country—Males and Females, Baumann-Süssmilch's, Churmark, Berlin—Males and Females, Leipzig—Males and Females, Nordhausen, Carlisle, Finlaison's Males and Females, Kerseboom's, Wargentin's Males and Females, Deparcieux's, Brune's Prussian Widows' Fund—Males and Females, and the Equitable Experience. He graphically represents the several tables by means of curves, taking for the abscissæ the years of life, and for the ordinates the corresponding number of living. As the radix of the tables he takes the number 4455, this being the number of survivors at age 50 under the Equitable Table. Accordingly at this age all the curves intersect. In this manner the author thinks he avoids the irregularities presented by the several curves in early and advanced life, and is enabled to obtain an average curve, which he considers best answers the requirements of life assurance, where the mortality of a mixed class of society is required. Dr. Scheffler professes to have adjusted the table derived from this average curve, but one may look in vain for a scientific process, such as we are accustomed to in this country. German writers altogether seem to have been remiss in the adjustment of their tables, and perhaps this may be the reason, that altho' they have numerous mortality tables of their own, the English tables—more particularly the Experience of the Seventeen Companies—are generally used, and their own tables are discarded. However, the author of the work we are reviewing cannot be accused of any want of loyalty to his pet table. With a view to adapt it to analytical investigations, he presents it to us in the shape of a quadratic function,  $f + gx + hx^2$ , the coefficients  $f, g, h$ , representing different constants

\* *Assurance Mag.*, vol. xiii., p. 14.

during different periods of life. From age 0 (birth) to age 6,  $y$  = the number of living =  $10000 - 11402.9x + 842.4x^2$ ; from age 6 to 80,  $y = 64440 - 395.86x - 4.331x^2$ ; and from age 80 to 100,  $y = 204364 - 4305.8x + 22.68x^2$ . Where the mortality of a select body is required, he makes use of Brune's Mortality of the Prussian Widows' Fund—one of the very tables he availed himself of in finding the mortality of a non-select body of men. This table of Brune's is perhaps the best of all the German tables, and resembles in many respects the Equitable Experience. In order to represent this mortality by a quadratic expression, the table had to be divided into four portions, and the interval from 6 to 80 had to be partitioned into two periods from 6 to 40, and 40 to 80. The corresponding constants are given.

Before making practical use of his tables, the author proceeds to consider questions of the accumulation of capital through the operation of interest. He rejects the assumption of periodical conversions, and justly argues that the growth of capital is not like that of vegetation subject to the action of the seasons, but is going on continuously.

Accordingly if  $y$  is the amount of  $c$  at the end of  $n$  years,  $i$  being the nominal rate of interest per unit per annum,

$$dy = i y dx$$

whence,

$$\frac{dy}{y} = i dx$$

and

$$\int_c^y \frac{dy}{y} = i \int_0^n dx$$

or,

$$y = ce^{in}.$$

If the capital  $c$  is placed out at interest at the end of  $x$  years from the present time for a term of  $n$  years—

$$\int_c^y \frac{dy}{y} = i \int_x^n dx$$

$$y = ce^{i(n-x)}$$

For the present value of  $y$  due  $n$  years hence, we have

and

$$c = ye^{-in}$$

Applying the principle to Annuities, let  $b$  represent the yearly payment, then the present value of  $b dx$  payable at the time  $x$  is  $\frac{b dx}{e^{ix}}$ ;

hence for the present value of the annuity we have to sum all these values up to  $n$  years, and  $c = \int_0^n \frac{b dx}{e^{ix}} = \frac{b}{i} (1 - e^{-in})$ , and since  $y = ce^{in}$ , we have

for the accumulations of the annuity  $y = \frac{b}{i} (e^{in} - 1)$ .

If the annuity be variable, the present value or amount can easily be found by considering  $b$  a function of  $x$ .

In Chapter 5 the author proceeds to show that the number living, instead of being subject to yearly decrements, may be considered to diminish continuously—and that this very idea is a necessary sequence of representing mortality by a curve. Hence the mean duration of life at age

$n$  for  $m$  years is  $\frac{1}{t_n} \int_n^{n+m} y dx$ , and applying the well-known first approximation which substitutes a polygon for the area of a curve, the ordinary expression for such mean duration of life is obtained. If instead of contenting himself with this first approximation he had taken a second approximation, he would have arrived at Mr. Woolhouse's useful and accurate formula for the value of the expectation of life.

In the following chapters formulæ are obtained for assurances and annuities for one and two lives on the assumption that both interest and mortality are acting "continuously."

The annual premium  $P$ , payable from age  $n$  to age  $m$ , for an assurance is found thus:

Since the increase of capital in respect of time  $dx$ , is  $Pdx$ , on the part of each of the  $y$  survivors, the total payment is  $P y dx$  payable  $x-n$  years from the present time, the present value of which is  $P y dx e^{-i(x-n)}$ , and the total value of the payment side is  $P e^{in} \int_n^m y e^{-ix} dx$ . The benefit side  $= -e^{in} \int_n^m \frac{dy}{dx} e^{-ix} dx$ , remembering that for the time  $dx$  the deaths are represented by  $-dy$ , the present value of which is  $-dy e^{i(x-n)}$ .

$$\text{Hence } P = - \frac{\int_n^m \frac{dy}{dx} e^{-ix} dx}{\int_n^m y e^{-ix} dx}; \text{ and similarly for other formulæ.}$$

Dr. Scheffler shows how approximate solutions may be arrived at by substituting for the integral  $\int_n^m y dx$  the expression  $\frac{1}{2} y_n + y_{n+1} + y_{n+2} + \dots$

$+ y_{m-1} + \frac{1}{2} y_m = S_n^m y - \frac{1}{2} (y_n + y_m)$ , and how the results thus obtained resolve themselves into those known already to us from text books, by assuming periodical payments and conversions of interest.

In § 12 and the succeeding chapters the author proceeds to the rigorous integration of the different expressions for annuities and assurances by substituting for  $y$  the quadratic function  $f + gx + hx^2$ , with its several sets of constants at the different divisions of life. Altho' the integrations offer no serious difficulty, still it is found necessary to have, for practical purposes, taking the rate of interest at 3 percent, no less than 30 values tabulated for each age. Several of the columns that are thus obtained involve as many as from 10 to 14 figures each, tho' it must be admitted that it is only those tabular values which refer to two lives which involve so large a number of places. The Commutation Tables for two lives, in the manner in which Jones, Chisholm, &c., present them to us, if not quite so cumbrous are sufficiently bewildering to the computer. On the other hand a complete set of tables, as given us by Dr. Scheffler, does not occupy more than 12 pages, whereas a set of tables for both one and two lives, by our existing methods, occupies about 40 pages. We find in the Appendix two sets of tables—the one on the basis of the Average Table of Mortality previously referred to, which are to be used in the calculation of annuities, or for risks upon non-select lives, and another set of tables based on Brune's Experience table for the calculation of assurance risks,

which appertain to select lives. The author goes still further, and urges that in certain calculations where two lives are involved, these tables should be combined:—as, for example, in an assurance payable when the wife survives the husband—since the Office satisfies itself as to the eligibility of the wife's life, she belongs to a selected body, whilst the ordinary mortality table applies to the husband. We really think this is overstraining the point. The author seems to forget the effects of selection as exercised by the public *against* an Assurance Society. Thus annuitants do not pass a medical examination before the risk is taken by the Office, but a process very nearly akin to it is practically undergone—that of self-examination—and accordingly, annuitants are as a rule not inferior lives to persons who take out life assurances. Besides, in such risks as survivorship or reversionary annuities, whilst the time when the annuity is entered upon is delayed, the term during which it is payable is also extended, and the two tendencies neutralize each other. Before practically applying his method to a number of examples, the author dwells upon the fact, that by considering both mortality and interest to be acting continuously, he obtains results which at the utmost differ by about 5 percent from the values obtained by the existing methods. He considers his mode of dealing with the subject the preferable one, as mortality, no doubt, exercises its influence in a continuous manner.

The succeeding chapters are devoted to an investigation into the proper modes of valuing the risks of Assurance and Annuity Societies, in order to arrive at an estimate of their financial position.\* Then are added very elaborate dissertations as to the most equitable mode of returning the profits to members of Assurance Societies. The author is of opinion that whilst a certain portion of the profits should be left unappropriated at each valuation, with a view to form a reserve fund to meet fluctuations and contingencies, the return of profits should be in proportion to the number and the amount of premiums paid during the interval between two valuations. We have lately heard so much about the “percentage” and “contribution” methods of dividing profits, that it is interesting to know that an impartial German writer, who throughout his work gives evidence of acute reasoning and analytical powers, should give his decided preference to the former.

Altho' the author is so earnest an advocate for the use of different tables of mortality for the various classes of assurance, he does not advise the formation of separate funds, but thinks the maintenance of one aggregate fund the best safeguard against the effects of fluctuation. On the other hand, he would not have any of the classes excluded from sharing in profits, as is often rather arbitrarily done.

After discussing the effects of using an erroneous table of mortality, and the correctives to be applied, he takes up the questions of Surrender Values, and under what conditions “Free Policies” should be issued.

Chapter 25 discusses the relations which should subsist between Participating and Non-participating rates. Little of real value has been written on this subject, and it would be a matter of interest to know how the different Companies have proceeded in evolving their non-participating rates, which are after all of comparatively recent introduction. Limiting his remarks to a Mutual Society, Dr. Scheffler is of opinion that the non-participating policyholders, by being excluded from their share in

\* Dr. Scheffler advocates a gross premium valuation with an adequate provision for future expenditure.

profits, leave the entire responsibility of the concern to the participating policyholders, who thereby become a species of shareholders. Hence the chief source of profit, viz., the difference between the two scales, should strictly speaking be made to correspond with the risks incurred, viz., the number and extent of assurances under the non-participating class as compared with the other business. But this proportion is continually varying, and moreover the rates have to be fixed at the starting of a Company, when it is difficult to estimate even the relative business that may ensue under the two classes. The author accordingly disapproves of the distinction altogether.

In the concluding chapter, he exposes the unsatisfactory character of institutions founded on the model of the Caisse Paternelle in Paris. Children of the same age form a class for themselves, and on their attaining the age of say 21 years, the funds are distributed among the survivors. Even if the several classes consisted of a large number of members, and the funds were distributed rateably, the system would not work so satisfactorily as that pursued in England under the name of endowments to children. The evil is increased when membership with equal privileges is secured irrespective of the amounts contributed. This is not even a tontine investment; it becomes a gambling transaction.

The treatise ends with a cursory glance at the advisability of writing off unduly large risks by means of reinsurance.

In the limited space at our command we are unable to remark upon and criticise all the different views propounded, and the many matters of interest referred to in the work.

Altogether, the book, which is copiously supplied with diagrams and tables, must be regarded as an important addition to the comparatively few original works on Life Assurance that have appeared in Germany. It may be doubted, however, whether it will find a permanent place among standard books on this subject. Woolhouse's excellent method of obtaining formulæ on the like assumption of continuity both of Mortality and Interest is immeasurably superior, and it is only surprising that our author should have ignored the valuable services of English writers, who with so much greater success have preceded him in a similar field. Against the absurdity of imagining that a table derived from an Olla Podrida of no less than 26 others, prepared by different authors under varying circumstances, would represent the mortality of a non-select body of men, we cannot too strongly protest.

On the other hand we are glad to be able to compliment Dr. Scheffler on the clearness of the notation and style which characterize his book.

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## CORRESPONDENCE.

ON THE CONSIDERING THE "PAID-UP" POLICY AS THE EQUIVALENT OF THE RATIO THE PREMIUMS PAID BEAR TO THE TOTAL NUMBER PAYABLE.

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—In a note appended to Mr. Younger's letter in the *Journal* for January last, on the subject of "Ten Years' Nonforfeiture Policies," you suggest the examination of the question, by means of a comparison of the amount of "paid-up" policy that the surrender value of the original

assurance would purchase, with the amount that would be given, by taking the ratio of the premiums already paid to the total number payable.

I have attempted in the following letter to carry out this suggestion; and the points to which more particularly my attention has been directed are, to find what conditions are required to make what may be called the "Empirical" method of finding the amount of "paid-up" policy true, and how far these conditions deviate from the real ones.

Thus, if a person assure at age  $x$  by  $n$  equal annual payments, and wish at age  $(x+t)$  to have a "paid-up" policy payable at the same time as the original policy, he will receive by the empirical method  $\frac{t}{n}$  per £1 of original assurance.

What is the relation between this value and the true one?

Mr. Sprague, in the *Journal of the Institute*, volume vii., page 58, has shown that, on the assumption that the single payments are sold by the same table as the policy is valued by,  $\left(1 - \frac{P_x}{P_{x+t}}\right)$  represents the real amount of "paid-up" policy. Though the formula is applied there only to ordinary assurances for the whole of life, it is equally applicable to other classes of assurances, if by  $P_x$  we mean the pure annual premium payable under the original contract, and by  $P_{x+t}$  the pure annual premium that would be payable at age  $(x+t)$  as the equivalent for the remaining time of the same contract.\* Thus, if  $P_x$  be the ten years' premium at age 30 to provide £1 at death, and if  $t=4$ ,  $P_{x+t}$  will be the 6 years' premium at age 34 to provide £1 at death.

Again, if  $P_x$  be the premium at age 30 for an Endowment Assurance payable at death or 60,  $P_{x+t}$  will be the premium at age 34 for an Endowment Assurance payable at death or 60. If  $P_x$  be the Survivorship Assurance premium, age 30 against age 70,  $P_{x+t}$  will be the Survivorship Assurance premium, age 34 against 74, and so on.

What we have to find, then, is, the relation between  $\frac{t}{n}$  and  $\left(1 - \frac{P_x}{P_{x+t}}\right)$ .

Now, since after  $n$  payments of premium, the policy-holder has discharged all his obligations, the Assurance Company is then liable for the total amount assured, i.e., 1, while at the same period the empirical formula will give  $\frac{n}{n}$ , i.e., 1, we find the empirical rule is correct on and after age  $(x+n-1)$ , (premium then due being paid). Thus the assumption

\* It may not be out of place to append here another proof of Mr. Sprague's formula. Using  $P_x$  and  $P_{x+t}$  in the extended sense given to them above, and supposing  $1 + \frac{n-t-1}{n} \alpha_{x+t}$  to mean the annuity payable from age  $(x+t)$  on to the expiry of the time named in the original contract, as also employing  $A_{x+t}$  to represent the single payment corresponding to the annual premium  $P_{x+t}$ , then  $\frac{V_{x|t}}{A_{x+t}}$  = the amount of "paid-up" policy.

Divide both numerator and denominator by  $1 + \frac{n-t-1}{n} \alpha_{x+t}$ , and we have

$$\frac{\frac{V_{x|t}}{1 + \frac{n-t-1}{n} \alpha_{x+t}}}{\frac{A_{x+t}}{1 + \frac{n-t-1}{n} \alpha_{x+t}}} = \text{"paid-up" policy.}$$

But  $\frac{V_{x|t}}{1 + \frac{n-t-1}{n} \alpha_{x+t}} = P_{x+t} - P_x$ , and  $\frac{A_{x+t}}{1 + \frac{n-t-1}{n} \alpha_{x+t}} = P_{x+t}$ , therefore  $\frac{P_{x+t} - P_x}{P_{x+t}} = 1 - \frac{P_x}{P_{x+t}} = \text{"paid-up" policy. Q. E. D.}$

made by this method is, that every premium paid previously to that age secures an equal part of the sum assured, *i.e.*, that the amount of single payment policy that could be purchased by the increase in the value of the policy, through one more payment of premium, is the same at every age. That is, the empirical method assumes

$$(1) \quad \frac{V_{x|1}}{A_{x+1}} - 0 = \frac{V_{x|2}}{A_{x+2}} - \frac{V_{x|1}}{A_{x+1}} = \frac{V_{x|3}}{A_{x+3}} - \frac{V_{x|2}}{A_{x+2}} = \dots \\ = \frac{V_{x|t}}{A_{x+t}} - \frac{V_{x|t-1}}{A_{x+t-1}} = \dots = \left(1 - \frac{V_{x|n-1}}{A_{x+n-1}}\right) = \frac{1}{n}$$

But, as we have seen,  $\frac{V_{x|t}}{A_{x+t}} = 1 - \frac{P_x}{P_{x+t}}$ .

Therefore the assumption is, that

$$(2) \quad 1 - \frac{P_x}{P_{x+1}} = \left(1 - \frac{P_x}{P_{x+2}}\right) - \left(1 - \frac{P_x}{P_{x+1}}\right) = \left(1 - \frac{P_x}{P_{x+3}}\right) - \left(1 - \frac{P_x}{P_{x+2}}\right) = \dots \\ = \left(1 - \frac{P_x}{P_{x+t}}\right) - \left(1 - \frac{P_x}{P_{x+t-1}}\right) = \dots = 1 - \left(1 - \frac{P_x}{P_{x+n-1}}\right) = \frac{1}{n}$$

*i.e.*,

$$(3) \quad 1 - \frac{P_x}{P_{x+1}} = \frac{P_x}{P_{x+1}} - \frac{P_x}{P_{x+2}} = \frac{P_x}{P_{x+2}} - \frac{P_x}{P_{x+3}} = \dots = \frac{P_x}{P_{x+t-1}} - \frac{P_x}{P_{x+t}} = \dots \\ = \frac{P_x}{P_{x+n-1}} = \frac{1}{n}$$

Dividing each of these terms by  $P_x$ , our assumption is

$$(4) \quad \frac{1}{P_x} - \frac{1}{P_{x+1}} = \frac{1}{P_{x+1}} - \frac{1}{P_{x+2}} = \frac{1}{P_{x+2}} - \frac{1}{P_{x+3}} = \dots \\ = \frac{1}{P_{x+t-1}} - \frac{1}{P_{x+t}} = \dots = \frac{1}{P_{x+n-1}} = \frac{1}{nP_x}$$

That is, a series in arithmetical progression, with a common difference of  $\frac{1}{nP_x}$ , is formed by the reciprocals of the premiums that would require

to be paid by persons entering at every succeeding age from  $x$  to  $(x+n)$  to place them in exactly the same position as that then held by the original assurer.

These premiums will, therefore, themselves form a harmonical series.

Therefore, in order that the amount of single payment policy, at any age, that could be purchased by the increase in the value of the policy, through one more payment of the premium, may be the same at every succeeding age, or, in other words, in order that the amount of "paid-up" policy at any age may be the same as the ratio the number of premiums paid bear to the total number payable, it is required that a harmonical series be formed by the premiums that would be charged at every succeeding age for a policy terminating with the original one, and under which all payments were to cease at the same time as under that contract.

The general expression for any term will thus be

$$P_{x+t} = \frac{2P_{x+t-1}P_{x+t+1}}{P_{x+t-1} + P_{x+t+1}}$$

In series (3) let us examine the two terms  $\frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}}$  and  $\frac{1}{n}$ . Which is greater? The two sides of the expression are

$$\begin{array}{cc} \text{True} & \text{Empirical} \\ \text{Method.} & \text{Method.} \\ n\mathbb{P}_x & \sim \mathbb{P}_{x+n-1} \end{array}$$

Now,  $\mathbb{P}_{x+n-1} = \frac{\mathbb{V}_{x|n-1}}{1} + \mathbb{P}_x$  (the denominator 1 of the fraction being the annuity-due at age  $(x+n-1)$  for the remainder of the term  $\mathbb{P}_x$  is payable). And substituting this for  $\mathbb{P}_{x+n-1}$ , we have

$$n\mathbb{P}_x \sim \mathbb{V}_{x|n-1} + \mathbb{P}_x$$

or

$$(n-1)\mathbb{P}_x \sim \mathbb{V}_{x|n-1}.$$

Now, *generally* speaking, the value of a policy of assurance, by annual payments not all exhausted, is less than the premiums that have been paid under it. Therefore, *generally*,

$$\frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}} > \frac{1}{n}, \text{ or } \left(1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}}\right) < \left(1 - \frac{1}{n}\right), \text{ or } < \frac{n-1}{n}.$$

That is, the amount of "paid-up" policy, at the age at which the last premium is payable, is *generally* greater by the empirical method than by the true method. Again, since generally,  $\frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}} > \frac{1}{n}$ , and the sum of

the whole series (3)  $= \frac{n}{n} = 1$ ; since one, at least of the terms is greater

than  $\frac{1}{n}$ , one or more of the others must be less than  $\frac{1}{n}$ . Let  $\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}$  —

$\frac{\mathbb{P}_x}{\mathbb{P}_{x+t+1}}$ , or  $\left(1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t+1}}\right) - \left(1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right)$ , be such a term; and assume that at age  $(x+t)$  the empirical and true methods give the same results.\*

Then

$$1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t+1}} - \frac{t}{n} < \frac{1}{n}$$

therefore

$$1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t+1}} < \frac{t+1}{n}$$

\* The empirical method does not necessarily give greater result than the true ones, so we are quite entitled to suppose a case accordingly. Thus,  $\frac{\mathbb{V}_{x|1}}{\mathbb{A}_{x+1}} - \frac{1}{n}$  represents the difference of the amounts by the two methods at end of one year; or  $\mathbb{V}_{x|1} - \frac{\mathbb{A}_{x+1}}{n}$  is the difference.

Now  $\mathbb{V}_{x|1}$  is a maximum when the current risk is a minimum. Say, then,  $l_x = l_{x+1}$ . In which case  $\mathbb{V}_{x|1} = \frac{\mathbb{P}_x}{v}$ , and  $\mathbb{A}_{x+1} = \frac{\mathbb{A}_x}{v}$ , and the expression will become  $\frac{\mathbb{P}_x}{v} - \frac{\mathbb{A}_x}{vn}$ , or  $n\mathbb{P}_x - \mathbb{A}_x$ . But  $\mathbb{A}_x = \mathbb{P}_x(1 + \frac{1}{n-1}a_x)$ ,  $\therefore$  our expression is  $n\mathbb{P}_x - \mathbb{P}_x(1 + \frac{1}{n-1}a_x)$ , or  $n - (1 + \frac{1}{n-1}a_x)$ . Now [except in the case when  $l_x = l_{x+1} = l_{x+2} = \dots = l_{x+n-1}$ , and  $v = 1$ , or  $i = 0$ ]  $n$  is greater than  $(1 + \frac{1}{n-1}a_x)$ . Therefore, when the mortality is a minimum, the amount of paid-up policy at the end of one year, by the true method, will be greater than by the empirical method. Q. E. D.

and the empirical method will give more than the real amount. So our only means of determining whether the "paid-up" policy, at any individual age, is greater or less by the empirical than by the real method is by actual experiment.

In the foot note in which the proof of the equation  $\frac{V_{x|t}}{A_{x+t}} = 1 - \frac{P_x}{P_{x+t}}$  is found, the  $A_{x+t}$  (besides being at the same rates as the policy is valued by) is assumed to be the *pure* single payment; that is, the assurance at age  $(x+t)$  is sold at cost price, and no allowance is made for expenses, nor any provision for future bonuses; so, should the amount of "paid-up" policy be calculated by this formula,\* the policyholder will not be entitled to any further share in the profit, and will only receive that which has already accrued in respect of the former payments of premium, and it will be seen from our examples that, in most of the classes of assurance to which what we have called the "empirical" method is likely to be applied, for calculating the "paid-up" policy, there is no such excess of amount by the real method as to warrant further bonuses being allocated.

The assumption then, that is made in considering the "paid-up" policy the equivalent of the ratio the premiums paid bear to the total number payable, is that the differences in the amounts of "paid-up" policy at each age are equal to each other (formula 3). If this were really the case,  $\frac{1}{n}$  would necessarily represent this equal difference.

But, as experiment will show, this is not so; and the differences on the whole form a series, commencing at some number greater or less than  $\frac{1}{n}$ , and terminating at some number less or greater than it. The less then the difference between these differences and  $\frac{1}{n}$  the nearer will the empirical method be to the actual.

The following examples will serve somewhat to show to what extent the required condition is fulfilled in various kinds of assurances. They have been calculated by the Carlisle and English Life (No. III., Males) Tables, interest being assumed at 3 per cent. Though, as was pointed out by you in your note to Mr. Younger's letter, other tables, such as the "Experience," may give less irregular results, yet, for the present object, viz., the estimating whether "paid-up" policies by this empirical mode will be, in the various kinds of assurance, greater or less than by the true method; any other table adopted will probably show the same *general* results, though the deficit or surplus in any individual case may not bear the same proportion to the correct amount at that age.

It must not, however, be understood that it is asserted that in every possible example, even by the tables adopted, the results will be of the same nature; so that in, say, the Limited Premiums, in every case, the

\* Should the full value of policy not be allowed, but say,  $\frac{1}{f}$  of it be deducted, and if some addition be made to the single payment, say,  $\frac{1}{g}$  of it, then the amount of paid-up

policy to be given would be 
$$\frac{1 - \frac{1}{f}}{1 + \frac{1}{g}} \left( 1 - \frac{P_x}{P_{x+t}} \right).$$

empirical method will give greater amounts than the real method. (We have already proved that theoretically this is not necessarily true.) On the whole, however, the results will be somewhat similar to those given, so that in, say, the "Limited Premiums" Assurance, the empirical will generally exceed the actual. This excess, however, as already shown by Mr. Younger, is not of such extent as to put any serious difficulty in the way of the adoption of the scheme, and the following examples will show that it might even more advantageously to the Offices be extended to Endowment Assurances.

*Endowment—payable at 60.*

Age at which Conversion takes place ( $x+t$ ).	AMOUNT OF PAID-UP POLICY.			EXCESS OVER EMPIRICAL OF		INCREASE IN AMOUNT FROM YEAR TO YEAR $\left(1 - \frac{P_x}{P_{x+t}}\right) - \left(1 - \frac{P_x}{P_{x+t-1}}\right) = \Delta$		$\Delta - \frac{1}{n}$	
	Carlisle.	English.	Empirical.	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
45						·08946	·09151	+·02279	+·02484
46	·08946	·09151	·06667	·02279	·02484	·08556	·08748	+·01889	+·02081
47	·17502	·17899	·13333	·04169	·04566	·08185	·08357	+·01518	+·01690
48	·25687	·26256	·20000	·05687	·06256	·07829	·07978	+·01162	+·01311
49	·33516	·34234	·36667	·06849	·07567	·07496	·07612	+·00829	+·00945
50	·41012	·41846	·33333	·07679	·08513	·07178	·07256	+·00511	+·00589
51	·48190	·49102	·40000	·08190	·09102	·06876	·06913	+·00209	+·00246
52	·55066	·56015	·46667	·08399	·09348	·06579	·06575	-·00088	-·00092
53	·61645	·62590	·53333	·08312	·09257	·06291	·36248	-·00376	-·00419
54	·67936	·68838	·60000	·07936	·08838	·06010	·05932	-·00657	-·00735
55	·73946	·74770	·66667	·07279	·08103	·05735	·05624	-·00932	-·01043
56	·79681	·80394	·73333	·06348	·07061	·05469	·05326	-·01198	-·01341
57	·85150	·85720	·80000	·05150	·05720	·05208	·05037	-·01459	-·01630
58	·90358	·90757	·86667	·03691	·04090	·04951	·04757	-·01716	-·01910
59	·95309	·95514	·93333	·01976	·02181	·04691	·04486	-·01976	-·02181

The premiums for endowments, as the time draws nearer for their enjoyment, increase very rapidly, so that  $P_{x+t}$  is large in proportion to  $P_x$ , and the empirical paid-up endowment is less than the true. At age  $(x+n-1)$ , since the value of an endowment is greater than the premiums paid under it, in the inequality  $(n-1)P_x \sim V_{x|n-1}$  on page 300, we have  $V_{x|n-1}$  the greater, and therefore the amount of paid-up endowment at the age immediately preceding the termination of the original contract is always less by the empirical than by the true method. It generally, though not necessarily, happens, whichever method in most assurance schemes gives the greater amount at age  $(x+n-1)$  will give the greater amount at all other ages as well. Were this universally true, we should have "paid-up" endowments by the empirical method always less than by the correct one; and, indeed, this will generally be found to be the case.

In the foregoing examples  $\left(\Delta \sim \frac{1}{n}\right)$  is always less by the Carlisle than by the English table, and therefore it follows that for endowments payable at 60, the conversion ages ranging from 45 to 59, the empirical method of calculating "paid-up" endowments will give results nearer the Carlisle than the English table.

## Temporary Assurances—till Age 60.

Age at which Conversion takes place ( $x+t$ ).	AMOUNT OF PAID-UP POLICY.			DEFICIT UNDER EMPIRICAL OF		INCREASE IN AMOUNT FROM YEAR TO YEAR		$\Delta - \frac{1}{n}$ .	
	Carlisle.	English.	Empirical.	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
45						·01029	·02531	-·05638	-·04136
46	·01029	·02531	·06667	·05638	·04136	·01163	·02514	-·05504	-·04153
47	·02192	·05045	·13333	·11141	·08288	·01460	·02499	-·05207	-·04168
48	·03652	·07544	·20000	·16348	·12456	·02101	·02489	-·04566	-·04178
49	·05753	·10033	·26667	·20914	·16634	·02579	·02489	-·04088	-·04178
50	·08332	·12522	·33333	·25001	·20811	·03155	·02502	-·03512	-·04165
51	·11487	·15024	·40000	·28513	·24976	·03087	·02249	-·03580	-·04418
52	·14574	·17273	·46667	·32093	·29394	·03051	·02192	-·03616	-·04475
53	·17625	·19465	·53333	·35708	·33868	·03063	·02154	-·03604	-·04513
54	·20688	·21619	·60000	·39312	·38381	·03344	·02123	-·03323	-·04544
55	·24032	·23742	·66667	·42635	·42925	·03633	·02102	-·03034	-·04565
56	·27665	·25844	·73333	·45668	·47489	·04265	·02089	-·02402	-·04578
57	·31930	·27933	·80000	·48070	·52067	·04821	·02105	-·01846	-·04562
58	·36751	·30038	·86667	·49916	·56629	·04673	·02026	-·01994	-·04641
59	·41424	·32064	·93333	·51909	·61269	·58576	·67936	+·51909	+·61269

The premiums for Temporary Assurances increase very slowly, so  $P_{x+t}$  is very little larger than  $P_x$ , and consequently the empirical "paid-up" temporary assurances are greatly in excess of the true amounts—so much so, indeed, as to make this method of calculating the "paid-up" policy quite inapplicable for this class of assurance. In the foregoing  $\left(\Delta \sim \frac{1}{n}\right)$  is at first much smaller by the English than by the Carlisle tables,

but latterly it is the reverse, so the empirical method of calculating "paid-up" temporary assurances for ages between 45 and 59, and terminating at 60, will give towards the beginning results nearer the English Life, and towards the end, nearer the Carlisle, table.

## Endowment Assurance—payable at 60 or Death.

Age at which Conversion takes place ( $x+t$ ).	AMOUNT OF PAID-UP POLICY.			EXCESS OVER EMPIRICAL OF		INCREASE IN AMOUNT FROM YEAR TO YEAR		$\Delta - \frac{1}{n}$ .	
	Carlisle.	English.	Empirical.	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
45						·06954	·07101	+·00287	+·00434
46	·06954	·07101	·06667	·00287	·00434	·06943	·07031	+·00276	+·00364
47	·13897	·14132	·13333	·00564	·00799	·06957	·06940	+·00290	+·00273
48	·20854	·21072	·20000	·00854	·01072	·06985	·06889	+·00318	+·00222
49	·27839	·27961	·26667	·01172	·01294	·06970	·06808	+·00303	+·00141
50	·34809	·34769	·33333	·01476	·01436	·06936	·06738	+·00269	+·00071
51	·41745	·41507	·40000	·01745	·01507	·06819	·06653	+·00152	-·00014
52	·48564	·48160	·46667	·01897	·01493	·06700	·06590	+·00033	-·00077
53	·55264	·54750	·53333	·01931	·01417	·06613	·06543	-·00054	-·00124
54	·61877	·61293	·60000	·01877	·01293	·06531	·06506	-·00136	-·00161
55	·68408	·67799	·66667	·01741	·01132	·06451	·06471	-·00216	-·00196
56	·74859	·74270	·73333	·01526	·00937	·06380	·06448	-·00287	-·00219
57	·81239	·80718	·80000	·01239	·00718	·06306	·06431	-·00361	-·00236
58	·87545	·87149	·86667	·00878	·00482	·06240	·06424	-·00427	-·00243
59	·93785	·93573	·93333	·00452	·00240	·06215	·06427	-·00452	-·00240

In Endowment Assurance premiums (they being made up of endowment premiums, which, as we have seen, give much greater "paid-up" policies by direct calculation than by the empirical method, and of temporary assurance premiums which give much less by the former than by the latter) the amounts will lie between these two extremes, and, as the examples will show, are, on the whole, more favourable to the Insurance Company adopting this new method of finding the "paid-up" policy than are those in the case of limited premium assurances.

*Assurances for the Whole of Life—by equal Annual Premiums, payable till Death occur.*

NOTE.— $n$  in this case will be the difference between the limiting age in the Table, or the year which the last survivor enters upon, but fails to complete, and the age at entry.

$n = \omega + 1 - x =$  in Carlisle Table 105 -  $x$ , and in English (Males) = 108 -  $x$ .

Age at which Conversion takes place ( $x+t$ ).	AMOUNT OF PAID-UP POLICY.				EXCESS OVER EMPIRICAL OF		INCREASE IN AMOUNT FROM YEAR TO YEAR		$\Delta - \frac{1}{n}$	
	Carlisle.		English.				$\left(1 - \frac{P_x}{P_{x+t}}\right) - \left(1 - \frac{P_x}{P_{x+t-1}}\right) = \Delta$ .			
	True.	Empirical.	True.	Empirical.	Carlisle.	English.	Carlisle.	English.		Carlisle.
45							03195	03540	+01528	+01528
46	03195	01667	03540	01587	01528	01953	03233	03479	+01566	+01566
47	06428	03333	07019	03175	03095	03844	03308	03415	+01641	+01641
48	09736	05000	10434	04762	04736	05672	03448	03355	+01781	+01781
49	13184	06667	13789	06349	06517	07440	03513	03294	+01846	+01846
50	16697	08333	17083	07937	08364	09146	03570	03234	+01903	+01903
51	20267	10000	20317	09524	10267	10793	03467	03105	+01800	+01800
52	23734	11667	23422	11111	12067	12311	03371	03034	+01704	+01704
53	27105	13333	26456	12698	13772	13758	03281	02970	+01614	+01614
54	30386	15000	29426	14286	15386	15140	03225	02908	+01558	+01558
55	33611	16667	32334	15873	16944	16461	03145	02845	+01478	+01478
56	36756	18333	35179	17460	18423	17719	03072	02784	+01405	+01405
57	39828	20000	37963	19048	19828	18915	02935	02720	+01268	+01268
58	42763	21667	40683	20635	21096	20048	02694	02653	+01027	+01027
59	45457	23333	43336	22222	22124	21114	02422	02583	+00755	+00755
60	47879	25000	45919	23810	22879	22109	.....	.....	.....	.....
.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
104	96892	98333	.....	.....	-01441	.....	03108	.....	+01441	+01441
.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
107	.....	.....	96567	98413	.....	-01846	.....	03433	.....	+01846

In the formula  $\left(1 - \frac{P_x}{P_{x+t}}\right) \sim \frac{t}{n}$ , should  $(n-t)$  be a very large quantity, as the probability of the life reaching the older ages has very little effect on the quantity  $\frac{P_x}{P_{x+t}}$ , which would remain at nearly the same value were the oldest age in the tables much less than it is, while, on the other hand, as the probability of reaching every age is assumed the same in the quantity  $\frac{t}{n}$ , if  $n$  be very large compared with  $t$ ,  $\frac{t}{n}$  will be a very small fraction,

and much less than  $\left(1 - \frac{P_x}{P_{x+t}}\right)$ , and consequently at the younger ages, and indeed at all the ages likely to occur in practice, the "paid-up" policy by the empirical method, to an assurer on this system, will be very much less than the correct amount.

In course of time, however the excess diminishes, and latterly turns the other way. Thus, at age  $(x+n-1)$ , or the oldest age in the table, the expression is  $\left(1 - \frac{P_x}{v}\right) \sim \frac{n-1}{n}$  or  $v \sim nP_x$ , of which the latter term, which corresponds to the result by the empirical method, is the greater.

The quantity  $\left(\Delta - \frac{1}{n}\right)$  is in the above examples at first less and afterwards greater by the Carlisle than by the English table, and therefore assurers for the whole of life, by equal annual premiums, will, should they between ages 45 and 59 change their policies into "paid-up" ones, get by the empirical method, at first, results nearer the Carlisle, and, afterwards, nearer the English tables.

*Assurances for the Whole of Life—by 5 Payments.*

Age at which Conversion takes place $(x+t)$ .	AMOUNT OF PAID-UP POLICY.			DEFICIT UNDER EMPIRICAL OF		INCREASE IN AMOUNT FROM YEAR TO YEAR $\left(1 - \frac{P_x}{P_{x+t}}\right) - \left(1 - \frac{P_x}{P_{x+t-1}}\right) = \Delta$ .		$\Delta - \frac{1}{n}$ .	
	Carlisle.	English.	Empirical.	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
30									
31	·19657	·19745	·20000	·00343	·00255	·19657	·19745	-·00343	-·00255
32	·39484	·39607	·40000	·00516	·00393	·19827	·19862	-·00173	-·00138
33	·59499	·59595	·60000	·00501	·00405	·24015	·19988	+·00015	-·00012
34	·79679	·79722	·80000	·00321	·00278	·20180	·20127	+·00180	+·00127
						·20321	·20278	+·00321	+·00278

*Assurances for the Whole of Life—by 10 Payments.*

Age at which Conversion takes place $(x+t)$ .	AMOUNT OF PAID-UP POLICY.			DEFICIT UNDER EMPIRICAL OF		INCREASE IN AMOUNT FROM YEAR TO YEAR $\left(1 - \frac{P_x}{P_{x+t}}\right) - \left(1 - \frac{P_x}{P_{x+t-1}}\right) = \Delta$ .		$\Delta - \frac{1}{n}$ .	
	Carlisle.	English.	Empirical.	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
30									
31	·09677	·09792	·10000	·00323	·00208	·09677	·09792	-·00323	-·00208
32	·19425	·19618	·20000	·00575	·00382	·09748	·09826	-·00252	-·00174
33	·29275	·29481	·30000	·00725	·00519	·09850	·09863	-·00150	-·00137
34	·39216	·39385	·40000	·00784	·00615	·09941	·09904	-·00059	-·00096
35	·49214	·49334	·50000	·00786	·00666	·09998	·09949	-·00002	-·00051
36	·59268	·59335	·60000	·00732	·00665	·10094	·10001	+·00054	+·00001
37	·69365	·69394	·70000	·00635	·00606	·10097	·10059	+·00097	+·00059
38	·79512	·79518	·80000	·00488	·00482	·10147	·10124	+·00147	+·00124
39	·89721	·89717	·90000	·00279	·00283	·10209	·10199	+·00209	+·00199
						·10279	·10283	+·00279	+·00283

*Assurances for the Whole of Life—by 15 Payments.*

Age at which Conversion takes place ( $x+t$ ).	AMOUNT OF PAID-UP POLICY.			DEFICIT UNDER EMPERICAL OF		INCREASE IN AMOUNT FROM YEAR TO YEAR		$\Delta - \frac{1}{s}$ .	
	Carlisle.	English.	Empirical.	Carlisle.	English.	$(1 - \frac{P_x}{P_{x+t}}) - (1 - \frac{P_x}{P_{x+t-1}}) = \Delta$ .		Carlisle.	English.
30						-06416	-06539	-00251	-00120
31	-06416	-06539	-06667	-00251	-00128	-06456	-06546	-00211	-00121
32	-12872	-13085	-13333	-00461	-00248	-06529	-06553	-00138	-00114
33	-19401	-19638	-20000	-00599	-00362	-06595	-06562	-00072	-00105
34	-25996	-26200	-26667	-00671	-00467	-06625	-06574	-00042	-00093
35	-32621	-32774	-33333	-00712	-00559	-06655	-06588	-00012	-00079
36	-39276	-39362	-40000	-00724	-00638	-06660	-06603	-00007	-00064
37	-45936	-45965	-46667	-00731	-00702	-06671	-06624	+00004	-00043
38	-52607	-52589	-53333	-00726	-00744	-06688	-06649	+00021	-00018
39	-59295	-59238	-60000	-00705	-00762	-06683	-06678	+00016	+00011
40	-65978	-65916	-66667	-00689	-00751	-06674	-06715	+00007	+00048
41	-72652	-72631	-73333	-00681	-00702	-06716	-06756	+00049	+00089
42	-79368	-79387	-80000	-00632	-00613	-06785	-06808	+00118	+00141
43	-86153	-86195	-86667	-00514	-00472	-06877	-06867	+00210	+00200
44	-93030	-93062	-93333	-00303	-00271	-06970	-06938	+00303	+00271

For the foregoing examples, then, the “paid-up” policy in this class of assurance is a little greater by the empirical than by the correct method—and other examples would have shown that, generally speaking, this will be the case.

The difference, however, is so small as not to render it at all hazardous for the Insurance Companies to grant “paid-up” policies calculated in this manner.

It will be observed from the column  $(\Delta - \frac{1}{n})$ , in the preceding cases,

that the “English” table gives results, on the whole, nearer those by the empirical method than does the “Carlisle” for these ages.

Finally, it has been shown that, in order that the ratio of premiums paid to the total number payable may express the correct amount of “paid-up” policy on the original status, it is necessary that the premiums that would require to be paid by persons entering at every succeeding age from  $x$  to  $(x+n)$ , to place them in exactly the same position as that then held by the original assurer, form a series in harmonical progression, and that, as  $P_x$  and  $P_{x+t}$  may be said to be independent of each other, it is not possible to prove in a general form whether the “paid-up” policy will be greater or less by using this mode of calculation than the correct amount; but that, on the whole, by this method, the “paid-up” policy granted would be much too large in the case of temporary assurances, and much too small for assurances by premiums payable till death and for endowments, the variation being so great as to render it inapplicable for any of these classes; and that for endowment assurances and policies by limited premiums, should the number payable, after change in the policy, not be very great, it will give results very close to the truth; in the first case, perhaps a little favourable to the Company; in the second, perhaps a little against it; but that, under all ordinary circumstances, this difference is so small as to permit Offices to adopt the system with perfect safety.

I am, Sir, your obedient servant,

*City of Glasgow Life Assurance Company,  
Glasgow, 10th July, 1869.*

JAMES R. MACFADYEN.

ON MR. WOOLHOUSE'S IMPROVED THEORY OF ANNUITIES  
AND ASSURANCES.

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—In Mr. Woolhouse's very able paper on the continuous method of obtaining the values of annuities, assurances, etc., a general formula for payments by instalments is demonstrated, of which considerable use is afterwards made. Let  $V_x$  denote a function depending for its value upon that of the variable quantity  $x$ , and let  $x$  denote an interval of time measured from some given epoch; then Mr. Woolhouse shows that

$$\begin{aligned} & \frac{1}{m}(V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} + \dots + V_\omega) \\ &= \int_0^\omega V dx + \frac{1}{2m}(V_0 + V_\omega) - \frac{1}{12m^2} \left\{ \left( \frac{dV}{dx} \right)_0 - \left( \frac{dV}{dx} \right)_\omega \right\} \\ & \quad + \frac{1}{720m^4} \left\{ \left( \frac{d^3V}{dx^3} \right)_0 - \left( \frac{d^3V}{dx^3} \right)_\omega \right\} - \dots \end{aligned}$$

Now let  $\frac{1}{m}(V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} + \dots + V_\omega)$  be represented by  $\Sigma^{(m)}V$ ; it is then shown that

$$\Sigma^{(m)}V = \Sigma^{(1)}V - \frac{m-1}{2m}V + \frac{m^2-1}{12m^2} \frac{dV}{dx} - \frac{m^4-1}{720m^4} \frac{d^3V}{dx^3} + \&c. \quad (1)$$

where  $V$ ,  $\frac{dV}{dx}$ ,  $\frac{d^3V}{dx^3}$ , etc. are all initial values, since in all matters connected with annuities and assurances  $V_\omega$ ,  $\left( \frac{dV}{dx} \right)_\omega$ ,  $\left( \frac{d^3V}{dx^3} \right)_\omega$  etc. vanish.

The formula (1) for payments by instalments is obtained by Mr. Woolhouse in a very simple and elementary manner; but, as it promises to become of great importance in the theory of life assurance, I have ventured to send you another demonstration of it, mainly due to Sir John Lubbock.

That gentleman, in a paper on annuities reprinted in the fifth volume of the *Journal of the Institute*, shows (p. 277) that if  $y$  be any variable, and  $y_0, y_1, y_2, \dots, y_{ni}, y_{(n+1)i}, \dots, y_{mni-i} (ni=1)$  its successive values, then

$$\begin{aligned} & y_0 + y_i + y_{2i} + \dots + y_{ni} + y_{(n+1)i} + \dots + y_{(mni-i)} \\ &= n(y_0 + y_1 + \&c. \dots + y_{m-1}) + \frac{n-1}{2}(y_m - y_0) - \frac{n^2-1}{12n}(\Delta y_m - \Delta y_0) \\ & \quad + \frac{n^2-1}{24n}(\Delta^2 y_m - \Delta^2 y_0) - \&c. \dots \dots \dots (2) \end{aligned}$$

where the coefficient of  $\Delta^2 y_m - \Delta^2 y_0$  is equal to the coefficient of  $x^{n-1}$  in the development of  $\frac{(1+x)^n - (1+x)}{1 - (1+x)^i}$ . This result is obtained, like

Mr. Woolhouse's, almost from first principles. If, now, we go another step, and calculate the coefficient of  $\Delta^3 y_m - \Delta^3 y_0$ , we find it to be  $-\frac{19n^4 - 20n^2 + 1}{720m^3}$ ; and since in life assurance calculations terminal values vanish, i.e.  $y_m = 0$ ,  $\Delta y_m = 0$ ,  $\Delta^2 y_m = 0$ , etc., the right-hand side of (2) becomes

$$= n(y_0 + y_1 + \dots + y_{m-1}) - \frac{n-1}{2} y_0 + \frac{n^2-1}{12n} \Delta y_0 \\ - \frac{n^3-1}{24n} \Delta^2 y_0 + \frac{19n^4-20n^2+1}{720n^3} \Delta^3 y_0 - \&c.$$

In this expression write  $(\epsilon^{\frac{d}{dx}} - 1)y_0$  for  $\Delta y_0$ ,  $(\epsilon^{\frac{d}{dx}} - 1)^2 y_0$  for  $\Delta^2 y_0$ , &c., the symbols of operation and quantity being separated. On doing this, expanding  $(\epsilon^{\frac{d}{dx}} - 1)y_0$ ,  $(\epsilon^{\frac{d}{dx}} - 1)^2 y_0$ ,  $(\epsilon^{\frac{d}{dx}} - 1)^3 y_0$ , &c., and collecting coefficients, it will be found that the coefficient of  $\frac{d^2 y_0}{dx^2}$  vanishes, and that (3) now becomes

$$= n(y_0 + y_1 + \dots + y_{m-1}) - \frac{n-1}{2} y_0 + \frac{n^2-1}{12n} \frac{dy_0}{dx} \\ + \frac{10n^4-10n^2-30n^4+30n^2+19n^4-20n^2+1}{720n^3} \frac{d^3 y_0}{dx^3} + \&c.$$

that is,  $y_0 + y_1 + y_2 + \&c. \dots + y_{ni} + y_{(n+1)i} + \dots + y_{(mni-i)} (ni=1)$

$$= n(y_0 + y_1 + \&c. \dots + y_{m-1}) - \frac{n-1}{2} y_0 + \frac{n^2-1}{12n} \frac{dy_0}{dx} \\ - \frac{n^4-1}{720n^3} \frac{d^3 y_0}{dx^3} + \&c.$$

or, dividing both sides by  $n$ ,

$$\frac{1}{n} \{y_0 + y_1 + y_2 + \&c. \dots + y_{ni} + y_{(n+1)i} + \dots + y_{(mni-i)}\} \\ = (y_0 + y_1 + \&c. \dots + y_{m-i}) - \frac{n-1}{2n} y_0 + \frac{n^2-1}{12n^2} \frac{dy_0}{dx} \\ - \frac{n^4-1}{720n^4} \frac{d^3 y_0}{dx^3} + \&c.$$

$$\text{or} \quad \Sigma^{(n)} y = \Sigma^{(1)} y - \frac{n-1}{2n} y_0 + \frac{n^2-1}{12n^2} \frac{dy_0}{dx} - \frac{n^4-1}{720n^4} \frac{d^3 y_0}{dx^3} + \&c.$$

which is Mr. Woolhouse's formula.

On my showing the above demonstration to Mr. Sprague, he remarked that it had occurred to him, that the method of separation of symbols might be made of more direct use in proving Mr. Woolhouse's formula. Acting upon that suggestion, I have obtained the following more direct proof, which will doubtless be of interest to some readers of the *Journal*.

Let  $u_x$  denote a function of  $x$ , and let  $D$  be put for  $1 + \Delta$ ; then, separating the symbols of operation and quantity,

$$D^n u_x = (1 + \Delta)^n u_x = u_{x+n} :$$

Thus we have

$$u_0 + \frac{u_1}{m} + \frac{u_2}{m} + \dots + \frac{u_{m-1}}{m} = (1 + D^{\frac{1}{m}} + D^{\frac{2}{m}} + \dots + D^{\frac{m-1}{m}}) u_0 \\ = \frac{1 - D^{\frac{m}{m}}}{1 - D^{\frac{1}{m}}} \cdot u_0 \\ = \frac{1}{1 - D^{\frac{1}{m}}} \cdot u_0$$

(since in life assurance calculations  $u_{\infty}=0$ )

$$\begin{aligned}
 &= -\frac{1}{\frac{1}{D^m}-1} \cdot u_0 \\
 &= (D^{\frac{1}{m}}-1)^{-1} u_0 \\
 &= (\epsilon^{\frac{d}{mdx}}-1)^{-1} u_0 \quad (\text{since } D=\epsilon^{\frac{d}{dx}}) \\
 &= m \int_0^{\infty} u_x dx + \frac{1}{2} u_0 - \frac{1}{12m} \left( \frac{du_x}{dx} \right)_0 + \frac{1}{720m^3} \left( \frac{d^3u_x}{dx^3} \right)_0 - \&c.
 \end{aligned}$$

$$\begin{aligned}
 \therefore \Sigma^{(m)}u &= \frac{1}{m} (u_0 + \frac{u_1}{m} + \frac{u_2}{m} + \&c. \dots + u_{\infty} - \frac{1}{m}) \\
 &= \int_0^{\infty} u_x dx + \frac{1}{2m} u_0 - \frac{1}{12m^2} \left( \frac{du_x}{dx} \right)_0 + \frac{1}{720m^4} \left( \frac{d^3u_x}{dx^3} \right)_0 - \&c.
 \end{aligned}$$

$$\text{But } \Sigma^{(1)}u = \int_0^{\infty} u_x dx + \frac{1}{2} u_0 - \frac{1}{12} \left( \frac{du_x}{dx} \right)_0 + \frac{1}{720} \left( \frac{d^3u_x}{dx^3} \right)_0 - \&c.$$

$\therefore$  subtracting,

$$\Sigma^{(m)}u - \Sigma^{(1)}u = -\frac{m-1}{2m} u_0 + \frac{m^2-1}{12m^2} \left( \frac{du_x}{dx} \right)_0 - \frac{m^4-1}{720m^4} \left( \frac{d^3u_x}{dx^3} \right)_0 + \&c.$$

or,

$$\Sigma^{(m)}u = \Sigma^{(1)}u - \frac{m-1}{2m} u_0 + \frac{m^2-1}{12m^2} \left( \frac{du_x}{dx} \right)_0 - \frac{m^4-1}{720m^4} \left( \frac{d^3u_x}{dx^3} \right)_0 + \&c.$$

This result will appear pretty obvious from the consideration that the  $x$ th payment in the one case will coincide with the  $m$ xth in the other; and so by putting  $mdx$  for  $dx$ , and dividing by  $m$ , in the expression for  $u_0 + u_1 + \dots + u_{\infty-1}$ , we shall obtain the value of

$$\frac{1}{m} (u_0 + \frac{u_1}{m} + \frac{u_2}{m} + \dots + u_{\infty} - \frac{1}{m}).$$

Mr. Woolhouse has, in his paper, deduced the values of annuities and assurances, when continuous. It is interesting in some respects to see how these results can be obtained from the corresponding values for yearly payments. Take the expression for  $A_x$ ; and we have  $A_x = \frac{1-ia_x}{1+i}$ , where  $i$  is the rate of interest, payable yearly, and  $a_x$  the present value of an annuity for the life of a person aged  $x$  years. This may be written

$$A_x = \frac{1}{1+i} (1-ia_x)$$

$$= \left\{ \begin{array}{l} \text{Present value of } \pounds 1 \text{ due} \\ \text{a year hence} \end{array} \right\} \times \left\{ \begin{array}{l} \pounds 1 - (\text{interest on } \pounds 1 \text{ for} \\ \text{a year}) \times a_x \end{array} \right\}$$

Now change years into moments: then  $i$  becomes  $\delta$ ,  $A_x$  becomes  $\bar{A}_x$ , and  $a_x$  becomes  $\bar{a}_x$ : and  $\bar{A}_x = 1 - \delta \bar{a}_x$ , since value of  $\pounds 1$  due instantly is  $\pounds 1$ .

(See Mr. Woolhouse's paper, page 115, in the July number of the *Journal*).

Again, the ordinary value of an assurance on the joint lives of  $x$  and  $y$  is given by the equation

$$\begin{aligned} A_{xy} &= \frac{l_{xy} - l_{x+1, y+1}}{l_{xy}} v + \frac{l_{x+1, y+1} - l_{x+2, y+2}}{l_{xy}} v^2 + \&c. \\ &= \frac{l_{xy}}{l_{xy}} v + \frac{l_{x+1, y+1}}{l_{xy}} v^2 + \&c. - \left\{ \frac{l_{x+1, y+1}}{l_{xy}} v + \frac{l_{x+2, y+2}}{l_{xy}} v^2 + \&c. \right\} \\ &= \frac{l_{x-1, y-1}}{l_{xy}} \left\{ \frac{l_{xy}}{l_{x-1, y-1}} v + \frac{l_{x+1, y+1}}{l_{x-1, y-1}} v^2 + \&c. \right\} \\ &\quad - \left\{ \frac{l_{x+1, y+1}}{l_{xy}} v + \frac{l_{x+2, y+2}}{l_{xy}} v^2 + \&c. \right\} \\ &= \frac{l_{x-1, y-1}}{l_{xy}} a_{x-1, y-1} - a_{xy} \end{aligned}$$

Now change the interval of time from a year to  $\tau$ , where  $m\tau$  = one year: then, instead of an assurance for £1, the above expression will give us the value of an assurance of £ $\tau$ , payable at the end of the interval  $\tau$  in which the life fails, so that we get, using the common notation,

$$\begin{aligned} \tau A_{xy}^{(m)} &= \frac{l_{x-\tau, y-\tau}}{l_{xy}} a_{x-\tau, y-\tau}^{(m)} - a_{xy}^{(m)} \\ &= \frac{l_{xy} - \Delta l_{x-\tau, y-\tau}}{l_{xy}} a_{x-\tau, y-\tau}^{(m)} - a_{xy}^{(m)} \\ &= a_{x-\tau, y-\tau}^{(m)} - \frac{\Delta l_{x-\tau, y-\tau}}{l_{xy}} a_{x-\tau, y-\tau}^{(m)} \\ \therefore A_{xy}^{(m)} &= \frac{a_{x-\tau, y-\tau}^{(m)} - a_{xy}^{(m)}}{\tau} - \frac{1}{l_{xy}} \cdot \frac{\Delta l_{x-\tau, y-\tau}}{\tau} a_{x-\tau, y-\tau}^{(m)}. \end{aligned}$$

Now diminish  $\tau$  indefinitely, and we have

$$\begin{aligned} \bar{A}_{xy} &= -\frac{d\bar{a}_{xy}}{dt} + \mu_{xy} \bar{a}_{xy} \\ &= \mu_{xy} \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dx} - \frac{d\bar{a}_{xy}}{dy}. \end{aligned}$$

From the value just obtained for  $\bar{A}_{xy}$ , we can obtain the value of  $\bar{A}_1$  in a similar way.

By the ordinary method,

$$A_1 = \frac{1}{2} \left\{ A_{xy} + \frac{a_{x-1, y}}{p_{x-1, 1}} - \frac{a_{x, y-1}}{p_{y-1, 1}} \right\}.$$

The way in which this is obtained is well known, and will be seen to apply when the interval of time is  $\tau$ , instead of a year. We shall thus have

$$\begin{aligned}
 \tau A_{xy}^{(m)} &= \frac{1}{2} \left\{ \tau A_{xy}^{(m)} + \frac{a_{x-\tau, y}^{(m)}}{p_{x-\tau, \tau}} - \frac{a_{x, y-\tau}^{(m)}}{p_{y-\tau, \tau}} \right\} \\
 &= \frac{1}{2} \left\{ \tau A_{xy}^{(m)} + \frac{l_x - \Delta l_{x-\tau}}{l_x} a_{x-\tau, y}^{(m)} - \frac{l_y - \Delta l_{y-\tau}}{l_y} a_{x, y-\tau}^{(m)} \right\} \\
 &= \frac{1}{2} \left\{ \tau A_{xy}^{(m)} + a_{x-\tau, y}^{(m)} - a_{xy}^{(m)} - \frac{\Delta l_{x-\tau}}{l_x} a_{x-\tau, y}^{(m)} - \left( a_{x, y-\tau}^{(m)} - a_{xy}^{(m)} \right) \right. \\
 &\quad \left. + \frac{\Delta l_{y-\tau}}{l_y} \cdot a_{x, y-\tau}^{(m)} \right\} \\
 \therefore A_{xy}^{(m)} &= \frac{1}{2} \left\{ A_{xy}^{(m)} + \frac{a_{x-\tau, y}^{(m)} - a_{xy}^{(m)}}{\tau} - \frac{1}{l_x} \cdot \frac{\Delta l_{x-\tau}}{\tau} \cdot a_{x-\tau, y}^{(m)} - \frac{a_{x, y-\tau}^{(m)} - a_{xy}^{(m)}}{\tau} \right. \\
 &\quad \left. + \frac{1}{l_y} \cdot \frac{\Delta l_{y-\tau}}{\tau} a_{x, y-\tau}^{(m)} \right\}
 \end{aligned}$$

Now, as before, diminish  $\tau$  indefinitely, and we get

$$\bar{A}_{xy} = \frac{1}{2} \left\{ \bar{A}_{xy} - \frac{d\bar{a}_{xy}}{dx} + \mu_x \cdot \bar{a}_{xy} + \frac{d\bar{a}_{xy}}{dy} - \mu_y \cdot \bar{a}_{xy} \right\},$$

which, after substituting for  $\bar{A}_{xy}$  from the equation

$$\bar{A}_{xy} = \mu_{xy} \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dx} - \frac{d\bar{a}_{xy}}{dy},$$

reduces to

$$\begin{aligned}
 \bar{A}_{xy} &= \frac{1}{2} \left\{ 2\mu_x \bar{a}_{xy} - 2 \frac{d\bar{a}_{xy}}{dx} \right\}, \\
 &= \mu_x \bar{a}_{xy} - \frac{d\bar{a}_{xy}}{dx}.
 \end{aligned}$$

which is Mr. Woolhouse's result.

Now, approximately,  $\bar{a}_{xy} = a_{xy} + \frac{1}{2}$ , and  $-\frac{d\bar{a}_{xy}}{dx} = \frac{1}{2}(a_{x-1, y} - a_{x+1, y})$ .

Making these substitutions, we get

$$\bar{A}_{xy} = \mu_x(a_{xy} + \frac{1}{2}) + \frac{1}{2}(a_{x-1, y} - a_{x+1, y}),$$

the very convenient working formula proposed by Mr. Woolhouse.

These demonstrations, although of no intrinsic value, are not, I venture to think, without interest, as showing the connection between the ordinary and continuous methods of obtaining annuities and assurances.

I am, Sir,

Your most obedient servant,

December 20th, 1869.

WILLIAM SUTTON.

P.S.—There is a misprint in Mr. Woolhouse's paper: the expressions on the right-hand side of equations (35) are the approximate values of  $-\frac{d\bar{a}_{xy}}{dx}$ ,  $-\frac{d\bar{a}_{xy}}{dy}$ ,  $-\frac{d\bar{a}_{xy}}{dt}$ , respectively, and not, as printed, of  $\frac{d\bar{a}_{xy}}{dx}$ ,  $\frac{d\bar{a}_{xy}}{dy}$ ,  $\frac{d\bar{a}_{xy}}{dt}$ .

## ON BRIGGS'S FORMULA FOR INTERPOLATION.

*To the Editor of the Assurance Magazine.*

Sir,—I read with very great interest the translation given in vol. xiv., p. 73, of Briggs's process of interpolation, and have to thank you and Mr. Williams for directing attention to an author and a method which had been almost entirely forgotten. It appears to me that Briggs's method is not only theoretically interesting, but that it may in calculating a table of successive values of a function prove practically useful, and possibly be found superior to any of the methods recommended by modern authors.

The law which the multipliers used in the process follow is not stated by Briggs, and is not obvious at first sight. After a few trials, I ascertained that it is as follows:—Referring to the Table X., on p. 79, I extract the following line—

$$8 \parallel 8(10) | 29\cdot6(12) | 67\cdot2(14) | 104\cdot72(16) | 118\cdot72(18) | 101\cdot248(20)$$

Here the coefficients are

$$8 \quad 29\cdot6 \quad 67\cdot2 \quad 104\cdot72 \quad 118\cdot72 \quad 101\cdot248,$$

and it will be found that these are the coefficients of  $x$ ,  $x^2$ ,  $x^3$ ,  $x^4$ ,  $x^5$ , and  $x^6$ , in the expansion of  $(1+x+2x^2)^8$ ; and, in general, that for any index  $n$ , the coefficients are those of  $x$ ,  $x^2$ ,  $x^3$ , . . . . ., in the expansion of  $(1+x+2x^2)^n$ . Briggs supposes, in his example, that all the differences after the 20th may be neglected, and therefore gives only the coefficients necessary for this case; but if higher differences are taken into account, then for indices greater than 4, other terms must be added to those given in his Table X.; and the coefficients of these terms, as well as the whole series of coefficients for indices greater than 18, may be found by the above rule.

In verifying this rule, I found that for the index 5, the coefficient of (19) should be  $\cdot560$ , but it is printed in the *Journal* as  $\cdot500$ . This led me to refer to the copy of Briggs's *Arithmetica Logarithmica*, in the Royal Library at Copenhagen, and I then found that your misprint, if indeed it may be called such, is very excusable. For, knowing that the coefficient ought to be  $\cdot560$ , I was able to see that the head of the 6 had been broken off; but if I had not had that knowledge, I should certainly have read it, as Mr. Williams has done, as 0. In the *Trigonometria Britannica*, p. 38, the figure is printed correctly.

There is a second misprint in the table X., as printed in the *Journal*, which was discovered by Mr. Sprague, while assisting me in the composition of this letter. The coefficient of (20), which is given correctly above as 101·248, is printed as 111·248 in Mr. Williams's translation.

I am, Sir,

Your obedient servant,

LUDVIG OPPERMANN.

London, 28th December, 1869.

# JOURNAL

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## INSTITUTE OF ACTUARIES

AND

### ASSURANCE MAGAZINE.

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*On General Numerical Solution.*

By W. S. B. WOOLHOUSE, F.R.A.S., &c.

[Reprinted from the "*Transactions of the London Mathematical Society*," with additions.]

THE ultimate object of the following paper is to elicit a practical and efficient method of resolving, numerically, problems of every description involving an unknown quantity. In order to accomplish this end with the utmost generality, the most important step is to determine the inversion of a given function in a suitable and convenient series, involving the differential coefficients which appertain to an approximate value of the variable.

An approximate solution being taken as the basis or origin, let the proposed function, when developed by Taylor's Theorem, be

$$U = U_0 + U'_0 x + U''_0 \cdot \frac{x^2}{2} + U'''_0 \cdot \frac{x^3}{2 \cdot 3} + \dots \quad (A),$$

where  $U$  is required to have a given value, and  $U_0$  and the differential coefficients  $U'_0$ ,  $U''_0$ ,  $U'''_0$ , ... are known from the form of  $U$ .

Then, if  $z = \frac{U - U_0}{U'_0}$ ;  $c_2 = \frac{U''_0}{2U'_0}$ ,  $c_3 = \frac{U'''_0}{3U'^2_0}$ , ...

(A) will become, simply,

$$z = x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots \quad (B);$$

and, according to the well-known formula for the reversion of an algebraic series, the evolved value of  $x$  will be expressed as follows:\*

$$\begin{aligned}
 x = & z - c_1 \cdot z^2 + (2c_2^2 - c_3) \cdot z^3 + (-5c_3^2 + 5c_2 \cdot c_3 - c_4) \cdot z^4 \\
 & + (14c_4^2 - 21c_2^2 c_3 + 6c_2 \cdot c_4 + 3c_3^2 - c_5) \cdot z^5 \\
 & + \{-42c_4^2 + 84c_2^2 c_3 - 28(c_2^2 c_4 + c_2 \cdot c_3^2) + 7(c_2 \cdot c_5 + c_3 \cdot c_4) - c_6\} \cdot z^6 \\
 & + \{132c_4^2 - 330c_2^2 c_3 + 60(3c_2^2 c_3^2 + 2c_2^2 c_4) \\
 & \quad - 12(c_3^2 + 3c_2^2 c_3 + 6c_2 \cdot c_3 \cdot c_4) + 8(c_2 \cdot c_6 + c_3 \cdot c_5) + 4c_4^2 - c_7\} \cdot z^7 \\
 & + \{-429c_4^2 + 1287c_2^2 c_3 - 495(2c_2^2 c_3^2 + c_2^2 c_4) \\
 & \quad + 165(c_2 \cdot c_3^2 + 3c_2^2 c_3 \cdot c_4 + c_2^2 c_5) \\
 & \quad - 45(c_2 \cdot c_4^2 + 2c_2 \cdot c_3 \cdot c_5 + c_2^2 c_6 + c_3^2 c_4) \\
 & \quad + 9(c_2 \cdot c_7 + c_3 \cdot c_6 + c_4 \cdot c_5) - c_8\} \cdot z^8 \\
 & \quad \&c. \qquad \qquad \&c. \qquad \qquad \&c. \quad \dots \quad (C).
 \end{aligned}$$

The reversion of the series in this form is, however, inconveniently diffuse. Its terms consist, not merely of an infinite series, but of an unlimited number of infinite series, determined by the varied algebraic combinations of the original coefficients contained in (B). By partitioning off the different series in such manner that the successive terms of each set shall differ only in the numeral factor and in the power of the principal coefficient, we shall find that the sums of these sets of infinite series of terms can be severally exhibited in finite algebraic functions of the minor root of the quadratic equation formed by taking only the three leading terms of (A), or two of (B). And by substituting these finite functions in place of the respective sets of series which they represent, we shall thereby obtain not only an abbreviated form of reversion, but one that is necessarily more rapidly convergent, because the terms of the latter are severally condensations of a corresponding infinite series of terms contained in the former, and the more significant terms have the precedence in both.

In order to ascertain the particular constitution of the various terms of the development (C) we shall first determine the general term which contains the factors  $c_1^q c_2^{q'} c_3^{q''} \dots z^n$ .

By Taylor's Theorem the terms of (C) which involve  $z^n$  are all comprised in the value of  $\frac{z^n}{n} \left( \frac{d^n x}{dz^n} \right)_0$ , that is, after making  $z=0$  (or  $x=0$ ) in the differential coefficient. But, when the variables  $z$  and  $x$  are evanescent it is evident that

\* The formula is correctly given, as far as the 11th power inclusive, under the article "Reversion" in the "*Penny Cyclopædia*," recently reprinted in the Division "Arts and Sciences" of the "*English Cyclopædia*."

$$\frac{dx}{dz} = \frac{x}{z}.$$

Also, by successive differentiation with respect to  $z$ ,

$$\begin{aligned}\frac{d^2x}{dz^2} &= \frac{d}{dz} \left( \frac{x}{z} \right) = \frac{d}{dz} \left( \frac{dx}{dz} \frac{x}{z} \right) = \frac{d}{dz} \left( \frac{x^2}{z^2} \right) \\ \frac{d^3x}{dz^3} &= \frac{d}{dz} \frac{d}{dz} \frac{x^2}{z^2} = \frac{d^2}{dz^2} \left( \frac{dx}{dz} \frac{x^2}{z^2} \right) = \frac{d^2}{dz^2} \left( \frac{x^3}{z^3} \right) \\ &\quad \&c. \qquad \qquad \&c. \qquad \qquad \&c.\end{aligned}$$

And, generally, 
$$\frac{d^n x}{dz^n} = \frac{d^{n-1}}{dz^{n-1}} \left( \frac{x^n}{z^n} \right).$$

Therefore the terms of (C) involving  $x^n$  are comprised in

$$\left[ \frac{d^n}{dz^n} \left( \frac{x^n}{z^n} \right) \right]_0 = \left[ \frac{d^{n-1}}{dz^{n-1}} \left( \frac{x^n}{z^n} \right) \right]_0 \dots \quad (n)$$

and are evidently to be found exclusively in those terms of  $\left( \frac{x}{z} \right)^n$  which contain  $x^{n-1}$ , since lower powers of  $x$  go out in the differentiation, and terms involving higher powers eventually vanish in making  $x=0$ .

Now, from (B), 
$$\frac{x}{z} = (1 + c_2 x + c_3 x^2 + c_4 x^3 \dots)^{-1}$$

and 
$$\frac{x^n}{z^n} = \{1 + (c_2 x + c_3 x^2 + c_4 x^3 \dots)\}^{-n}.$$

If we put  $s = q + q' + q'' \dots$  and suppose the last expression to be expanded by the binomial theorem, the term exhibiting the factors  $c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots$  must obviously be contained in the multiple term

$$(-1)^s \frac{n-1+s}{n-1} \left[ \frac{s}{n-1} \right] (c_2 x + c_3 x^2 + c_4 x^3 \dots)^s$$

and must therefore be found in

$$(-1)^s \frac{n-1+s}{n-1} \left[ \frac{s}{n-1} \right] (c_p x^{p-1} + c_{p'} x^{p'-1} + c_{p''} x^{p''-1} \dots)^s$$

which presents the involved coefficients exclusively.

For brevity let the last factor be denoted by  $(y + y_1 + y_2 \dots)^s$ . Then, by successively applying the binomial theorem, we find that in  $\{y + (y_1 + y_2 + y_3 \dots)\}^s$  the terms which contain  $y^q$  are collectively represented by

$$\left[ \frac{s}{q} \right] \frac{s}{s-q} y^q \{y_1 + (y_2 + y_3 \dots)\}^{s-q};$$

the terms which contain also  $y_1^{q'}$  are hence

$$\frac{\frac{s}{q} \frac{1}{s-q}}{\frac{q'}{q} \frac{1}{s-q-q}} \cdot y_1^{q'} \{y_2 + (y_3 \dots)\}^{s-q-q'};$$

&c. &c. and, after cancelling identities from numerator and denominator there finally results the general term

$$\frac{\frac{s}{q} \frac{1}{q'} \frac{1}{q''} \dots}{\frac{q}{q'} \frac{1}{q''} \dots} y_1^{q'} y_2^{q''} \dots$$

Therefore restoring  $c_p x^{p-1}$ ,  $c_{p'} x^{p'-1}$ ,  $c_{p''} x^{p''-1}$ , ... respectively in place of  $y$ ,  $y_1$ ,  $y_2$ , ... and substituting in the foregoing expression, the term in  $\left(\frac{x}{z}\right)^n$  which contains  $c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots$  is

$$\pm \frac{\frac{n-1+s}{n-1} \frac{1}{s}}{\frac{q}{q'} \frac{1}{q''} \dots} c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots x^{n-1}$$

$$\text{or} \quad \pm \frac{\frac{n-1+s}{n-1} \frac{1}{q} \frac{1}{q'} \frac{1}{q''} \dots}{\frac{q}{q'} \frac{1}{q''} \dots} c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots x^{n-1}$$

where  $n=1+(p-1)q+(p'-1)q'+\dots$

Hence, according to (m), the required term of the development (C) is

$$\begin{aligned} & \pm \frac{z^n d^{n-1}}{n dx^{n-1}} \frac{\frac{n-1+s}{n-1} \frac{1}{q} \frac{1}{q'} \frac{1}{q''} \dots}{\frac{q}{q'} \frac{1}{q''} \dots} c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots x^{n-1} \\ & = \pm z^n \frac{\frac{n-1+s}{n} \frac{1}{q} \frac{1}{q'} \frac{1}{q''} \dots}{\frac{q}{q'} \frac{1}{q''} \dots} c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots \\ & = \pm z^n \cdot \left\{ \frac{pq + p'q' + p''q'' \dots}{n \frac{q}{q'} \frac{1}{q''} \dots} \cdot c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots \right\} \dots \dots \quad (a) \end{aligned}$$

the algebraic sign being  $(-1)^{q+q'+q'' \dots}$

Consider the terms which involve powers of the principal coefficient  $c_2$ ; these are hence of the form

$$\pm z^n \cdot \left\{ \frac{2m + pq + p'q' + p''q'' \dots}{n \frac{m}{m} \frac{q}{q'} \frac{1}{q''} \dots} c_2^m \cdot c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots \right\} \dots \quad (\beta)$$

and the terms which do not contain  $c_2$  are likewise included by beginning with  $m=0$ .

Put  $r=pq+p'q'+p''q'' \dots$

Then  $n=m+(r-s+1)$ ; and the general term is

$$\pm \frac{|2m+r|}{\overline{m+r-s+1} \overline{m} \overline{q} \overline{q'} \overline{q''} \dots} (c_2 z)^m z^{r-s+1} c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots;$$

and putting  $v=c_2 z$ , and collecting these terms for all values of  $m$ , we get

$$\frac{c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots}{\overline{q} \overline{q'} \overline{q''} \dots} z^{r-s+1} \Sigma \frac{|2m+r|}{\overline{m+r-s+1} \overline{m}} (-v)^m \dots \quad (\gamma)$$

As a preliminary to the finite evaluation of this infinite series, first take the initial case in which the terms exhibit powers of  $c_2$  only; then  $q=0$ ,  $q'=0$ , &c., and the complete sum of this set of terms is

$$z \cdot \Sigma \frac{|2m|}{\overline{m+1} \overline{m}} (-v)^m = z \frac{\sqrt{1+4v}-1}{2v},$$

since in this last expression the coefficient of  $v^m$  is obviously half the coefficient of  $v^{m+1}$  in  $\sqrt{1+4v}$ .

The expression just found is a root of the quadratic  $z=x+c_2 x^2$ , as it evidently ought to be.

$$\text{Let } a = \frac{\sqrt{1+4v}-1}{2v}, \quad \beta = \frac{1}{\sqrt{1+4v}}, \quad \gamma = \frac{1-\beta}{2}, \quad \omega = az,$$

$$h_2 = \beta c_3, \quad h_4 = \beta c_4, \quad h_5 = \beta c_5, \quad \&c.$$

Then, since  $a^2 = \frac{1-a}{v}$ , if  $\phi$  be an arbitrary function,

$$a^{r+2} \phi = \frac{1}{v} (a^r \phi - a^{r+1} \phi).$$

Therefore, if  $f_{r,m}$  denote the coefficient of  $v^m$  in  $a^r \phi$ , and  $r$  take consecutive integer values, the coefficients will be thus related:

$$f_{r+2,m} = -\Delta f_{r,m+1}.$$

This is satisfied by either of the forms of coefficients  $r \frac{|2m+r-1|}{\overline{m+r} \overline{m}}$

and  $\frac{|2m+r|}{\overline{m+r} \overline{m}}$ . To deduce the corresponding values of  $\phi$ , make  $r=1$ , and the former gives  $a\phi=a$  or  $\phi=1$ ; also, when  $r=0$ , the latter gives  $\phi=\beta$ .

Hence in  $a^r$  the coefficient of  $v^m$  is  $r \frac{|2m+r-1|}{\overline{m+r} \overline{m}}$ ,

$$\text{,, } a^r \beta \text{ ,, ,, } \frac{|2m+r|}{\overline{m+r} \overline{m}};$$

therefore  $a^r = \Sigma r \frac{|2m+r-1|}{|m+r| |m|} (-v)^m$ ,  $a^r \beta = \Sigma \frac{|2m+r|}{|m+r| |m|} (-v)^m$ .

We shall now determine the functions exhibited in  $(\gamma)$ , viz.,

$$\psi_{r,s} = \Sigma \frac{|2m+r|}{|m+r-s+1| |m|} (-v)^m.$$

In the first two cases,  $s=0$ ,  $s=1$ , we have

$$\psi_{r,0} = \Sigma \frac{|2m+r|}{|m+r+1| |m|} (-v)^m = \frac{a^{r+1}}{r+1},$$

$$\psi_{r,1} = \Sigma \frac{|2m+r|}{|m+r| |m|} (-v)^m = a^r \beta.$$

$$\text{Put } R = \frac{(av)^{r+1}}{r+1} = \Sigma \frac{|2m+r|}{|m+r+1| |m|} (-v)^{m+r+1};$$

$$\begin{aligned} \text{then } \frac{d^s R}{dv^s} &= \Sigma \frac{|2m+r|}{|m+r-s+1| |m|} (-v)^{m+r-s+1} \\ &= v^{r-s+1} \psi_{r,s}. \end{aligned}$$

Now, taking the expression  $R = \frac{(av)^{r+1}}{r+1}$  and differentiating successively, observing that  $\frac{d(av)}{dv} = \beta$ ,  $\frac{d\beta}{dv} = -2\beta^3$ ; and also  $2\beta av = 1 - \beta$ , we get

$$\frac{dR}{dv} = (av)^r \beta,$$

$$\frac{d^2 R}{dv^2} = r(av)^{r-1} \beta^2 - 2(av)^r \beta^3$$

$$= (av)^{r-1} \beta^2 (r - 2\beta av) = (av)^{r-1} \beta^2 (r - 1 + \beta),$$

$$\frac{d^3 R}{dv^3} = (r-1)(av)^{r-2} \beta^3 (r-1+\beta) - 2(av)^{r-1} \beta^3 \{2(r-1)\beta + 3\beta^2\}$$

$$= (av)^{r-2} \beta^3 \{(r-1)^2 + (r-1)\beta - (1-\beta)[2(r-1) + 3\beta^2]\}$$

$$= (av)^{r-2} \beta^3 \{(r-1)(r-3) + 3(r-2)\beta + 3\beta^2\},$$

$$\frac{d^4 R}{dv^4} = (r-2)(av)^{r-3} \beta^4 \{(r-1)(r-3) + 3(r-2)\beta + 3\beta^2\}$$

$$- 2\beta^3 (av)^{r-2} \{3(r-1)(r-3)\beta^2 + 12(r-2)\beta^3 + 15\beta^4\}$$

$$= (av)^{r-3} \beta^4 [(r-1)(r-2)(r-3) + 3(r-2)^2 \beta + 3(r-2)\beta^3 - (1-\beta)\{3(r-1)(r-3) + 12(r-2)\beta + 15\beta^2\}]$$

$$= (av)^{r-3} \beta^4 \{(r-1)(r-3)(r-5) + (6r^2 - 36r + 45)\beta + 15(r-3)\beta^2 + 15\beta^3\}; \text{ \&c. \&c.}$$

After putting  $1-2\gamma$  for  $\beta$  in the last factor of each expression, we obtain

$$\begin{aligned}\psi_{r,1} &= a^r \beta, \\ \psi_{r,2} &= a^{r-1} \beta^2 \{r-2\gamma\}, \\ \psi_{r,3} &= a^{r-2} \beta^3 \{r(r-1)-6r\gamma+12\gamma^2\}, \\ \psi_{r,4} &= a^{r-3} \beta^4 \{r(r-1)(r-2)-12r(r-1)\gamma+60r\gamma^2-120\gamma^3\}, \\ &\quad \&c. \qquad \qquad \qquad \&c. \qquad \qquad \qquad \&c.\end{aligned}$$

And generally,

$$\begin{aligned}\psi_{r,s} &= a^{r-s+1} \beta^s \left\{ (r \dots r-s+2) - \frac{s(s-1)}{1} (r \dots r-s+3) \gamma \right. \\ &\quad + \frac{s+1 \dots s-2}{1.2} (r \dots r-s+4) \gamma^2 \\ &\quad \left. - \frac{s+2 \dots s-3}{1.2.3} (r \dots r-s+5) \gamma^3 \dots \right\} \dots \quad (\delta).\end{aligned}$$

Hence, observing that  $a^{r-s+1} z^{r-s+1} = \omega^{r-s+1}$ ,

$$\beta^s \cdot c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots = \beta^{q+q'+q'' \dots} c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots = h_p^q h_{p'}^{q'} h_{p''}^{q''} \dots,$$

the sum of the terms ( $\beta$ ), viz.

$$\Sigma \pm z^n \cdot \left\{ \frac{|2m+pq+p'q'+p''q'' \dots|}{|n| |m| |q| |q'| |q''| \dots} c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots \right\},$$

when  $m$  takes all values  $0, 1, 2, \dots \infty$ , is

$$\begin{aligned}z^{r-s+1} \frac{c_p^q c_{p'}^{q'} c_{p''}^{q''} \dots}{|q| |q'| |q''| \dots} \psi_{r,s} &= \omega^{r-s+1} \cdot h_p^q h_{p'}^{q'} h_{p''}^{q''} \dots \frac{|s-1|}{|q| |q'| |q''| \dots} \\ &\times \left\{ \frac{r \dots r-s+2}{1 \dots s-1} - \frac{s}{1} \frac{r \dots r-s+3}{1 \dots s-2} \gamma + \frac{s(s+1)}{1.2} \frac{r \dots r-s+4}{1 \dots s-3} \gamma^2 \right. \\ &\quad \left. - \frac{s(s+1)(s+2)}{1.2.3} \frac{r \dots r-s+5}{1 \dots s-4} \gamma^3 + \&c. \right\} \dots \quad (\epsilon),\end{aligned}$$

which is finite and consists of  $s$  terms. The numeral coefficients within the brackets are simply formed by writing down  $s$  terms of the series  $1, s, \frac{s(s+1)}{2}, \frac{s(s+1)(s+2)}{2.3}, \&c.$ , and placing under-

neath them  $s$  terms of the series  $1, r, \frac{r(r-1)}{2}, \frac{r(r-1)(r-2)}{2.3}, \&c.$ ,

but in a reverse order, and then multiplying together the corresponding numbers vertically.

It will now be easy to write down *ad libitum* the collective values for any proposed sets of terms of the development (C).

We shall here enumerate some of the principal cases, and then state the complete formula.

The leading set of terms are those which involve  $c_2$  only, viz.,

$$z - z^2 c_2 + 2z^3 c_2^2 - 5z^4 c_2^3 + \dots,$$

which are of the form  $\lambda z^n c_2^m$ . In this case  $r=0$ ,  $s=0$ , and the sum of all the terms referred to has before been found  $= az = \omega$ .

The next set

$$-z^3 \cdot c_3 + 5z^4 c_2 \cdot c_3 - 21z^5 c_2^2 \cdot c_3 + \dots$$

comes under the form  $\lambda z^n c_2^m c_3$ . Here  $p=3$ ,  $q=1$ ; therefore  $r=3$ ,  $s=1$ , and the sum of the terms  $= z^3 \frac{c_3}{1} \psi_{3,1} = z^3 c_3 \cdot \alpha^3 \beta = \omega^3 h_3$ .

Take now the set

$$7z^6 c_3 c_4 - 72z^7 c_2 c_3 c_4 + 495z^8 c_2^2 c_3 c_4 - \dots$$

These are of the form  $\lambda z^n c_2^m \cdot c_3 c_4$ , in which  $r=7$ ,  $s=2$ ; and therefore the sum  $= z^6 c_3 c_4 \cdot \psi_{7,2} = z^6 c_3 c_4 \cdot \alpha^6 \beta^2 (7-2\gamma) = \omega^6 \cdot h_3 h_4 (7-2\gamma)$ .

As one more example, take the terms which are of the form  $\lambda z^n c_2^m \cdot c_3^2 c_4$ . In this case,  $c_2^q c_3^q = c_2^3 c_4^1$ ; therefore  $r=10$ ,  $s=3$ , and the sum of all such terms  $= \omega^8 \cdot h_3^2 h_4 (45-30\gamma+6\gamma^2)$ .

By taking in this way every requisite form of combination of the coefficients, and afterwards uniting the several values, the development (C) becomes concentrated into the following comprehensive result:

$$\begin{aligned} x = & \omega \\ & - \omega^3 \cdot h_3 \\ & - \omega^4 \cdot h_4 \\ & - \omega^5 \{ h_5 - h_3^2 (3-\gamma) \} \\ & - \omega^6 \{ h_6 - h_3 h_4 (7-2\gamma) \} \\ & - \omega^7 \{ h_7 - (2h_3 h_5 + h_4^2) (4-\gamma) + h_3^3 (12-9\gamma+2\gamma^2) \} \\ & - \omega^8 \{ h_8 - (h_3 h_6 + h_4 h_5) (9-2\gamma) + h_3^2 h_4 (45-30\gamma+6\gamma^2) \} \\ & - \quad \quad \quad \&c. \quad \quad \quad \&c. \quad \quad \quad \&c. \dots \dots \dots (D). \end{aligned}$$

This formula, which may easily be extended if required, is complete to the 8th order of  $\omega$ , and not only includes the values of all the terms exhibited in the development (C), but embodies all the cognate terms to infinity. Now the value of  $\gamma$  depends upon the coefficient  $h_2$ , and is

$$\gamma = \beta a v = \beta c_2 \omega = h_2 \omega$$

Therefore, by substitution and retention of all terms to the 8th order, we obtain

$$\begin{aligned}
 x = & \omega - \omega^3 \cdot h_3 \\
 & - \omega^4 \cdot h_4 \\
 & - \omega^5 \cdot (h_5 - 3h_3^2) \\
 & - \omega^6 \cdot (h_6 - 7h_3 h_4 + h_3^2 h_3^2) \\
 & - \omega^7 \cdot (h_7 - 8h_3 h_5 - 4h_4^2 + 2h_3 h_3 h_4 + 12h_3^3) \\
 & - \omega^8 \cdot (h_8 - 9h_3 h_6 - 9h_4 h_5 + 45h_3^2 h_4 + 2h_3 h_3 h_5 + h_3 h_4^2 \\
 & \quad - 9h_3 h_3^2) \\
 & \&c. \qquad \qquad \&c. \qquad \qquad \&c. \dots\dots\dots (E).
 \end{aligned}$$

By means of ( $\epsilon$ ) we find that the general term of (E) is

$$(-1)^{r-m} \omega^n \cdot \frac{|s+m-1|}{\boxed{m} \boxed{s-m-1}} \frac{|r|}{\boxed{n} \boxed{q} \boxed{q'} \dots} h_2^m h_3^q h_4^{q'} \dots$$

where  $r = pq + p'q' + \dots$ ,  $s = q + q' + \dots$ ,  $n = m + (r - s + 1)$ ;

and where  $h_2$  is absent, or  $m = 0$ , the terms are invariably the same as the analogous terms contained in the ordinary formula (C).

The formula, however, yet admits of further concentration and simplification. Referring back to the original equation

$$z = x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots,$$

we perceive that  $\omega$  is a solution of the quadratic portion

$$z = \omega + c_2 \omega^2,$$

and that this would be an accurate solution of the entire equation if we could assume that the residue  $c_3 \omega^3 + c_4 \omega^4 + \dots = 0$ , or  $h_3 \omega^3 + h_4 \omega^4 + \dots = 0$ . Thus the expression for  $\omega - x$ , or the collected terms of the development which involve exclusively the coefficients  $h_3, h_4, \&c.$ , must vanish with  $h_3 \omega^3 + h_4 \omega^4 + \dots$ , and therefore contain the latter as a factor. Hence, putting

$$\mu = h_3 \omega^3 + h_4 \omega^4 + h_5 \omega^5 + \dots,$$

the foregoing expression is readily transformed into

$$\begin{aligned}
 x = & \omega - \mu - h_2 \mu^2 \\
 & + 3h_3 \mu \omega^2 \\
 & + 4h_4 \mu \omega^3 \\
 & + (5h_5 - 12h_3^2) \mu \omega^4 \\
 & + (6h_6 - 33h_3 h_4 + 9h_3^2 h_3^2) \mu \omega^5 \dots\dots\dots (F).
 \end{aligned}$$

In practice, however, we may simply take  $x = \omega - \mu$ ; the other

terms are of the fifth and higher orders, and may be omitted as immaterial.

It may be useful here to recapitulate the principal conditions and formulæ for practical application. The symbol  $U$  denoting the function to be reverted, the equation to be solved with respect to  $x$  is  $U=0$ , Then  $U'$ ,  $U''$ ,  $U'''$ , &c. being the successive differential coefficients, the formulæ for calculation will be

$$\left. \begin{aligned} z &= -\frac{U_0}{U'}; \quad c_2 = \frac{1}{2} \frac{U''}{U'}, \quad c_3 = \frac{1}{3} \frac{U'''}{U'}, \quad \&c. \\ v &= c_2 z, \quad q = \sqrt{1 + 4v}, \\ p &= \frac{1+q}{2}, \quad \omega = \frac{z}{p}, \\ h_3 &= \frac{c_3}{q}, \quad h_4 = \frac{c_4}{q}, \quad \&c. \\ x &= \omega - h_3 \omega^3 - h_4 \omega^4 \dots \end{aligned} \right\} \dots (\lambda).$$

As the development proceeds in powers of  $\omega$  or  $z$ , the efficacy of the approximation requires that  $x$  may be a small fraction, or that  $U_0$  may be small as compared with  $U'_0$ . That is, the zero or starting-point of the calculation should be taken as near as practicable to the sought value.

Should the function be of such a nature that its differential coefficients are of a complicated character, it will be more convenient to compute a series of values of the function for assumed equidistant values of  $x$ , and thence to determine the differential coefficients as depending upon the several orders of differences of these quantities. In this method the convergency of the differences will indicate if the interval be taken sufficiently small; and that quantity which comes nearest to the prescribed result being taken as the origin or zero-point of the calculation, the accuracy of the process will be satisfactorily established. It will be a sufficient guide, in the course of the work, to observe that if the progression of the differences be close enough for purposes of interpolation, they will effectually supply all that is requisite for the case in hand. The computation of five values of the function, so as to include two preceding and two following the origin, will in general supply all the necessary data. If these values and their differences be

$$\begin{array}{c|c|c|c|c} U_{-2} & a_{-2} & b_{-1} & c_{-1} & d_0 \\ U_{-1} & a_{-1} & b_0 & c_0 & \\ U_0 & a_0 & b_1 & c_1 & \\ U_{+1} & a_{+1} & b_2 & c_2 & \\ U_{+2} & a_{+2} & b_3 & c_3 & \end{array}$$



The time of new moon, or conjunction in longitude, is therefore May 11<sup>d</sup> 1717415, that is, on May 11th at 4<sup>h</sup> 7<sup>m</sup> 19<sup>s</sup> P.M.

The terms  $-h_3\omega^3 - h_4\omega^4$  would not sensibly affect the result, which they would increase by only three-hundredth parts of a second.

*Example 2.\**—The Long Annuities, which have 30 years to run, are sold at 19 years' purchase; what rate of interest does the purchaser obtain for his money?

In this example the equation for solution is

$$U = \frac{1 - (1 + \rho)^{-30}}{\rho} - 19 = 0,$$

which does not admit of direct algebraic solution by any known method. But if we take any specified rate of interest, it is easy to calculate the corresponding year's purchase from the formula  $\frac{1 - (1 + \rho)^{-30}}{\rho}$ . In fact, these results are already calculated and registered in extensive tables to be found in all works on Compound Interest and Annuities. From these the following values, for each half per cent. of interest, are extracted and differenced:

$\rho$	Years' Purchase	Differences			
2	22.396456	-1.466163			
2½	20.930298	1.329852	+136311	-14855	
3	19.600441	1.208396	121456	-13072	+1783
3½	18.392045	-1.100012	+108384		
4	17.292033				

We have to calculate from these the rate of interest at which the year's purchase becomes exactly 19; and for this calculation they supply the following data:

$$a_0 = -1.269124, \quad b_0 = +.121456,$$

$$c_0 = -13963, \quad d_0 = +1783;$$

$$-U_0 = -.600441, \quad U' = -1.266797, \quad U'' = +.121307,$$

$$U''' = -.013963, \quad U'''' = +.001783.$$

From these data we have the following calculation;

log $U''$ + 9.0838859		log $(-U_0)$ - 9.7784703	
,, $U'$ - 0.1027071		..... - 0.1027071	
,, $2c_2$ - 8.9811788		log $z$ + 9.6757632	
,, 2 0.3010300		.... ,, $4c_2$ - 9.2822088	
Nat. Nos.	{ - .0907762		,, $4v$ - 8.9579720
	{ + .9092238		,, $1 + 4v$ 9.9586708
	{ .9535322		,, $q$ 9.9793354
	{ .9767661		,, $p$ 9.9897906
	{ .4852579		,, $\omega$ + 9.6859726

\* From David Jones's "Annuities," §c., p. 42, wherein, however, the result is correct to only two places of decimals.

log U'	-0.1027071		
„ q	9.9793354		
<hr/>			
{ „	-0.0820425		
<hr/>			
Comp.	-9.9179575	.....	-9.9179575
log U'''	-8.1449787	log U''''	+7.2511513
Co. „ 6	9.2218487	Co. „ 24	8.6197888
<hr/>			
„ h <sub>3</sub>	+7.2847849	„ h <sub>4</sub>	-5.7888976
„ (-ω <sup>3</sup> )	-9.0579178	„ (-ω <sup>4</sup> )	-8.7438904
<hr/>			
{ „	-6.3427027	{ „	+4.5327880
{ numb.	- .0002201	{ numb.	+ .0000034

Hence by the formula (λ) we find

$$\begin{aligned}
 x &= \omega - h_3 \omega^3 - h_4 \omega^4 = .4852579 \\
 &\quad - .2201 \\
 &\quad + .34 \\
 &\quad \hline
 &= .4850412
 \end{aligned}$$

This being expressed in parts of the interval of half per cent., we must take one-half of it, so that the required rate of interest is 3.2425206 per cent. per annum, which is true to the last place of figures.

The calculation is here carried out with logarithms to seven places in order to show the accuracy of the method; but ordinarily more concise data and five or even four figure logarithms will be practically sufficient; and U''', U''', and the corrections depending upon them may be dispensed with, as in Example 1.

*Example 3.*—As a concluding example, take a cubic equation, given in most treatises on Algebra, viz.,  $x^3 - 2x - 5 = 0$ .

By successive differentiation, we have

$$\begin{aligned}
 U &= x^3 - 2x - 5, \\
 U' &= 3x^2 - 2, \\
 U'' &= 6x, \\
 U''' &= 6.
 \end{aligned}$$

A real root of the equation evidently lies near to the number 2; and for  $x=2$  we have

$$\begin{aligned}
 -U &= 1, & U' &= 10, & U'' &= 12, & U''' &= 6; \\
 \therefore z &= \frac{1}{10}, & c_2 &= \frac{3}{5}, & c_3 &= \frac{1}{10}; \\
 v &= \frac{8}{50}, & q &= \frac{\sqrt{31}}{5}, & p &= \frac{\sqrt{31} + 5}{10}, \\
 \omega &= \frac{\sqrt{31} - 5}{6}, & h_3 &= \frac{1}{2\sqrt{31}};
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= 2 + \omega - h_2 \omega^3 \\
 &= 2 + \frac{\sqrt{31}-5}{6} - \frac{1}{2\sqrt{31}} \left( \frac{\sqrt{31}-5}{6} \right)^3 \\
 &= \frac{1}{216} \left( \frac{1411}{81} \sqrt{31} + 199 \right) \\
 &= 2.0945513, \text{ which is true to the last figure.}
 \end{aligned}$$

These examples will serve to show the practical utility of the formulæ, which may with unfailing efficacy be similarly applied to all kinds of problems. It is, however, in the numerical solution of complicated and otherwise unmanageable equations that the value of the method will be most conspicuous. By the latter process, in which the differences are used, all complication is entirely got rid of so soon as the successive numerical values are obtained by direct computation, and the most difficult inverse problems are effectually worked out by an invariable and universal method.

NOTE.—The formula (E) may be readily deduced by the following elementary process:—

What is required is to find the development of  $x$  in powers of  $\omega$ , when

$$\omega + c_2 \omega^2 = x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots \quad (a)$$

This may be effected after expressing  $\omega$  in powers of  $x$ , for  $x$  will then be the reversion of the series so obtained.

The equation (a) gives, after transposition,

$$(1 + 2c_2 \omega)(\omega - x) - c_2(\omega - x)^2 = c_3 x^3 + c_4 x^4 + \dots$$

and, after dividing by  $1 + 2c_2 \omega$ ,

$$(\omega - x) - h_2(\omega - x)^2 = h_3 x^3 + h_4 x^4 + \dots$$

Putting  $M = h_3 x^3 + h_4 x^4 + \dots$ , this quadratic gives

$$\omega - x = \frac{1 - \sqrt{1 - 4h_2 M}}{2h_2} = M + h_2 M^2 + 2h_2^2 M^3 + \dots$$

or, substituting for  $M$ , two terms being sufficient,

$$\begin{aligned}
 \omega = x &+ h_3 x^3 + h_4 x^4 + h_5 x^5 + (h_6 + h_2 h_3^2) x^6 \\
 &+ (h_7 + 2h_2 h_3 h_4) x^7 + (h_8 + h_2 h_4^2 + 2h_2 h_3 h_5) x^8 + \dots \quad (b)
 \end{aligned}$$

which does not contain  $x^2$ .

Now the formula (C) when the second term is absent, or  $c_2 = 0$ , becomes

$$\begin{aligned}
 x = z &- c_3 z^3 - c_4 z^4 - (c_5 - 3c_3^2) z^5 - (c_6 - 7c_3 c_4) z^6 \\
 &- (c_7 - 8c_3 c_5 - 4c_4^2 + 12c_3^3) z^7 \\
 &- (c_8 - 9c_3 c_6 - 9c_4 c_5 + 45c_3^2 c_4) z^8 - \dots \quad (c)
 \end{aligned}$$

Applying this formula to (b) gives immediately, without any reduction, the required formula, viz.

$$\begin{aligned}
 x = & \omega - h_2 \omega^3 - h_4 \omega^4 - (h_5 - 3h_3^2) \omega^5 - (h_6 + h_2 h_3^2 - 7h_3 h_4) \omega^6 \\
 & - (h_7 + 2h_2 h_3 h_4 - 8h_3 h_5 - 4h_4^2 + 12h_3^3) \omega^7 \\
 & - (h_8 + h_2 h_4^2 + 2h_2 h_3 h_5 - 9h_3 h_6 - 9h_2 h_3^3 - 9h_4 h_5 + 45h_3^2 h_4) \omega^8 - \dots (E)
 \end{aligned}$$

Also, by applying (c) to (E) we revert back again to the formula (b) and thus check (b) and (E) both.

*Remark on the preceding Paper.* By A. DE MORGAN, Esq., F.R.A.S.

Having seen Mr. Woolhouse's Paper before it was publicly read, I arrived at an independent establishment of the development which is, in the Paper, deduced from the known series for reversion. This mode of establishment is also an extension; though it is hardly to be expected that any extended application will be found desirable. At the request of Mr. Woolhouse, I subjoin a short notice.

Let  $\phi\omega = \phi x + \psi x$ , where  $\psi x$  will be small; and write  $f x$ , when convenient, for  $\phi x + \psi x$ . Assume  $x = \omega + t$ , which gives

$$-\psi\omega = f'\omega \cdot t + f''\omega \cdot \frac{t^2}{2} + \dots$$

Find  $t$  in powers of  $-\psi\omega$  by common inversion, omitting  $\omega$  after  $f$  for abbreviation, and writing  $f_n$  for  $f^{(n)}$ :  $2.3 \dots n$ ,

$$\begin{aligned}
 x = \omega - \frac{1}{f'} \cdot \psi\omega - \frac{f''}{f_i^3} \cdot (\psi\omega)^2 - \frac{2f''^2 - f' f_{ii}}{f_i^5} \cdot (\psi\omega)^3 \\
 - \frac{5f''^3 - 5f' f_{ii} f_{ii} + f_i^2 f_{ii}}{f_i^7} \cdot (\psi\omega)^4 - \dots,
 \end{aligned}$$

a representation of  $f^{-1}\phi\omega$ , or of  $f^{-1}(f\omega - \psi\omega)$ . Three terms will be more than sufficient.

Let  $\phi x + \psi x$  be  $ax + bx^2 + \dots + kx^m + (p + qx + \dots + sx^n)x^{n+1}$ ; let  $a = \phi x + \psi x$  be the equation to be solved, and let  $a = \phi x$  give  $x = \omega$ . The root of  $a = \phi\omega$  being known,  $\phi'\omega$  is known; call it  $l$ . Expanding the above in powers of  $\omega$ , it is clear that the first three terms will give all short of  $\omega^{3m+3}$ ; Mr. Woolhouse, by an entirely different method, stops at  $\omega^8$  when  $m=2$ . Taking this case, we find  $a = a\omega + b\omega^2$ ,  $a + 2b\omega = \sqrt{(a^2 + 4ba)} = l$ . And, writing down no more than necessary for  $\omega^8$  inclusive,

$$\begin{aligned}
 x = \omega - \frac{c + e\omega + f\omega^2 + g\omega^3 + h\omega^4 + k\omega^5}{l + 3c\omega^2 + 4e\omega^3 + 5f\omega^4 + 6g\omega^5 + 7h\omega^6 + 8k\omega^7} \omega^3 \\
 - \frac{(b + 3c\omega + 6e\omega^2 + \dots + 28k\omega^6)(c + e\omega + f\omega^2)^2}{(l + 3c\omega^2 + \dots + 8k\omega^7)^3} \omega^6 - \dots
 \end{aligned}$$

For  $l$  write 1; and, remembering that in this result  $at^{-1}$ ,  $bt^{-1}$ , &c. must be written for  $a$ ,  $b$ , &c., we have

$$\begin{aligned} x = & \omega - c\omega^2 - e\omega^4 - (f - 3c^2)\omega^5 \\ & - (g - 7ce + bc^2)\omega^6 - (h - 8cf - 4e^2 + 12c^2 + 2bce)\omega^7 \\ & - (k - 9cg - 9ef + 45c^2e - 9bc^2 + 2bcf + be^2)\omega^8 - \dots \end{aligned}$$

This result agrees entirely with that of Mr. Woolhouse.

*On the Rate of Mortality prevailing among Assured Lives, as influenced by the length of time for which they have been assured.*  
By THOMAS B. SPRAGUE, M.A. Vice-President of the Institute of Actuaries.

[Read before the Institute, 20th December, 1869.]

TO say that the subject of the rate of mortality among assured lives and annuitants is at present very imperfectly understood, and that the process usually adopted to deduce the rate of mortality from observations upon these classes, is not a proper one for the purpose, and leads to erroneous and misleading results, may seem at first sight to be very rash assertions; but I believe that upon consideration they will be found strictly correct.

It is universally acknowledged that the rate of mortality among assured lives is very light during the first few years that follow the grant of the assurance; being extremely small in the first year, and gradually increasing, until after the lapse of a greater or less number of years, the mortality becomes, according to some authorities, equal to that indicated by tables deduced from the population at large, and according to others, still heavier. This is of course satisfactorily explained by the medical examination of the lives proposed for assurance, which has the effect of eliminating those persons who are suffering from such acute or chronic diseases, dangerous to life, as can be detected by the medical officers of the Assurance Companies.

It seems further probable that the consequent advantage to the Assurance Companies, or the *benefit of selection*, as it is called, will be lost after a certain number of years. But those who have given most attention to the subject, are far from agreed as to the length of time for which the benefit of selection endures. Mr. Spens, in his "Tables of Mortality Experience of Scottish Amicable Life Assurance Society from 1826-1860," arrives at the conclusion that the effect of selection lasts for six years from the date of the assurance, and is then exhausted, the mortality being subsequently nearly the same as that indicated by the English

Life Table No. 2. Mr. Higham, on the other hand, has arrived at the conclusion that the effect of selection lasts for a term equal to half the difference between the age at entry and 80; this term therefore being no less than 30 years for the age at entry, 20. Mr. Farren, who has treated the subject in his "Life Contingency Tables" published in 1850, with what appears to me a needless parade of science, says nothing as to this important point, stating only in general terms that "The value of selection is scarcely traceable after the first years being nearly wholly neutralized by attending causes, particularly by discontinuances on the part of healthy lives, soon after selection. The deterioration ultimately merges into about the same rates of mortality as those occurring among the male population" (p. xii.).

If we consider how the medical selection of the lives proposed for assurance operates, we shall see that there is nothing impossible in the supposition that its effect may endure and be traceable, for any number of years up to 20, 30, or 40, or even to the extreme limit of life. Among the population at large, or among a very large number of persons chosen by lot, there will be persons in every possible state of health. Some will be on their death-beds, suffering under diseases which will certainly kill them within a few hours, or it may be days, or weeks. Some will be in the last stage of lingering illnesses, such as consumption, of which they will certainly die within the course of two or three years. Others, again, will be suffering under the effects of acute diseases, such as fevers; or from the effect of accidents; from which they may either entirely recover within a few weeks or months, or recover only with impaired constitutions and diminished prospects of life, or from which on the contrary they may die. There will lastly be some who are suffering from chronic diseases, such as heart disease; which will certainly on the average shorten their lives, but will not prevent individuals among them from attaining to extreme old age. In fact, it is easy to conceive that a body of such lives may throughout be subject to a much heavier mortality than that prevailing among the population at large; and yet, that if sufficiently numerous, they may not all die before the extreme age in the Table of Mortality is reached. The original medical selection will more or less completely weed out from the persons proposed for insurance those who belong to any of the above classes. In this way it is undeniable that the Assurance Office will escape many premature deaths, by the weeding out of persons suffering from acute diseases. And it seems very probable that the effect of the

medical selection, through weeding out the persons labouring under chronic diseases, will have the effect of slightly reducing the rate of mortality even to the extremity of life. Whether its effect, supposing it to exist to the extremity of life, would be of sufficient magnitude to be traceable; or for how many years it would be traceable; are questions with regard to which *à priori* reasoning can give no assistance. We must deduce the answers from careful examination of observed facts; and the more numerous the facts we can obtain, the more satisfactorily shall we be able to answer these questions.

Let us now trace the progress of a large body of assured lives, on the supposition that none of the members are withdrawn from observation otherwise than by death. We shall then see that there is a constant tendency for the constitution of the body to assimilate itself to that of a body selected by lot from the class of persons to which the members belong. A very short time will suffice to introduce into the body, acute diseases and accidents, which run their course rapidly towards death or recovery. In other words, assured lives may die of acute diseases, such as fevers, or from accident, within a short time from the date of the assurance; and the experience of every Assurance Company could probably furnish examples of this kind. A few years will suffice to develop in some of the persons who were passed as first class lives, consumption and other diseases, of which they will die within two or three years. As time progresses, a larger proportion of the surviving members will be suffering from chronic diseases, until ultimately after the lapse of a sufficient number of years, the constitution of the body, supposed to be still sufficiently large, would be precisely the same as that of a body of the same number of members selected by lot from the class of society to which the assured lives belong.

Much light would be thrown upon this subject by a properly conducted examination of the causes of death among insured lives; if it could be ascertained what diseases or what classes of diseases are most fatal in the early years of the policy, and in each successive interval of, say, five or ten years; if, in other words, the mortality from each class of diseases were compared with the standing of the policies. Several elaborate nosological tables have been published by different Companies; and in some, an attempt has been made to compare the deaths from each disease with the numbers living at various ages; but the enquiry now suggested appears to me to be likely to be attended with far more useful results.

To use Mr. Higham's expressive phrase, the above described body will ultimately consist of "mixed" lives, some being in full vigour and health, others in fair health, some in feeble, and some in very

bad health. The average vitality of the members being now less than that of a number of recently selected persons of the same ages, they are sometimes inaccurately spoken of as "deteriorated" lives; but it cannot be too strongly stated, or too carefully borne in mind, that the deterioration is only *on the average*. Many of the lives may be in even better health than formerly; and the great proportion of the whole would probably be still regarded as select lives, insurable at the ordinary rate. This ultimate constitution of a body of assured lives has very important effects when we come to consider the valuation of policies for surrender, but it will be more convenient to defer for the present the consideration of this branch of the subject.

The above remarks would exhaust the theoretical part of the subject, on the assumption that the medical selection is the only cause operating to disturb the rate of mortality and make it deviate from its normal value; but there is another cause in operation, the influence of which we must consider, viz., the constant withdrawal from observation of a greater or less number of the members, by the lapse or surrender of their policies. It is now generally believed that after the lapse of a considerable number of years the mortality among insured lives becomes greater than that among the population at large; and this is attributed to the above-mentioned withdrawals. Mr. Higham appears to have been the first to draw attention to this point. He says (*Assurance Magazine*, vol. i., p. 190), "On comparing the probabilities of living a year in " the separate classes with those of Mr. Farr, given in the reports of " the Registrar-General, one is struck with the fact that assured lives " are, for some time after selection, much better than the community " at large, but that after a while they become much worse. . . . " . . . . This can arise from no other cause than *the selection* " *which the assured exercise against the Companies* by dropping " policies on healthy lives and retaining those on lives which have " become bad or doubtful." But these views have not been accepted universally. On the contrary, opinions have been greatly divided as to the effect produced by the withdrawals. It is clear that if those who abandon their policies have on the average a superior vitality to those who retain them, the rate of mortality among the lives remaining insured will be increased thereby, or will be larger than it would have been if there had been no withdrawals. But some authors have given it as their opinion that policies are generally abandoned in consequence of inability to pay the premium, and that consequently the persons who abandon their

policies, being mostly in embarrassed circumstances, will on the average possess, not a greater, but a less vitality than those who keep them up. It would serve no good purpose to quote in full the various opinions on this subject expressed by different authors. They no doubt have their value, as founded on examination, more or less complete, of observed facts; but in the absence of any information as to the nature and number of those facts, it is impossible to assess that value correctly. The effect of the withdrawals, which undoubtedly take place, is a question of fact; and as such, is to be decided by an appeal to facts, and not by quoting any authority, however eminent. It may here suffice to anticipate so far as to say that I myself fully agree with those who attribute to these withdrawals a powerful influence in increasing the rate of mortality.

Supposing the withdrawals to exercise an influence on the subsequent rate of mortality among the lives left under observation, our facts will show us only the combined effect of such withdrawals and of the original medical selection; and it will be extremely difficult, if not quite impossible, to distinguish the effects of the two causes. If, however, we had observations made upon a sufficiently large number of persons among whom no withdrawals take place, for instance, among a body of annuitants, we should get rid of the disturbing cause and probably be able to obtain most valuable results. Of course, the selection exercised by the annuitants in buying annuities on the lives of themselves or other nominees, differs widely from the selection exercised by an Office granting assurances; but it is probably on the average quite as efficacious, inasmuch as it seldom happens that a person is willing to sink money on a life, unless he believes it to have at least the full average prospect of longevity. It is greatly to be desired that the valuable experience accumulated by the Government Annuity Office should be re-collated and published in such a form as to throw light upon this point. The results could scarcely fail to be of the highest importance to the Insurance Companies. If this were done, the results could at the same time be easily arranged so as to throw much light upon another disputed point, viz.—the alleged decrease in the rate of mortality since the last century.

The recent publication of the Mortality Experience of Life Assurance Companies collected by the Institute of Actuaries, appeared to afford a suitable opportunity for pursuing the investigation of the influence of selection beyond the point at which Mr. Higham had left it. The facts now available for examination are far more numerous than those which he employed; and in the later years of insurance more particularly, the facts are so much more numerous

that we shall be able to speak with certainty as to the ultimate rate of mortality, where he could do little more than conjecture. My first step on resolving to enter on this investigation was to study very carefully Mr. Higham's paper in the first volume of the *Assurance Magazine*; and one result of that examination was to confirm my opinion that that paper contains the most complete examination of the effect of selection which has hitherto appeared. I nevertheless soon arrived at the conclusion that, for reasons which I will state presently, Mr. Higham's method was not the one best adapted to bring out into prominence the true law which the facts follow, and I decided to employ a method more resembling that adopted by Mr. Spens in his *Tables of the Mortality Experience of the Scottish Amicable Life Assurance Society*, published in 1861.

I have confined my investigations at present to the experience of the mortality of the healthy male assured lives, excluding, on account of the paucity of numbers, those insured under the age of 15, and above the age of 75. The facts dealt with are shown in the following Table (A), where it is to be noticed that the *age at entry* is not the *current age*, or *age next birthday*, but the *age last birthday*.

TABLE (A).

Age at Entry.	Entered.	Existing.	Dis-continued.	Died.	Age at Entry.	Entered.	Existing.	Dis-continued.	Died.
15	282	158	101	23	46	2156	1111	497	548
16	370	200	150	20	47	2014	1007	470	537
17	497	254	202	41	48	1884	923	460	501
18	675	367	265	43	49	1806	904	403	499
19	1086	557	470	59	50	1943	721	387	435
20	1555	860	612	83	51	1451	616	380	455
21	2497	1391	932	174	52	1261	579	296	386
22	3069	1779	1099	191	53	1084	472	265	347
23	3546	2094	1166	286	54	1027	427	226	374
24	4213	2634	1265	314	55	883	353	200	330
25	4631	2908	1326	397	56	777	282	170	325
26	4937	3092	1428	417	57	702	236	147	319
27	5304	3397	1439	458	58	656	216	135	305
28	5239	3358	1401	480	59	657	229	139	289
29	5791	3677	1491	623	60	446	130	118	198
30	5321	3327	1439	555	61	403	124	93	186
31	5289	3267	1393	629	62	354	97	82	175
32	5181	3225	1322	634	63	304	94	66	144
33	5183	3245	1278	660	64	261	70	61	130
34	4930	3047	1229	654	65	209	37	61	111
35	4759	2902	1210	647	66	162	33	39	90
36	4248	2567	1101	580	67	151	33	31	87
37	4342	2575	1117	650	68	94	20	23	51
38	3880	2260	978	642	69	86	22	21	43
39	4062	2352	1016	694	70	58	6	17	35
40	3571	2048	866	657	71	63	14	20	29
41	3154	1792	773	589	72	32	3	9	20
42	2943	1672	702	569	73	43	7	8	28
43	2778	1509	703	566	74	20	4	9	7
44	2676	1482	621	573	75	9	2	2	5
45	2420	1312	574	534					

These facts are summarized in Table (B), in which are added the percentages of the Existing, Discontinued, and Died, to the Entered, in each group of ages.

TABLE (B).

Ages at Entry.	Entered.	EXISTING.		DISCONTINUED.		DIED.	
		Number.	Percent.	Number.	Percent.	Number.	Percent.
15-20	4465	2396	53·66	1800	40·31	269	6·02
21-25	17956	10806	60·18	5788	32·23	1362	7·59
26-30	26592	16851	63·37	7198	27·07	2543	9·56
31-35	25342	15686	61·90	6432	25·38	3224	12·72
36-40	20103	11802	58·71	5078	25·26	3223	16·03
41-45	13971	7767	55·59	3373	24·14	2831	20·26
46-50	9403	4666	49·62	2217	23·58	2520	26·80
51-55	5706	2447	42·88	1367	23·96	1892	33·16
56-60	3238	1093	33·76	709	21·90	1436	44·35
61-65	1531	422	27·56	363	23·71	746	48·73
66-70	551	114	20·69	131	23·77	306	55·54
71-75	167	30	17·97	48	28·74	89	53·29
Total . .	129025	74080	57·42	34504	26·74	20441	15·84

I first had the numbers of the "exposed to risk" as stated in table H<sup>m</sup> (pp. 143 to 163) recomputed. Then arranging in two vertical columns the numbers at risk and the deaths, in each year of assurance, as shown in Table (C), I obtained the total number exposed to risk and the total deaths, in each year of assurance. The probable numbers of deaths for each age at entry and in each year of assurance, were then calculated, and the results entered by the side of the actual deaths as shown. For example, out of the 5321 persons insured at the age of 30 last birthday there were 648 at risk in the year of assurance 20; and these were taken as of the present age 50, and their number multiplied into the probability of dying in a year at that age ( $=\cdot 01594$  by the Experience of the 17 Offices) thus giving 10·33 as the number of probable deaths for age at entry 30 and year of assurance 20. It will be noticed that after 40 years of insurance the facts become too few to admit of separate treatment and are therefore combined in a single total. These probable deaths have been calculated separately from three tables of mortality:—the "17 Offices' Experience," the "Peerage Males," as adjusted by Mr. Berridge, (*Journal*, vol. xii., p. 225,) and lastly according to a table adjusted by Mr. Woolhouse from the observations on the male lives assured contained in the volume already mentioned, which last table I will call the "New Experience" Table.

TABLE (C.)

YEAR OF INSURANCE, 0.						
Age at Entry.	No. at Risk.	Actual Deaths.	Attained Age.	COMPUTED DEATHS.		
				17 Offices.	Peerage Males.	New Experience.
15	137·5	..	15	·95	·64	·46
16	181	..	16	1·27	1·02	·63
17	241·5	1	17	1·71	1·80	·93
18	328·5	1	18	2·34	2·86	1·56
19	523	1	19	3·77	5·07	3·01
20	751	3	20	5·48	7·75	4·76
21	1217·5	5	21	8·99	12·97	8·19
22	1498	3	22	11·17	16·15	10·24
23	1738·5	5	23	13·14	18·69	11·74
..	....	..	..	....	....	....
..	....	..	..	....	....	....
..	....	..	..	....	....	....
..	....	..	..	....	....	....
69	42	3	69	2·52	2·16	2·40
70	28·5	2	70	1·85	1·64	1·77
71	30	..	71	2·11	1·94	2·04
72	15·5	..	72	1·18	1·12	1·16
73	21	1	73	1·72	1·70	1·74
74	9·5	..	74	·84	·87	·87
75	4	..	75	·38	·39	·39
Total	63,644·5	290		687·91	699·66	650·32

YEAR OF INSURANCE, 1.						
Age at Entry.	No. at Risk.	Actual Deaths.	Attained Age.	COMPUTED DEATHS.		
				17 Offices.	Peerage Males.	New Experience.
15	255	1	16	1·79	1·44	·91
16	321·5	3	17	2·27	2·39	1·24
17	442·5	4	18	3·16	3·87	2·12
18	586	6	19	4·22	5·70	3·37
19	915·5	5	20	6·68	9·44	5·79
20	1263·5	4	21	9·32	13·46	8·50
21	2144·6	14	22	16·01	23·14	14·65
22	2646	17	23	20·02	28·47	17·89
23	3105	25	24	23·81	32·94	20·61
..	....	..	..	....	....	....
..	....	..	..	....	....	....
..	....	..	..	....	....	....
..	....	..	..	....	....	....
69	74	2	70	4·81	4·26	4·60
70	49·5	2	71	3·47	3·19	3·36
71	54	2	72	4·09	3·91	4·05
72	27·5	2	73	2·25	2·23	2·27
73	39	4	74	3·45	3·56	3·56
74	18	2	75	1·72	1·77	1·77
75	8	..	76	·83	·83	·85
Total	116,565·0	891		1,310·63	1,309·51	1,246·93

The probabilities according to the last table not having been yet published, I give them in the subjoined Table (D); and I have added, in order to save my readers the trouble of reference, the probabilities of dying in a year according to the other tables of mortality.

It will be noticed that the "Peerage Males" and the "New Experience" tables agree in presenting a secondary maximum in the mortality at the age 22; that is to say, that according to both tables, the mortality at that age is greater than it is at the ages which immediately follow as well as those which precede. In this respect there can, I think, be no doubt that they agree with the observed and ascertained facts of human mortality. The probability of dying in a year at the above age, 22, is much less in the "New Experience" than in the "Peerage Males"; but this may be fairly attributed to the medical examination to which assured lives are subjected. The two tables are far from agreeing as to the age at which the secondary minimum of mortality falls; the "Peerage Males" placing it at 34, while the "New Experience" places it at 25. Of these results, the former appears by far the more probable. It is scarcely necessary to point out that in comparing the two tables, it must be constantly borne in mind that the medical selection at the time of assurance exercises an influence in the direction of diminishing the rate of mortality throughout nearly the whole of the "New Experience" table. Notwithstanding this influence, the rates of mortality in the two tables very closely agree on the whole. The very rapid increase in the rate of mortality in the "Peerage Males" from the age 83 to the end of life, is probably owing to faulty graduation. The general agreement of the two tables, combined with the effect of the medical selection already pointed out, is a very strong argument in favour of the existence of some influence tending to raise the rate of mortality in the subsequent years of the insurance.

TABLE (D).

PROBABILITIES OF DYING IN A YEAR.							
Age.	Peerage Males.	17 Offices.	New Experience, adjusted by Mr. Woolhouse.	Age.	Peerage Males.	17 Offices.	New Experience, adjusted by Mr. Woolhouse.
15	·00473	·00694	·00340	58	·02122	·02639	·02563
16	·00564	·00700	·00353	59	·02236	·02825	·02754
17	·00743	·00706	·00388	60	·02369	·03034	·02968
18	·00876	·00713	·00479	61	·02523	·03261	·03204
19	·00970	·00721	·00575	62	·02703	·03512	·03464
20	·01031	·00729	·00633	63	·02914	·03784	·03749
21	·01065	·00738	·00673	64	·03161	·04083	·04041
22	·01079	·00746	·00684	65	·03450	·04408	·04343
23	·01076	·00756	·00676	66	·03787	·04761	·04657
24	·01061	·00767	·00664	67	·04179	·05147	·04989
25	·01037	·00777	·00663	68	·04634	·05563	·05323
26	·01009	·00789	·00669	69	·05159	·06009	·05734
27	·00978	·00801	·00690	70	·05762	·06493	·06219
28	·00947	·00814	·00717	71	·06452	·07016	·06805
29	·00919	·00828	·00743	72	·07237	·07580	·07494
30	·00894	·00842	·00772	73	·08127	·08188	·08286
31	·00873	·00858	·00792	74	·09129	·08847	·09120
32	·00859	·00875	·00810	75	·09850	·09556	·09836
33	·00851	·00892	·00829	76	·10413	·10318	·10637
34	·00850	·00910	·00850	77	·10890	·11147	·11469
35	·00856	·00929	·00877	78	·11381	·12044	·12321
36	·00869	·00948	·00911	79	·11978	·13006	·13306
37	·00889	·00969	·00946	80	·12769	·14041	·14465
38	·00915	·00991	·00978	81	·13838	·15144	·15804
39	·00948	·01013	·01008	82	·15261	·16319	·17135
40	·00985	·01036	·01031	83	·17108	·17591	·18585
41	·01027	·01061	·01049	84	·19432	·18968	·19888
42	·01074	·01089	·01073	85	·22271	·20510	·20989
43	·01124	·01125	·01113	86	·25644	·22248	·21965
44	·01176	·01170	·01156	87	·29544	·24223	·23123
45	·01231	·01221	·01219	88	·33940	·26527	·23930
46	·01287	·01284	·01294	89	·38775	·29238	·25320
47	·01344	·01352	·01370	90	·43967	·32373	·27945
48	·01403	·01426	·01444	91	·49415	·36099	·31274
49	·01462	·01506	·01522	92	·54999	·40526	·35131
50	·01522	·01594	·01595	93	..	·45723	·41578
51	·01582	·01690	·01667	94	..	·51630	·50730
52	·01645	·01795	·01755	95	..	·58427	·63704
53	·01710	·01909	·01860	96	..	·64865	·81633
54	·01779	·02031	·01973	97	..	·69231	1·00000
55	·01852	·02166	·02103	98	..	·75000	
56	·01932	·02313	·02245	99	..	1·00000	
57	·02021	·02468	·02399				

The totals of table (C) were then collected, as shown in Table (E), which exhibits for each separate year of insurance, but for all ages collectively, the numbers at risk, the actual deaths, and the expected, or probable, deaths according to the three tables, with the percentages that the actual deaths are of the expected in each case.

TABLE (E).

Year of Insurance.	Number at Risk.	Actual Deaths.	COMPUTED DEATHS (a), AND PERCENTAGE OF ACTUAL DEATHS TO COMPUTED (b).					
			17 Offices.		Peerage Males.		New Experience.	
			$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$
0	63644.5	290	687.91	42.2	699.66	41.4	650.32	44.6
1	116565	891	1310.63	68.0	1309.51	68.0	1246.93	71.5
2	103312.5	1028	1208.71	85.0	1186.64	86.6	1156.45	88.9
3	92759	1075	1135.11	94.9	1094.12	98.2	1089.77	98.6
4	83589.5	1106	1066.97	103.7	1015.42	108.9	1030.96	108.4
5	75943	996	1013.70	98.3	952.92	104.5	983.35	101.3
6	69422	991	969.47	102.2	902.11	109.9	943.79	105.0
7	63136	946	923.29	102.5	852.44	111.0	901.66	104.9
8	57170.5	882	874.56	100.9	802.24	109.9	856.22	103.0
9	51890	851	833.12	102.1	760.02	112.0	817.41	104.1
10	46797.5	845	789.44	107.0	716.99	117.9	775.98	108.9
11	41907	753	746.30	100.9	675.39	111.5	734.64	102.5
12	37676	753	706.89	106.5	637.20	118.2	696.57	108.1
13	33761	725	667.33	108.6	599.60	120.9	658.23	110.1
14	30281.5	697	632.49	110.2	566.88	123.0	624.18	111.7
15	26903.5	661	594.23	111.2	531.41	124.4	586.79	112.6
16	24128	618	563.44	109.7	502.54	123.0	556.51	111.0
17	21411	516	531.16	97.1	472.93	109.1	524.67	98.3
18	19162.5	549	504.92	108.7	448.86	122.3	498.78	110.1
19	16844	482	471.18	102.3	418.48	115.2	465.36	103.6
20	15014.5	449	443.99	101.1	393.61	114.1	438.46	102.4
21	13423.5	449	417.61	107.5	369.09	121.7	412.50	108.8
22	11892.5	412	389.68	105.7	343.47	120.0	384.89	107.0
23	10527	359	364.58	98.5	321.13	111.8	360.28	99.6
24	9192.5	345	338.82	101.8	298.98	115.4	335.05	103.0
25	7904	312	307.59	101.4	271.65	114.9	304.13	102.6
26	6863.5	320	280.97	113.9	247.98	129.0	277.94	115.1
27	5962	244	256.86	95.0	226.35	107.8	254.30	95.9
28	5151.5	236	235.48	100.2	208.29	113.3	233.23	101.2
29	4428	228	213.00	107.0	188.68	120.8	211.09	108.0
30	3827.5	182	193.80	93.9	172.17	105.7	192.10	94.7
31	3316.5	198	176.78	112.0	157.70	125.6	175.38	112.9
32	2759	157	155.59	100.9	140.30	111.9	154.60	101.6
33	2316.5	141	136.91	103.0	124.30	113.4	136.28	103.5
34	1963	135	121.25	111.3	110.68	122.0	120.50	112.0
35	1602.5	96	104.92	91.5	96.90	99.1	104.48	91.9
36	1353	98	92.54	105.9	85.86	114.1	92.32	106.2
37	1106	70	77.03	90.9	71.19	98.3	76.89	91.0
38	919.5	68	68.02	100.0	63.49	107.1	67.96	100.1
39	706	49	55.61	88.1	52.05	94.1	55.56	88.2
40	534	42	44.52	94.3	42.09	99.8	44.52	94.3
41-63	1890	196	195.60	100.2	190.73	102.8	196.13	99.9
Total	1188958.5	20441	20900.00	97.80	19322.05	105.79	20427.16	100.07

It will be seen that the columns of percentages show, as was to be anticipated, considerable irregularities; and in order to draw any conclusion, it is necessary to combine the facts into a series of groups, as shown in the following Table (F):—

TABLE (F).

Year of Insurance.	Number at Risk.	Actual Deaths.	COMPUTED DEATHS (a), AND PERCENTAGE OF ACTUAL DEATHS TO COMPUTED (β).					
			17 Offices.		Peerage Males.		New Experience.	
			a	β	a	β	a	β
0	63644·5	290	687·91	42·2	699·66	41·45	650·32	44·6
1	116565	891	1310·63	68·0	1309·51	68·04	1246·93	71·5
2	103812·5	1028	1208·71	85·0	1186·64	86·63	1156·45	88·9
3-5	252291·5	3177	3213·78	98·86	3062·46	103·74	3104·08	98·64
6-10	288416	4515	4389·88	102·85	4033·80	111·93	4295·06	105·12
11-15	170529	3589	3347·24	107·22	3010·48	119·21	3300·41	108·74
16-20	96560	2614	2514·69	103·95	2236·42	116·89	2483·78	105·24
21-25	52939·5	1877	1818·28	103·23	1604·32	116·99	1796·85	104·46
26-30	26232·5	1210	1180·11	102·53	1043·47	115·96	1168·66	103·54
31-63	18468	1250	1228·77	101·73	1135·29	110·10	1224·62	102·07

The columns of percentages now exhibit such a regular progression as to afford the strongest possible presumption that we have here the true law of the results indicated. Stating some of these results in words, we may say that the mortality in the year 0 of insurance is 42·2 percent\* of that computed by the "17 Offices' Table"; the mortality in the year 1 of the insurance is 68 percent of that similarly computed; and in the year 2, 85 percent. In the years 3, 4 and 5 together the percentage amounted to 99. In the group including years 6 to 10 inclusive the actual deaths were 102·9 percent of the computed; and in the years 11 to 15, 107·2 percent of the computed. Here however a maximum has been reached, and the ratio of the actual deaths to the computed diminishes from this point. In the years 16 to 20 the percentage is 104; in 21 to 25 it is 103·2; in 26 to 30 it is 102·5; and for the subsequent years it is 101·7.

The above result at first sight is so much opposed to all our preconceived ideas that we feel disposed to reject it as incredible. Such was certainly my own inclination; for I had quite expected that the ratio of the actual to the computed deaths would increase throughout, and be greatest in the latest years of insurance. It occurred to me that the fact of grouping together persons of all ages at entry might have a disturbing effect upon the results, as would clearly be the case if the table of mortality from which the probable deaths are computed, gave much too high a mortality at

\* It seems time that this phrase, of such constant occurrence, should be no longer printed *per cent.*, and regarded as a mere abbreviation of *per centum*, but be finally adopted into the English language by being written as a single word *percent*, as has been done in some other languages, as German and Danish. The fact that the phrase has already been virtually adopted is proved by the use of its derivative *percentage*.

the early or middle ages; and that if the examination were confined to persons of nearly the same age, the result might be very different. I therefore had the results distributed into smaller groups, containing respectively the persons whose ages *at the time of observation* were 15 to 20, 21 to 25, 26 to 30, . . . . . and so on up to 91 to 95. Before giving the results of this grouping, however, I will draw attention to the figures given by Mr. Spens in his Tables of the Scottish Amicable Experience, already mentioned. Table EM<sub>A</sub> of that work shows the deaths actual and expected by the English Life Table No. 2, arranged in quinquennial groups of ages, in each year of the assurances. From this table, combined with the totals of Table MA, which shows the lives at risk and deaths in each year of the insurances, I deduce the following results:—

TABLE (G).

Years of Insurance.	Numbers at Risk.	Actual Deaths.	Expected Deaths.	Percentage of Actual to Expected Deaths.
1	8,482	47	105·641	44·49
2	7,079	61	91·998	66·30
3, 4	11,004	115	151·593	75·86
5	4,069½	50	59·956	83·39
6	3,365	49	52·146	93·96
7, 8, 9	6,698½	122	114·678	106·39
10, 11, 12	3,018	64	60·939	105·02
13, 14, 15	1,895	42	45·025	93·28
16 to 22	2,084	57	63·848	89·27
23 to 34	653	25	31·169	80·21
Total . . . . .	48,348	632	776·993	81·34

These results in their general character fully agree with those given above. The expected deaths having been calculated from a different table of mortality, the actual values of the percentages are of course different; but there is the same rapid increase to a maximum, and then the same slower diminution. The principal point of difference would seem to be in the position of the maximum, which here comes at the 9th year, while in the previous table it is about the 13th. It will be noticed that there is no “year of insurance, 0,” entered in the above table. This arises from Mr. Spens having adopted a different mode of estimating the years of insurance from that adopted by the Committee of the Institute. The latter have based their calculations on *calendar* years, assuming that on the average six months elapse between

the effecting of each insurance and the following 31st December, and considering these six months as the "year of insurance, 0." Mr. Spens has followed what appears to me the much preferable course of considering the 12 months which follow the grant of the insurance, as constituting the "year of insurance, 1"; the next 12 months, the "year of insurance, 2"; and so on.

The above series of figures are deduced from Mr. Spens's facts without the least violence being done to them, and may almost be said to lie on the surface of the subject. It seems indeed strange that Mr. Spens should have missed them so completely as he has done. He says (*Journal*, vol. x., p. 70) that (having followed a certain process of reasoning, the nature of which he has not explained) he "believes that after six years it may be assumed " that the mortality is in accordance with that of the English " Life Table No. 2"; and adds (p. 71) "I am persuaded that " any correction which may be made can but little affect the " values of annuities calculated on this assumption." I cannot bring myself to agree in this conclusion. It may possibly be true; but if so, its truth should certainly be proved and not assumed. It is true that the actual deaths after six years are 310 against 315·659 expected, so that the difference is not large; but of these actual deaths no less than 186 occurred in the years 7 to 12 inclusive, against 175·717 expected, showing an excess of 10 deaths; while the actual deaths are less than the expected by only 3 in the years 13 to 15; and it is not till we arrive at the years 16 to 22, that the balance is restored by the actual deaths falling short of the expected by 7 nearly.

I have now to call attention to the following tables (H), in which the total number at risk, the actual deaths, and the computed deaths in each year of insurance, are given for persons whose present ages are 15 to 20, 21 to 25, 26 to 30, 31 to 35, &c. Since persons assured under the age of 15 are excluded from consideration, it is clear that in the first group (present ages 15 to 20) the facts will entirely fail us after five years of insurance. In the next group (ages 21 to 25) the facts will extend to 10 years of insurance; while it is not until we arrive at the present ages 51 to 55, that the facts will extend up to the fortieth year of insurance.

TABLES (H).

Years of Insurance.	Number at Risk.	Actual Deaths.	COMPUTED DEATHS AND PERCENTAGE OF ACTUAL DEATHS TO COMPUTED.					
			17 Offices.		New Experience.		Peerage Males.	
			Number.	Per-centage.	Number.	Per-centage.	Number.	Per-centage.
Present ages 15-20.								
0	2162.5	6	15.52	38.7	11.35	52.9	19.14	31.3
1, 2	3921.5	25	28.21	88.6	21.07	118.7	35.94	69.6
3-5	1311.	11	9.51	115.7	7.75	141.9	12.93	85.1
	7,395	42	53.24	78.9	40.17	104.6	68.01	61.8
Present ages 21-25.								
0	8814	24	66.97	35.8	59.09	40.6	93.50	25.7
1	12857.5	76	97.89	77.6	86.16	88.2	136.36	55.7
2, 3	13698.5	97	104.57	92.8	91.61	105.9	144.97	66.9
4	3124.5	29	23.89	121.4	20.89	133.8	32.96	88.0
5, 6	2858.5	31	21.80	142.2	19.04	162.8	30.21	102.6
7-10	1217	15	9.34	160.6	8.05	186.3	12.85	116.7
	42,570	272	324.46	83.8	284.84	95.5	450.85	60.3
Present ages 26-30.								
0	13106.5	39	106.86	36.5	94.26	41.4	124.21	31.4
1	23172	132	189.11	69.8	167.06	79.0	219.25	60.2
2	18888.5	142	154.23	92.1	136.30	104.2	178.51	79.5
3	15260.5	121	124.77	97.0	110.44	109.6	143.87	84.1
4	11854	115	97.04	118.5	86.00	133.7	111.48	103.2
5- 8	22178.5	220	182.28	120.7	162.26	135.6	207.13	106.2
9-15	3681	38	30.40	125.0	27.09	140.3	34.00	111.8
	108,141	807	884.69	91.2	783.41	103.0	1,018.45	79.2
Present ages 31-35.								
0	12510	66	111.58	59.2	103.86	63.5	107.31	61.5
1	23480.5	141	209.50	67.3	195.07	72.3	201.40	70.0
2	21583.5	164	192.49	85.2	179.32	91.5	185.19	88.6
3	19333.5	169	172.56	97.9	160.72	105.2	165.82	101.9
4- 7	56488.5	505	505.28	99.9	470.85	108.8	484.16	104.3
8-11	24229.5	237	217.86	108.8	203.28	116.6	207.37	114.3
12-20	5607.5	60	50.82	118.1	47.37	126.7	47.80	125.5
	163,233	1,342	1,460.09	91.9	1,360.47	98.6	1,399.05	95.9

TABLES (H)—continued.

Years of Insurance.	Number at Risk.	Actual Deaths.	COMPUTED DEATHS AND PERCENTAGE OF ACTUAL DEATHS TO COMPUTED.					
			17 Offices.		New Experience.		Peerage Males.	
			Number.	Per-centage.	Number.	Per-centage.	Number.	Per-centage.
Present ages 36-40.								
0	9920·5	39	98·18	39·7	96·46	40·4	91·12	42·8
1	19408·5	119	192·11	61·9	188·71	63·1	178·35	66·7
2	18108·5	150	179·10	83·8	175·91	85·3	166·20	90·3
3	17465	170	172·76	98·4	169·69	100·2	160·34	106·0
4, 5	32037	332	317·12	104·7	311·62	106·5	294·40	112·8
6-9	50889·5	566	504·63	112·2	496·14	114·1	468·80	120·7
10-16	37115	438	369·99	118·4	364·43	120·2	344·34	127·2
17-25	3042	36	30·65	117·5	30·28	118·9	28·53	126·2
	187,986	1,850	1,864·54	99·2	1,853·24	100·9	1,732·08	106·8
Present ages 41-45.								
0	6925	29	78·13	37·1	77·34	37·5	77·56	37·4
1	13904	110	156·85	70·1	155·24	70·9	155·63	70·7
2	13619·5	138	153·29	90·0	151·69	91·0	152·01	90·8
3	13172	141	148·42	95·0	146·86	96·0	147·27	95·7
4-7	49349·5	556	557·34	99·8	551·63	100·8	553·37	100·5
8-12	48833·5	581	552·51	105·2	546·85	106·2	548·76	105·9
13-15	18948·5	229	215·23	106·4	213·12	107·5	214·11	107·9
16-19	12863·5	182	147·08	123·7	145·66	124·9	146·50	124·2
20-30	4247·5	64	49·19	130·1	48·71	131·4	49·07	130·4
	181,863	2,030	2,058·04	98·6	2,037·10	99·7	2,044·28	99·3
Present ages 46-50.								
0	4660	30	66·18	45·3	66·79	44·9	64·99	46·2
1	9430	93	133·97	69·4	135·19	68·8	131·52	70·7
2	9243	115	131·07	87·7	132·28	86·3	128·71	89·3
3	9062	129	128·68	100·2	129·84	99·4	126·33	102·1
4, 5	17618·5	253	250·55	101·0	252·82	100·1	245·88	102·9
6, 7	17489	253	248·14	102·0	250·36	101·1	243·66	103·8
8-10	24958	376	355·20	105·9	358·46	104·9	348·63	107·9
11-13	21600	338	307·90	109·8	310·60	108·8	301·99	111·9
14-18	26265	414	375·63	110·2	378·98	109·2	368·22	112·4
19-25	14416·5	231	208·99	110·5	210·66	109·7	204·13	113·2
26-35	1438·5	23	21·41	107·4	21·53	106·8	20·77	110·7
	156,180·5	2,255	2,227·72	101·2	2,247·51	100·3	2,184·83	103·2

TABLES (H)—continued.

Years of Insurance.	Number at Risk.	Actual Deaths.	COMPUTED DEATHS AND PERCENTAGE OF ACTUAL DEATHS TO COMPUTED.					
			17 Offices.		New Experience.		Peerage Males.	
			Number.	Per-centage.	Number.	Per-centage.	Number.	Per-centage.
Present ages 51-55.								
0	2832	23	53.53	43.0	52.28	44.0	48.09	47.8
1	5896	56	111.55	50.2	108.96	51.4	100.17	55.9
2	5959	92	112.65	81.7	110.00	83.6	101.17	90.9
3	6015	104	114.07	91.2	111.36	93.4	102.34	101.6
4-6	18004.5	323	342.08	94.4	333.92	96.7	306.64	105.3
7-10	23104.5	428	439.17	97.5	428.68	99.8	393.62	108.8
11-14	21280	430	404.35	106.3	394.70	108.9	362.43	118.6
15-18	17338	354	330.73	107.0	322.73	109.7	296.01	119.6
19-22	12765.5	271	244.61	110.8	238.64	113.6	218.52	124.0
23-40	10834	221	210.52	105.0	205.13	107.7	187.11	118.1
	124,028.5	2,302	2,363.26	97.4	2,306.40	99.8	2,116.10	108.8
Present ages 56-60.								
0	1604.5	19	41.97	45.3	40.81	46.6	33.86	48.5
1, 2	6763	137	177.22	77.3	172.46	79.4	143.02	95.8
3, 4	6945	156	181.45	86.0	176.55	88.4	146.52	106.5
5-7	11192	270	292.82	92.2	284.91	94.8	236.34	114.2
8-11	15255.5	395	400.13	98.7	389.36	101.4	322.75	122.4
12-17	20338	565	534.21	105.8	519.81	108.7	430.76	131.2
18-22	13800.5	383	363.71	105.3	353.91	108.2	293.02	130.7
23-29	11540	305	306.24	99.6	298.08	102.3	246.28	123.8
30-end	3318.5	82	89.97	91.1	87.59	93.6	71.79	114.2
	90,757	2,312	2,387.72	96.8	2,323.48	99.5	1,924.34	120.1
Present ages 61-65.								
0	759.5	7	28.24	24.8	27.86	25.1	21.84	32.1
1	1652.5	38	61.73	61.6	60.90	62.4	47.73	79.6
2	1795.5	48	66.53	72.1	65.61	73.2	51.45	93.3
3-5	5897	215	219.64	97.9	216.73	99.2	169.88	100.5
6-11	13168.5	495	493.24	100.4	486.65	101.7	381.69	129.7
12-15	9142.5	354	343.02	103.2	338.48	104.6	265.49	133.3
16-21	11720.5	463	440.95	105.0	435.12	106.4	341.21	135.7
22-25	6698	257	252.17	101.9	248.79	103.3	195.10	131.7
26-31	6705	258	253.74	101.7	250.28	103.1	196.36	131.4
32-end	3239.5	123	124.71	98.6	122.95	100	96.55	127.4
	60,778.5	2,258	2,283.97	98.9	2,253.37	100.2	1,767.30	127.8

TABLES (H)—*continued.*

Years of Insurance.	Number at Risk.	Actual Deaths.	COMPUTED DEATHS AND PERCENTAGE OF ACTUAL DEATHS TO COMPUTED.					
			17 Offices.		New Experience.		Peerage Males.	
			Number.	Per-centage.	Number.	Per-centage.	Number.	Per-centage.
Present ages 66-70.								
0	270	7	14.52	48.2	14.02	49.9	12.02	58.2
1	638.5	21	34.40	61.0	33.16	63.3	28.51	73.7
2	717.5	26	38.64	67.3	37.27	69.8	32.02	81.2
3-5	2626	127	142.82	88.9	137.66	92.3	118.77	106.9
6-8	3153.5	173	171.59	100.8	165.43	104.6	142.76	121.2
9-13	5974	332	327.38	101.4	315.45	105.2	273.07	121.6
14-16	3817	213	208.93	101.9	201.36	105.8	174.28	122.2
17-26	11404	612	628.64	97.4	605.44	101.1	525.52	116.5
27-end	7682.5	409	426.55	95.9	410.34	99.7	357.44	114.4
	36,283	1,920	1,993.47	963	1,920.13	100.0	1,664.39	115.4
Present ages 71-75.								
0	80	1	6.23	16.1	6.20	16.1	6.02	16.6
1	188	12	14.98	80.1	15.01	79.9	14.66	81.9
2,3	420	30	33.38	89.9	33.38	89.9	32.60	92.0
4,5	573.5	45	45.19	99.6	45.09	99.8	43.90	102.5
6-9	1707	137	135.27	101.3	135.28	101.3	131.95	103.8
10-17	4692.5	397	376.75	105.4	377.88	105.1	369.40	107.5
18-21	2699.5	229	217.02	105.5	217.71	105.2	212.85	107.6
22-32	5998	499	484.80	102.9	486.93	102.5	476.64	104.7
33-end	2654	198	216.78	91.3	218.27	90.7	214.05	92.5
	19,012.5	1,548	1,530.40	101.2	1,535.75	100.8	1,502.07	103.1
Present ages 76-80.								
1-5	204	12	22.71	52.8	23.32	51.5	22.14	54.2
6,7	184.5	19	21.67	87.7	22.25	85.4	20.74	91.6
8-11	536.5	59	61.94	95.3	63.62	92.7	59.73	98.8
12-16	843	98	98.67	99.3	101.35	96.7	94.63	103.6
16-19	1037	124	120.56	102.9	123.82	100.1	115.90	107.0
20-26	1945	238	228.78	104.0	235.00	101.3	219.03	108.7
27-36	2117	268	249.79	107.3	256.65	104.4	239.04	112.1
37-end	941.5	119	112.54	105.7	115.44	103.1	107.16	111.0
	7,808.5	937	916.66	102.2	941.45	99.5	878.37	106.7

TABLES (H)—*continued.*

Years of Insurance.	Number at Risk.	Actual Deaths.	COMPUTED DEATHS AND PERCENTAGE OF ACTUAL DEATHS TO COMPUTED.					
			17 Offices.		New Experience.		Peerage Males.	
			Number.	Per-centage.	Number.	Per-centage.	Number.	Per-centage.
Present ages 81-85.								
6-17	443.5	72	73.73	97.7	77.14	93.3	70.41	102.3
18-21	304.5	51	52.17	97.8	54.52	93.5	50.79	100.4
22-26	441	75	74.87	100.2	78.30	95.8	72.45	103.5
27-31	436	82	74.03	110.8	77.40	105.9	71.66	114.4
32-end	750	147	128.70	114.2	134.49	109.3	125.30	117.3
	2,375	427	403.50	105.8	421.85	101.2	390.61	109.3
Present ages 86-90.								
11-23	139	29	34.32	84.5	32.31	89.8	42.26	68.6
24-33	169	39	42.81	91.1	39.94	97.6	53.15	73.4
34-end	173	44	43.76	100.5	40.78	107.9	54.53	80.7
	481	112	120.89	92.6	113.03	99.1	149.94	74.7
Present ages 91-96.								
All	66	27	27.35	98.7	24.96	108.2	31.38	86.1

The general results of these tables may be stated as follows:—

(1) *Present ages 15 to 20.* In the year 0 of insurance, the actual deaths are  $38\frac{1}{2}$  percent of those to be expected according to the 17 Offices' table. But in years 1 and 2 together they are  $88\frac{1}{2}$  percent of the expected; while in the remaining years 3 to 5, they are  $115\frac{1}{2}$  percent of the expected; or the mortality is considerably in excess of the expectation, notwithstanding the very short time which has elapsed from the date of the assurances.

(2) *Present ages 21 to 25.* In the year 0 of insurance the actual deaths are 36 percent of the anticipated according to the 17 Offices' table. In the year 1, they are  $77\frac{1}{2}$  percent of the same; in the years 2 and 3, 93 percent; in the year 4,  $121\frac{1}{2}$  percent; in years 5 and 6, 142 per cent; and in the remaining years,  $160\frac{1}{2}$  percent of the same. The mortality among persons of the same present age thus rapidly increases with the standing

of the assurance, until, when no more than  $6\frac{1}{2}$  years have elapsed from the date of assurance, the deaths are 60 percent in excess of those to be anticipated according to the experience of the 17 Offices. Comparing the mortality with that to be anticipated according to the "Peerage Males" table, we see that in the years 5 and 6, the two very closely agree, while in the subsequent years, the actual mortality is  $16\frac{1}{2}$  percent in excess of the expectation.

These results appear fully to confirm the opinion held by many actuaries, that, excluding the effect of quite recent selection, the mortality at these ages (21 to 25) is greater than at the succeeding ages.

(3) *Present ages 26 to 30.* Here the mortality in the year 0 of insurance is  $36\frac{1}{2}$  percent of the expectation—almost exactly the same as in group (2). It then increases rapidly until in the year 3 it is 97 percent of the expectation, and in year 4 it is  $118\frac{1}{2}$  percent, ultimately rising to 125 percent of the expectation.

(4) *Present ages 31 to 35.* Here there is a singular irregularity in the year 0 of insurance, the percentage of the actual to the anticipated deaths being much greater than in any other of the groups, and no less than 59 percent. By the year 4 to 7, the deaths amount to 100 percent of the expectation, and become ultimately 118 percent of the same.

(5) *Present ages 36 to 40.* The mortality in the year 0 of insurance is  $39\frac{1}{2}$  percent of the expectation. By the year 3 it has risen to  $98\frac{1}{2}$  percent, becoming ultimately  $117\frac{1}{2}$  percent of the expectation.

(6) *Present ages 41 to 45.* The progression here is very similar to the last. The mortality in the year 0 is 37 percent of the expectation. In the years 4 to 7 inclusive it is 100 percent of the expectation, and increases throughout, until after 19 years of insurance it is 130 percent of the expectation.

Hitherto, the rate of mortality has increased throughout with the standing of the insurance, with one trifling exception in the ages 36 to 40; but now a change begins to appear in the progression of the figures.

(7) *Present ages 46 to 50.* The mortality at these ages increases from 45 percent of the expectation in the year 0, to 100 percent in the year 3, and to  $110\frac{1}{2}$  percent in the years 19 to 25. But in the subsequent years, 26 to 35, it is slightly less, viz.,  $107\frac{1}{2}$  percent. The difference here is not great; but it is a decrease instead of an increase.

(8) *Present ages 51 to 55.* The mortality in the year 0, is

43 percent of the expectation, and increases until in the years 7 to 10 it becomes  $97\frac{1}{2}$  percent. It continues to increase until in the years 19 to 22 it is 111 percent; but in the following years it is less, being equal to 105 percent of the expectation. There is nothing in these figures, or in those relating to the next period 56—60, to bear out Mr. Farren's conclusion that the effect of selection is at its minimum at the "break of life" 51—60.

The change in the law which the figures follow, now becomes more marked. In the next group, *present ages 56 to 60*, the mortality in the year 0 is 45 percent of the expectation. It increases to 92 percent in the years 5 to 7, and 106 percent in the years 12 to 17. This is its maximum value. In the next five years, 18 to 22, it is rather less, and gradually decreases until after 29 years of insurance it is only 91 percent. A similar law is observable in the next three groups.

(10) *Present ages 61 to 65.* The mortality in the year 0 is here only 25 percent of the expectation, and in the years 0 and 1 combined 50 percent. It then increases until in the years 16 to 21 it is 105 percent of the expectation, which is its maximum value, as it subsequently gradually decreases until after 31 years of insurance it ultimately becomes  $98\frac{1}{2}$  percent of the expectation.

(11) *Present ages 66 to 70.* The mortality increases from 48 percent of the expectation in the year 0, to 101 percent in the years 6 to 8, and 102 percent in the years 14 to 16, then diminishing until ultimately after 26 years it is 96 percent of the expectation.

(12) *Present ages 71 to 75.* Here out of 80 persons at risk for a full year in the year 0 of insurance, one only died, the mortality thus being only 16 percent of the expectation; but in the years 0 and 1 combined, it is 61 percent of the expectation. It increases to  $99\frac{1}{2}$  percent in years 4 and 5, and  $105\frac{1}{2}$  percent in the years 18 to 21, becoming ultimately after 32 years of insurance, 91 percent.

At this point the law again begins to show symptoms of change, and the percentages ultimately exhibit a regular increase throughout as the length of the assurance increases.

(13) *Present ages 76 to 80.* The figures now become so few at starting that it is useless to take single ages. In the years of insurance 1 to 5 the mortality is 53 percent of the expectation. It increases to 99 percent in the years 12 to 15, and still further to 107 percent in the years 27 to 36, being subsequently rather less,  $105\frac{1}{2}$  percent of the expectation.

(14) *Present ages 81 to 85.* Here the mortality in the years 6 to 17 is  $97\frac{1}{2}$  percent of the expectation, and becomes after 31 years of insurance 114 percent of the same.

The figures at higher ages are so few that it is scarcely worth while to refer to them.

On a review of the above results we observe that the law exhibited in table (F), of the actual mortality as compared with the expected increasing to a maximum, and then again decreasing, is retained in a very marked manner throughout the interval of 20 years, from age 56 to 75 ; and it will be seen further that the figures involved are so large, and the results are so consistent with each other, that we are forced to admit the phenomenon as a true deduction from the facts. At both extremities of the above period, viz., from the age 46 to 55 at the one end, and from 76 to 80 at the other end, the same law appears in a less prominent manner ; while under the age of 46, and above the age of 80, the rate of mortality increases continually with the standing of the assurance.

Our next duty will be to endeavour to find some explanation of the apparent anomaly. This we shall find in the combined effect of the two causes pointed out above, viz. (1) the gradual wearing out of the beneficial effect of the medical examination at entry, and (2) the effect produced by the withdrawal of healthy lives.

The effect of the former of these upon the mortality has been already fully considered.

If we now consider the way in which the withdrawal of healthy lives must affect the rate of mortality among the persons remaining under observation, we shall see that it may be expected to produce precisely the effect shown in the above tables. We shall find that the effect produced upon the rate of mortality by the withdrawal of a number of healthy persons, will be precisely the same as would be produced by the addition of a number of unsound lives. To prove this, let us suppose 1000 persons insure their lives at the same time, and that they are divided into two bodies of 500 each ; then after the lapse of a year, a certain proportion of the lives in each body, say, for example, to fix the ideas, five percent of the original number, *i.e.* 25, will have had diseases developed in them, which render them no longer insurable at the ordinary rate, while the other survivors of each body may be regarded as still insurable at the ordinary rate. If, now, we suppose that at the end of the year all these insurable lives in one of the bodies lapse their policies, there will remain assured of the one body only the 25 lives now uninsurable, and of the other body,

all the survivors of the original 500. Doubling these numbers, the rate of mortality is of course unaltered, and we see that the rate of mortality among the persons still remaining under observation out of the original 1000, will be the same as if to the survivors of the original 1000 there were added after the lapse of a year, 50 unsound lives. To express the result in a more general form, we may say that the withdrawal of the good lives increases the proportion which the bad lives bear to the whole. Now the greatest number of withdrawals take place within a few years from the date of the policy, and after the lapse of, say, 15 years, they become so few as to produce no appreciable effect upon the mortality. In order therefore to trace the effect of the withdrawals upon the rate of mortality, we may suppose that in each year of insurance up to the 15th, a diminishing number of unsound lives is added to those surviving out of the original number insured. These unsound lives will experience a very much heavier mortality than the mixed lives which survive from the original number insured; but as their numbers are by supposition continually recruited by fresh additions during the early years of the insurance, there will be a rapidly increasing number of deaths among them, which will have the effect of causing the rate of mortality among the whole body to increase faster than it would otherwise do. But when we come to the subsequent years of the insurance, when the unsound lives receive no new additions to their numbers, the heavy rate of mortality prevailing among them will cause their number to bear a continually diminishing ratio to the whole, and these unsound lives will consequently exercise each year a less influence on the rate of mortality. As far as they are concerned, there will therefore be a tendency for the rate of mortality to improve; but as the survivors of the original 500 in the above illustration are probably yearly getting, on the average, worse, there is a tendency on that account for the general mortality to get worse. We have thus two forces now operating in opposite directions, and it depends on which is the more powerful whether the mortality will improve or get worse; and there will be found no difficulty in explaining the whole of the observed results, by the combined operation of these two causes.

There can, I think, remain no doubt after a study of the figures in the above tables, that the withdrawals do exert a very great influence in increasing the rate of mortality. Mr. Higham, in the passage quoted above, (p. 331) draws this conclusion very

fairly from the circumstance that the mortality among insured lives after the lapse of some years from the date of selection, is much heavier than that among the population at large. It might be objected however that this reasoning is not conclusive, inasmuch as it is conceivable that the mortality among insured lives might be ultimately greater than that among the population at large, in consequence of their consisting mostly of persons who work harder than the average, and thus exhaust their vital energies prematurely. But no such argument can apply to explain the phenomena shown above, viz.—first the increase of the mortality and then its subsequent diminution.

The foregoing observations not only demonstrate that the withdrawals produce a powerful effect in increasing the rate of mortality among the lives remaining under observation, but enable us to see clearly the way in which this effect is produced, and, to a certain extent, to measure its magnitude. It is obvious that the greater the number of withdrawals, the greater will be the effect produced upon the mortality; and when any special circumstance causes an unusually large number of withdrawals, it may be expected to produce a most disastrous effect upon the mortality of the remaining lives. A conspicuous instance of this is afforded in the transfer of the business of one Office to another of inferior credit; and to this cause I believe may be principally attributed the recent lamentable failure of the Albert Life Office. On the other hand, it may be expected that those Offices which have the fewest withdrawals, will experience the most favourable mortality. Mr. Higham's remarks upon these points (pp. 191, 2) appear to me extremely judicious.

It is now time that I should point out what appear to me to be the objections to Mr. Higham's course of proceeding, and wherein my conclusions differ from his. His method of proceeding is as follows.

He groups together in separate classes the lives who entered at the ages  $5m-2$ ,  $5m-1$ ,  $5m$ ,  $5m+1$ ,  $5m+2$ , where  $m$  is an integer. He then says: "With a view to determine the length of  
" time during which selection continues to exercise an influence,  
" and during which consequently the classes must be kept apart,  
" the numbers exposed to risk, and the numbers who survived,  
" were combined for each class in quinary sets, and the probabilities  
" of living a year were thence deduced. The following is an  
" example of the resulting table:—

*"Years of Risk, Ages 70, 71, 72, 73 and 74."*

			Number at Risk.	Number who Survived.	Probability of Living a Year.
Lives selected at age 20			4	2	.50000
"	"	25	4	2	.50000
			16	16	1.
"	"	30	20	18	.90000
			286	260	.90908
"	"	35	306	278	.90849
			978	912	.93250
"	"	40	1284	1190	.92679
			1604	1470	.91647
"	"	45	2883	2660	.92104
			2126	1950	.91721
"	"	50	5014	4610	.91943
			2615	2402	.91854
"	"	55	7629	7012	.91912
			2855	2650	.92820
"	"	60	10484	9662	.92159
			3390	3156	.93098
"	"	65	13874	12818	.92389
			3197	2981	.93244
"	"	70	17071	15799	.92549
			1252	1171	.93532
			18323	16970	.92615

On considering the construction of this Table, it will be seen that the 978 persons entered as having been selected at the age 35, comprised all the persons who entered at the ages 33, 34, 35, 36, 37, and who had remained under observation 35, 36, 37, 38 or 39 years. They will therefore include persons who entered at the age of 33, and had been insured for 35 years, and who had therefore attained the age of 68; and persons who entered at the age of 37, and had been insured 39 years, and had consequently attained the age of 76. The mean age of the persons in each group may therefore vary between wide limits, which it is impossible to predict beforehand, and which will depend partly upon the relative numbers insured at the several ages, and partly upon the relative numbers remaining under observation to the five ages which are grouped together. It is therefore clear that when the numbers assured at consecutive ages vary rapidly, there may result an alteration in the value of the probability of living a year, which Mr. Higham's method would attribute to the effect of selection. The following, however, is a more weighty objection to Mr. Higham's method. He says "Assuming that there *is* a virtue in selection, "it would be found, but for the irregularities which affect all "observations of the kind, that commencing at the bottom of the

“ table and reading upwards, the probabilities would continually  
 “ diminish until we should reach the class, or the first of the classes,  
 “ the lives in which have been assured so long that the effect of  
 “ selection has been exhausted from among them ; and from this  
 “ point, still reading upwards, the probabilities would be all alike.”

It will be seen that Mr. Higham compares the probability of living a year, as deduced from observations on persons insured at the mean age, say 50, with that obtained from observations on all the persons, of the same mean age, who have been longer insured. In this way, he arrives at the result that the effect of selection among persons insured at the mean age 50 is exhausted by the time they have reached the mean age 72, inasmuch as the probability of living a year, .91854, deduced from the 2,615 persons who are insured at the mean age 50, is less than the probability, .91943, deduced from the 5,014 persons who were insured at younger ages ; or, putting the question in another way, the effect of selection among persons of the present mean age 72 is not traceable further back than to persons who entered at the mean age 50. This is correct according to the assumption Mr. Higham has made. But we have seen that the true law deducible from the observations is that the probability of living a year decreases as the standing of the policy is greater, down to a minimum ; and then increases, as the standing of the policy becomes still greater ; and it is obvious that Mr. Higham's method of procedure is calculated rather to disguise and keep out of sight this law than to assist in bringing it to light. On examining more closely Mr. Higham's figures, this is seen to be actually the case. Thus, referring to the table quoted above, we see that the probability of living a year at the mean age 72, is

.91854	as deduced from	2,615	persons insured at the mean age	50,
.91721	”	2,161	”	” 45,
.91647	”	1,604	”	” 40.

In this way, we see that the effect of selection is traceable back to persons entering at 40 instead of 50 ; for the probability of living a year steadily diminishes with the length of the insurance, until we arrive at the persons insured at the mean age 40, when it has a minimum value, the value of the probability as deduced from the 1,284 persons insured at younger ages than 40, being .92679. It will here be noticed how much our judgment as to the weight to be attached to each probability, is assisted by having the numbers at risk stated ; and it is much to be regretted that Mr. Higham has not given these particulars in every case.

As already mentioned, Mr. Higham deduces from his facts the simple rule that the effect of selection lasts for half the length of time intervening between the age at entry and 80; but on examining the table given by Mr. Higham on pp. 184, 5, it will be seen that from the mean ages at entry 25 to 45 inclusive, comprising 41,758 persons assured out of the total number 56,942, the effect of selection lasts for a longer time than the rule would indicate; the excess being  $2\frac{1}{2}$  years at 25, 5 years at 30,  $17\frac{1}{2}$  years at 35, 10 years at 40, and  $2\frac{1}{2}$  years at 45. I am thus led to the conclusion that, much as Mr. Higham's estimate of the duration of the effect of selection exceeds that of some authors, it still falls short of the truth at the middle and most important ages of life. Mr. Higham states that his tables of mixed mortality are framed by a combination of all the data which supplied the probabilities on the left side of the bar; but he does not state distinctly whether the dotted or continuous bar is here meant. If the former, it seems probable that the omission of the important portion of the facts corresponding to the figures included between the continuous and the dotted bars, may have caused some material error in the values of annuities as given in the subsequent part of the paper.

(*To be continued.*)

*On the Proper Method of Loading the Premiums required for the Assurance of Sums at Death, &c.* By WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

OUR predecessors adopted a ready and (all things considered) a sufficiently effectual method of regulating the charges to be made for granting assurances on lives. They satisfied themselves on the one hand that the mortality table adopted allowed for a much higher mortality than would probably be experienced, and on the other they assumed in their calculations a rate of interest below that which they knew would be realized on their investments. With these precautions they could rely pretty safely upon realizing a considerable profit, and this consideration not unnaturally proved sufficient for the scientific requirements of the time.

The chief defect of this system, as is the case with all the so-called "practical" methods, not founded upon true principles, consisted in the fact that while it answered sufficiently well in ordinary cases, there were others in which it proved totally

inadequate to the end in view. In dealing with assurances effected upon one life against another, for instance, it was quite impossible to say in which direction the assumed exaggerated mortality might operate; and thus an opportunity was afforded, as soon as more correct tables became accessible, for speculations disadvantageous to the Companies; opportunities of which there would be no lack of shrewd calculating men ready to take advantage.

The defect in question proved fatal to the system as soon as the direction of opinion fell to men possessed of the requisite scientific attainments. The writings and practice of Milne, Gompertz, De Morgan, Babbage and others, quickly abolished a practice, not only in the highest degree unphilosophical, but fraught with danger, not the less real because it was not recognised by the practical men by whom the system had been upheld.

The difference between the two schools, if I may so term them, is clearly explained in Gompertz's paper, read before the Royal Society on the 29th June, 1820. He says, "It appears a point of great importance to decide what ought to be the demands of those Companies, so that the public may reap the greatest benefit from them. And the only means of answering this question is the possession of the mathematical and philosophical principles by which those Institutions ought to be guided. In the present improved state of the science of life contingencies it is not sufficient for a proper regulation to follow old customs and calculations drawn from a less perfect experience than we have now the means of obtaining. . . . The tables should be as accurate as they can possibly be made, and the interest should be calculated at that rate which shall appear to be the average interest to be made for money; but such additional demands should be made by the Company or Institution as to leave an adequate portion for its security, profit and expenses, for it does not seem possible, in the various beneficial applications which can be made from a proper knowledge of this branch of the mathematics, to judge invariably how to adapt tables which are not correct in themselves, connected with a rate of interest which is not the average rate made in reality, so that the advantage may tend in one direction. In granting assurances on lives it is a practice to use a certain table of mortality and to calculate at a certain rate of interest without making any additional charge; in the presumption that the tables are in themselves incorrect but that their deviation from the truth is in favour of the Society; and that the interest of money is less

"than they can really make; but such a plan does not appear  
"to me sufficiently scientific to be followed by Companies  
"concerned with life contingencies generally."

Thus originated the method now universally acknowledged (if not universally acted upon) of basing the rates of premium upon what is termed a "true" table of mortality and a "true" rate of interest, and making a direct and independent provision for profit and expenses. The recognition of the *principle* was the great end attained; the method of determining the amount of such premium in each case, technically called the "loading" of the premium, being a matter of secondary although by no means inconsiderable importance.

To the principle thus introduced no exception can be taken. Like many other instances which occur in the progress of science, its truth is so obvious that we find it difficult to understand how it could ever have been disputed. The mode adopted of carrying it out, however—which I believe consisted invariably in the first instance of a percentage addition to the premium—was not, I venture to think, by any means so happily devised. Indeed it soon became apparent that the adoption of the new system involved us in difficulties and inconsistencies from which the old system was free. The most prominent was the excessively high scale of premiums required for assurances effected at advanced ages,—more especially if made by a single payment,—an evil which Mr. Jellicoe proposed to remedy by the ingenious suggestion of combining a proportional with a constant addition to the net premium. This palliative sufficiently accomplished the particular end in view, but left other—less obvious but not less real—anomalies untouched.

It is undoubtedly true that an Assurance Office is an establishment opened for the sale of reversions, and the great fundamental law of exchange that price varies so as to maintain the equality between supply and demand, applies just as much as in any other business. At first sight, therefore, it might seem perfectly reasonable that the margin for profit and contingencies should, as in most other cases, be proportional to the price of the thing brought to market.

But although it may sometimes be expedient to view an Assurance Office in this light, yet it must not be overlooked that the nature of its business differs in an essential point from the general case. Generally speaking, the price of anything in the market represents its *cost*,—*i.e.*, the outlay incurred in producing and

bringing it to market,—plus the *profit*, the expectation of which is the inducement for bringing it there. It follows that the amount of the profit properly consists of a percentage on the price. Now, in our case we have nothing analogous to cost, but the current expenses of the establishment and the capital at risk, neither of which, as I shall show, is proportional to the price of the benefits secured.

Let us take a simple instance. Suppose the chance of a given individual dying before the expiration of one year to be  $\cdot 1$ , or one-tenth—the rate of interest on the investments of the Company, 4 per cent,—and first let the question be to determine the premium to be charged for an assurance against the contingency mentioned. The true value of the risk for the sum of £100 is  $\cdot 1 \times (1\cdot 04)^{-1} \times 100 = 9\cdot 615$ . Now, seeing that the Office may be a loser to the extent of 86·539 by the transaction, it will, I think, be admitted that the margin or loading in this case should not be much less than 2·404, or 25 per cent of the net premium, or 20 per cent of the premium charged. For the sake of argument, I will assume that this is the proper ratio of profit for this case, according to the existing state of supply and demand.

I will now suppose that the benefit to be secured consists of the same sum payable in the event of the individual *surviving* the year. Here the true premium is  $\cdot 9 \times (1\cdot 04)^{-1} \times 100 = 86\cdot 539$ , and adding a loading of 25 per cent, as before, we arrive at the singular conclusion that £108·174 is the proper premium per cent to be charged in this case.

The explanation of this absurdity is easily seen to be that the capital at risk in the two cases instead of being *directly* is *inversely* proportional to the price. In the first case, the amount risked is 86·539, and the premium, 9·615; while in the second the figures are reversed, the amount risked being 9·615, and the premium, 86·539.

This effectually disposes of the question as regards the capital at risk. With respect to expenses it is evident that there is only one item in the charges of an Office—viz., the commission allowed to agents, which is proportional to the amount of the premiums received. I think it is very desirable that the present practice of allowing the commission on the entire premium should be discontinued, and that a higher rate, computed on the loading only, should be substituted for it,—but in the meantime of course the charge must be provided for by a proportional addition to the premium. This particular item therefore is excluded from the conclusions hereafter arrived at.

It will now, I think, be admitted that the percentage method of loading the premium is not at all applicable to the case, and that I have made good my assertion that the choice of this method was not a happy one. It now however devolves upon me to propose another which, while sufficiently convenient for practical use, shall at least be less obnoxious to censure than that which it is proposed to supersede.

It is an obvious and well-known fact that actuarial calculations are based upon two perfectly distinct and almost equally important elements,—viz. the rate of mortality at successive ages, and the rate of interest for money. These two elements enter into all the calculations connected with the business of life assurance,—sometimes in nearly equal proportions, at other times in proportions differing greatly from each other.

To resort to an extreme case,—which is generally the best for elucidating a principle,—let us suppose one in which the element of mortality is entirely eliminated,—leaving the element of interest only to be dealt with. For instance let us suppose that the transactions of the Office consist exclusively of deposits made from time to time by its policyholders as premiums to secure a sum of money at the expiration of a given time, independently of any contingency. Upon what principle should the relation between the deposits and the sum assured be determined?

This question I think admits of but one answer. An Office transacting business of the description referred to would first ascertain what rate of interest it might expect to realize upon its investments, and then calculate its premiums at a rate proportionately lower.

So far, then, our course seems clear enough. But let us now suppose that it is the element of interest which is annihilated,—the element of mortality remaining only to be dealt with.

I will first take the case of a simple Immediate Life Annuity. Now it is a peculiarity attaching to this benefit that the two elements of mortality and interest affect the value in precisely the same way. Thus, if we suppose a constant addition to be made to the force of mortality at all ages, the effect upon the calculated value of the annuity will be the same as if the addition had been made to the force of interest instead. Again, a table of the mean duration of life may be used as a table of the values of annuities certain (in perpetuity) at a variable force of interest equal to the force of mortality at each age.

This circumstance points out the way to deal with this case.

If the effect of mortality is identical with that of interest, then in loading a table in which the former is substituted for the latter, we have only to operate upon the mortality precisely as we should have done upon the interest,—that is, we must reduce the force of mortality in a constant proportion at all ages.

Reasoning by analogy, we might perhaps proceed at once to apply the principle thus deduced to the case of a life assurance, or sum payable at death,—of course *increasing* the force of mortality in a constant ratio instead of *diminishing* it. I prefer, however, to deduce this result independently from an entirely different point of view. It was, indeed, by the following course of reasoning that the principle itself was first suggested.

To take a simple case,—suppose an assurance effected for the term of one year only. Here the transaction between the Company and the policyholder is precisely of the nature of a wager, and the question resolves itself into an investigation of the proper mode of determining the odds to be taken or given by a gamester who pursues his avocation as a matter of business, and accordingly regulates his procedure upon strictly commercial principles.

Let  $a$  represent the chances in favour of the event upon which the claim depends, and  $b$  the chances against it;—the probability of the event consequently being  $\frac{a}{a+b}$ . Now, if we multiply this probability by a constant factor (which would correspond to our percentage loading on the premium), we involve ourselves in this dilemma. Let  $1+\lambda$  be the constant multiplier, then for all values of  $\lambda$  greater than  $\frac{b}{a}$  the probability of the event exceeds unity. This absurdity may, however, be avoided by a simple contrivance,—viz., by multiplying not the probability of the event but the chances in favour of it. The probability would then become  $\frac{a(1+\lambda)}{a(1+\lambda)+b} = \frac{a}{a + \frac{b}{1+\lambda}}$ , a quantity which, however great  $\lambda$  may

be taken, will always be less than unity.

Applying this to the case of an assurance payable at death, we have for and against the event of death in one year, at age  $x$ ,  $d_x$ , and  $l_{x+1}$  respectively. If therefore, in lieu of these, we assume the chances in question to be  $(1+\lambda)d_x$ , and  $l_{x+1}$ , we shall have for the probability of dying in the year,

$$\frac{d_x(1+\lambda)}{l_{x+1}+d_x(1+\lambda)} = \frac{d_x+\lambda d_x}{l_x+\lambda d_x} \text{ instead of } \frac{d_x}{l_x}.$$

The probability of living a year becomes

$$1 - \frac{d_x + \lambda d_x}{l_x + \lambda d_x} = \frac{l_{x+1}}{l_x + \lambda d_x},$$

and the probability of living  $n$  years,

$$\frac{l_{x+1}}{l_x + \lambda d_x} \times \frac{l_{x+2}}{l_{x+1} + \lambda d_{x+1}} \times \dots \times \frac{l_{x+n}}{l_{x+n-1} + \lambda d_{x+n}}$$

from which expression a table of annuity values might be computed, and the premium for a whole life assurance determined therefrom in the usual way.

Instead of proceeding by yearly intervals, however, it will be better to resort to the "continuous" method—which one may safely predict will form the basis of the future system of life contingencies. The expression  $\frac{d_x + \lambda d_x}{l_x + \lambda d_x}$  then becomes  $-\frac{dl_x}{l_x}(1 + \lambda)$ , for  $\lambda d_x$  becomes insignificant in comparison with  $l_x$ , as  $d_x$  is diminished without limit, but not so in comparison with  $d_x$ . Hence the effect of the loading is to convert the force of mortality from  $-\frac{dl_x}{l_x dx}$  to  $-\frac{dl_x}{l_x dx}(1 + \lambda)$ , that is, the force of mortality is increased proportionately at all ages.

In consideration, then, of the foregoing results, the method which I would propose for loading premiums is to operate upon the forces of interest and mortality, by dividing the former by  $1 + \lambda$  in all cases, and by multiplying or dividing the latter by the same quantity, according to the nature of the contingency upon which the claim depends,—the value of  $\lambda$  being determined according to the judgment of the computer and the circumstances of the case.

If we double the force of mortality at all ages, and then calculate a table of annuities therefrom, the result will be the same as if we computed a table of annuities on two joint lives of equal ages. And if we do the same with the force of mortality trebled at all ages, we get a table of annuities on three joint lives of equal ages. Consequently, if the loading in respect of mortality were supposed to be 100 per cent, and it were desired to find the premium to be charged for an assurance on a given life, we should merely have to calculate the premium for an assurance on two joint lives of the same age, and so on for other rates of loading.

Hence it appears that if we have tables of annuities calculated for equal ages, say, up to four lives, at various rates of interest, we should be able to determine the premium to be charged according

to any given rate of loading, in respect both to interest and mortality; for by taking out the results corresponding to  $\lambda=0, 1, 2, 3,$  and  $4,$  the value corresponding to any other value of  $\lambda$  could be found by interpolation. However, as it would be desirable to have a shorter method of attaining the required results, I proceed to show how this object may be accomplished.

The conclusions hitherto deduced are applicable to all tables of mortality, quite irrespective of their mode of construction. I shall now however assume that the law of mortality is expressed by the formula  $F_x = a + bq^x$ ; i.e. that the total force of mortality is composed of two partial forces, the one a constant, and the other increasing according to the Gompertzian theory of a geometrical progression.

Waiving on this occasion the question whether the formula referred to is a true exponent of the normal law of mortality, it will be allowed to be at least sufficient for practical purposes, if the tables at present in use be taken as the standard. No one I imagine will contend that anything is gained by retaining the irregularities exhibited by the Carlisle Table, or that a reconstructed table derived by means of the formula in question will not be at least as trustworthy as the original table. Such being the case there can be no reason why we should not avail ourselves of the extreme simplicity which the formula exhibits to deduce useful and important general principles to guide us in our professional pursuits.

As a case in point,—let us suppose that the quantity  $q$  is unity. The force of mortality, and therefore the annual premiums for assurances, are the same at all ages, and consequently no reserve fund would be required for the liabilities of an Office beyond what might be considered necessary to provide for an unusual mortality, as in the case of assurances against fire. But the series represented by  $a + bq^x$  may be made as nearly constant as we please by increasing the constant  $a$ . Hence it follows that a constant addition to the force of mortality at all ages, while it will necessarily increase the annual premium, will at the same time diminish the reserve required,—not *relatively* to the increased premium only,—but *absolutely*. This consideration shows very forcibly the fallacy of judging of the relative positions of Offices by the number of years premiums in hand, even when (as can rarely be the case) the circumstances are in all other respects parallel.

A consideration of the expression  $a + bq^x$  enables us also to

expose the popular fallacy that the reserve is required because the event of death must, sooner or later, take place. Death would be just as certain if the force of mortality were constant, precisely as if a coin be continually tossed it must, by the law of chances, *eventually* fall head, notwithstanding that the chance of its falling head is the same at each throw. Nevertheless, in the case of a constant mortality, as we have seen, no accumulation would be necessary; and hence we see that it is not the certainty of the event, but the continual increase in the force of mortality from age to age which alone necessitates the reserve.

Further, the expression  $a + bq^x$  shows us how it is that the rates of premium increase so much faster at higher ages than in youth. At the earlier ages the constant  $a$  is much the greater of the two terms, and hence the force of mortality at these ages is nearly constant. At the higher ages the term  $bq^x$  exceeds the constant, whence the force of mortality increases rapidly.

To return from this digression to the subject in hand, let us see what useful deductions can be drawn respecting the effect of multiplying the force of mortality by the constant loading  $1 + \lambda$ .

In the expression  $a(1 + \lambda) + b(1 + \lambda) \cdot q^x$ , which the force of mortality becomes, the addition  $a\lambda$  made to the constant term is equivalent to the introduction of the element of interest,— $a\lambda$  corresponding precisely to the force of discount. Hence we see that this part of the mortality loading merges into the loading on account of interest.

Again, the multiplication of the term  $bq^x$  has the effect of a constant addition to the age. Thus, putting  $b(1 + \lambda) \cdot q^x = b \cdot q^{x+d}$  we have  $1 + \lambda = q^d$ , whence  $d$  (the addition to the age)  $= \frac{\log(1 + \lambda)}{\log q}$  a constant quantity.

We thus see that the method of loading proposed resolves itself, in the case of tables constructed by means of the formula  $F_x = a + bq^x$ , into a constant addition (positive or negative) to the rate of interest, together with a constant addition to the age.

Having thus arrived at a very simple method for loading assurance premiums—such method resulting directly (in the case of the particular law of mortality referred to) from a rational and consistent principle—I would suggest that the method in question may with propriety be adopted as the proper one in all cases, whether the table of mortality be constructed precisely according to the given law, or not.

The following are specimens of the application of the method. They are the single and annual premiums for an ordinary life assurance, calculated from the Carlisle Table, upon the supposition that the true rate of mortality is increased in the ratio of 1:1.4, and the true rate of interest (supposed to be about  $4\frac{1}{2}$  per cent.) diminished in the ratio of 1.4:1.

Age.	Single Premium.	Annual Premium.
20	37.107	1.833
30	42.899	2.335
40	49.741	3.075
50	59.017	4.475
60	69.208	6.984

In reference to other descriptions of assurances, it may be sufficient to refer here to one or two of the more obvious results of the application of the theory. In the case of term assurances we should have an addition to the premium nearly equal to the addition made to the force of mortality. Again, in the case of an assurance on life against life, the principle of the method would require that the age of the one life should be increased, and that the age of the other should be diminished, the effect of which would doubtless be to increase the loading annually applied to these cases.

I propose on another occasion to go somewhat fully into these matters, when a complete set of specimen tables, embracing every description of benefit, will be given. In the meantime I will conclude this portion of my subject with the observation that in applying the theory to the business of a Reversionary Society, or an Office established for the *purchase* of interests depending upon the contingencies of life, we should of course have to reverse the process. The rate of interest, instead of being diminished, would have to be increased; and the force of mortality increased in the case of a life annuity, and diminished in that of a sum payable at death. So far as regards the rate of interest, this agrees with the course suggested by Mr. Sprague in his recent paper on this subject. The plan now proposed of operating upon the force of mortality also, by means of a constant addition to or deduction from the age, will, I think, be found desirable, more especially where advanced ages are concerned, in which cases the addition to the rate of interest has generally but little effect upon the result.

## HOME AND FOREIGN INTELLIGENCE.

## LAW UNION FIRE AND LIFE INSURANCE COMPANY.

*Established 1854.*

## REPORT OF THE DIRECTORS.

The Directors beg to submit to the Shareholders the Balance Sheet for the Tenth Year of the Company's operations.

On reference thereto it will be seen that in the Life Department, the total Assets amount to £487,789. 18s. 10d., and the Liabilities, including the present value of sums assured, and Annuities, and all outstanding claims, &c., amount to £414,155. 5s. 8d., leaving a surplus of £73,634. 8s. 2d., out of which sum the Directors recommend, under the advice of Mr. JELlicoe the eminent Actuary, the division of the sum of £11,830. 9s. 4d. among the Shareholders, and the Policy-holders entitled to participate in profits. The balance remaining will then be £61,803. 18s. 10d., which will be held in reserve.

The Proportion of Profits to be appropriated to the Shareholders, according to the terms of the Deed of Settlement, amounts to £5,000.

\* \* \* \*

The proportion of the Profits in the Life Department, belonging to the Policy-holders entitled to participate, will be appropriated amongst them as a Reversionary Bonus, in reduction of their Premiums, or in Cash, as they may elect. A notice of the amount allotted on each of such Policies will be sent in due course.

The total number of Life Policies in force on the 30th of September last (1864), (exclusive of Annuity Policies,) was 1,778, insuring the sum of £954,919.

\* \* \* \*

The Valuation of the Company's Assets and Liabilities has been made in the strictest manner, and at Three per cent Interest.

## STATEMENT of ASSETS and LIABILITIES on the 30th of September, 1864.

## LIFE DEPARTMENT.

Dr.

	£	s.	d.
To present value of sums Insured (including Endowments, Endowment Insurances and Annuities), less Re-insurances .....	394,890	18	11
„ Ditto ditto of Reversionary Bonus declared in 1859 .....	4,482	9	2
„ Proportion of Shareholders' Capital, being one fourth of the amount paid .....	12,500	0	0
„ Claims admitted, but not yet payable .....	1,925	0	0
„ Proportion of Balance of Dividend due to Shareholders .....	134	5	5
„ Annuities due, but not paid .....	98	14	10
„ Sundry Outstanding Accounts (including Re-insurances due) ..	173	17	4
„ Balance .....	73,634	8	2
Total .....	£ 487,789	13	10

## Cr.

	£	s.	d.
By present value of future Premiums, payable upon Life and Endowment Policies, less Re-insurances .....	373,801	6	11
" Proportion of Government Stock .....	19,491	11	6
" Ditto ditto Loans on Mortgage .....	77,730	7	2
" Amount invested in purchase of Reversions .....	3,951	16	2
" Ditto ditto Policies of other Offices .....	583	0	0
" Proportion of Capital invested in Building of No. 126, Chancery Lane (including Furniture and Fittings) .....	3,407	6	10
" Credit Premiums due upon Policies .....	841	14	5
" Premiums due, but not paid before 30th September .....	1,051	0	6
" Proportion of Interest due on Loans and Investments .....	2,368	13	10
" Balance in hands of Agents .....	£2,352	8	6
" " with Bankers (on Current and Deposit Account) .....	2,169	4	1
" " with the Secretary .....	41	3	11
	4,562	16	6
Total .....	£487,789	13	10

From the statement of Income and Expenditure it appears that the amount of Premiums received in the year ending 30th Sept., 1864, was £80,973 11s. 6d.—*Ed. J. I. A.*

## AUSTRALIAN MUTUAL PROVIDENT SOCIETY.

*Established 1849.*

## FOURTH QUINQUENNIAL REPORT.

## REPORT OF THE DIRECTORS.

The Directors submit to the Members the results of the Fourth Quinquennial Investigation, which has been made into the affairs of the Society, as at 28th February, 1869, and in connection therewith, beg to lay before the Meeting:—

## I. Report by the Actuary with the following appendices, viz:—

1. Balance Sheet of the Society.
2. Statement of Receipts and Expenditure of the Assurance Branch, from 1st March, 1864, to 28th February, 1869, and Balance Sheet of same Branch, at 28th February, 1869.
3. Do. do. of the Endowment Branch.
4. Do. do. of the Annuity Branch.
5. Summary and Valuation of the Policies in force at 28th February, 1869.
6. Summary of the Receipts and Disbursements of the Assurance, Endowment, and Annuity Branches, from the commencement of the Society, in 1849, to 28th February, 1869.
7. Summary shewing the state of the Assurance, Endowment, and Annuity Funds in each year, from the commencement of the Society, and at each Quinquennial Investigation.
8. Table of the adjustment of Interest between the Assurance, Endowment, and Annuity Branches, from the commencement of the Society, and at each Quinquennial Investigation.

9. Statement of the position of the Branches at 1859 and 1864, with and without adjustments.
10. Table shewing the division of the Total Expenses between the Branches from the commencement of the Society.
11. Table of Policies granted, void, and in force.
12. Table shewing the comparative rates of Bonus declared on Policies at the present and preceding Investigations.

II. Letter from PROFESSOR PELL, the Consulting Actuary, declaring the results arrived at by the Actuary, to be correct, safe, and equitable to all the Members in terms of the By-laws.

These documents were finally considered at a Special Meeting of the Board, held on the 13th July, when the following Resolutions adopting the views expressed by the Actuary, and confirming the appropriation of the surplus of the Assurance Branch, as recommended by him, and approved of by the Consulting Actuary, were recorded on the Minutes, viz:—

- 1.—That the Actuary's Report on the Fourth Quinquennial Investigation of the affairs of the Society be adopted.
- 2.—That of the ascertained surplus of the Assurance Branch, amounting to £153,437 10s. 8d., the sum of £97,333 be divided amongst the Members of that Branch, in terms of XXIX By-law.
- 3.—That £6,104 10s. 8d. be reserved for prospective Bonuses, in accordance with the XXX By-law, and that the prospective Bonus, which shall be allotted to every Policy of the Assurance Branch which shall become a claim, shall be at the rate of £1 10s. 0d. per cent. per annum on the sum assured, for the first two years of the new Quinquennium.
- 4.—That £50,000 be retained as a Guarantee Fund.

The Members will perceive from the documents submitted, that the Endowment and Annuity Funds are not in so satisfactory a condition as they were represented to be on former occasions, the reasons for which are dealt with at length by the Actuary, whose report is so complete and exhaustive, that your Directors feel there is little occasion for further remark from them.

The Assurance Fund yields a large surplus, and the announcement amongst other facts illustrative of its progress, that the realised assets of the Fund amount to 70 per cent. of the entire Assurance Premiums received by the Society, will be regarded as conclusive evidence of the highly satisfactory state of this Branch.

The formation of a Guarantee Fund, in accordance with the provisions of the XXIX By-law, is regarded by your Directors as matter for congratulation. The sum of £50,000 set apart for this purpose, may be practically viewed as having gradually accumulated, and is not entirely chargeable upon the profits of the past Quinquennium.

For the reasons assigned in the Actuary's Report, there will be a considerable difference between the Bonuses now declared, and those declared to Policies of corresponding durations at former Investigations.

Your Directors beg to express their opinion that the provisions contained in the By-laws regulating the distribution of surplus amongst the various Tables, require amendment. Under the present system, the excessive preponderance of Bonus in favor of Tables **B** (Limited Payments) and **J** (Endowment Assurances), over Table **A** (Ordinary Whole Life), under the Assurance Branch, is inequitable. It is, therefore, the intention of the

Board to propose an alteration in the By-laws, to secure a more equitable division of the Profits.

\* \* \* \* \*

On the 28th February last, the Society attained the *twentieth* year of its duration, a befitting occasion, in the estimation of your Directors, on which to lay before you so full and comprehensive a statement of your affairs.

The rate of Bonus being now confirmed, the calculations for the distribution of the surplus will be rapidly proceeded with, but they will necessarily occupy a considerable time. As soon as completed a certificate of the amount of Bonus allotted will be sent to each member.

### REPORT BY THE ACTUARY.

*Sydney, 6th July, 1869.*

BEFORE proceeding to report on the result of the Fourth Quinquennial Investigation of the Society's affairs, it will be proper for me to refer to the principles on which it has been conducted.

At the three preceding Quinquennial Investigations, the Carlisle rate of mortality was used with 4 per cent. interest. On this occasion the same table and rate of interest have been adopted.

The nett or pure premiums have alone been taken into account. That portion of each premium commonly called the loading, which provides for expenses of management and for future profit, has been wholly reserved, and therefore none of the profits to be hereafter realised have been anticipated.

The value of each policy has been computed separately. This was necessary, inasmuch as the By-laws require that the Profit "shall be rateably allotted to every policy \* \* \* in proportion to the actual value of the total amount thereof." With a view to test the accuracy of the separate values, I have carried out a second valuation of all the policies under Table A; this has been done by classifying the policies according to the existing ages of the lives assured, and ascertaining the total values at each age. This class valuation, embracing the bulk of the business, confirms the result of the detailed valuation.

With these explanations, I now submit the result of the Fourth Quinquennial Investigation, shewing the position of the Society on the 28th February last.

\* \* \* \* \*

#### TOTAL ASSETS:—

The Balance Sheet shows, that on the 28th February  
the Assets amounted to . . . . . £602,646 12 7

#### TOTAL LIABILITIES:—

Present value of Liabilities under Assurance, Endowment, and Annuity Policies, including Bonus Additions, Claims emerged, but not yet paid, Commission due to Agents, and all other outstanding accounts . . . . . £453,000 12 5

Balance . . . . . £149,646 0 2

\* \* \* \* \*

## ASSURANCE BRANCH:—

The Balance Sheet of this Branch shows, that on the 28th February the Assets amounted to	£536,901	0	2
Present value of Liabilities under Assurance Policies, including Bonus Additions and Claims emerged, but not yet paid	£383,463	9	6
	<hr/>		
Balance, being amount of Surplus on 28th February, 1869	£153,437	10	8
	<hr/>		

Of this Surplus, I recommend that £97,333, be divided among the members, in terms of the XXIX By-law, that £6,104 10s. 8d. be reserved for Prospective Bonuses in accordance with the XXX By-law, and that £50,000 be retained as a Guarantee Fund.

The surplus of £153,437 10s. 8d. as shewn above, correctly represents the state of the Assurance Fund on 28th February, 1869. But it does not show the total profits that have resulted from the operations of the Quinquennial period. To explain this, it will be necessary for me to refer to the alteration made in the By-laws, since the last Division of Profits, regulating the period of participation by members in the profits of the Society. The former practice was, to divide profits not only at a Quinquennial Investigation, but also in the intermediate years between Investigations, among those members whose policies successively attained a duration of five years. This yearly division of profit was at the same rate as that declared at the last Investigation. The system continued in operation until February, 1868, and the result is, that Bonuses of the value of £23,427 10s. 2d. have been already declared out of the profits of the five years ending 28th February, 1869. This sum of £23,427 10s. 2d. represents the abatement in the Bonuses to be now declared, on those policies that have participated during the past five years.

The account would then stand thus:—

Balance of the Assurance Fund, being the surplus at 28th February, 1869	£153,437	10	8
Profit divided during the Quinquennium	23,427	10	2
	<hr/>		
Total	176,865	0	10
Less Reserve Fund at 29th February, 1864	15,096	13	2
	<hr/>		
Gross Profit of the Assurance Branch for the five years ending 28th February, 1869	£161,768	7	8
	<hr/>		

The £97,333 to be apportioned among the members, together with the sum of £23,427 10s. 2d. already divided, represents reversionary bonuses amounting to £286,444, and if applied as a uniform bonus to all the policies now entitled to participate, would on an average give an addition of £1 10s. per cent. per annum on the sums originally assured, equivalent to 27 per cent. on the premiums paid. Or if the number of policies now entitled to participate were restricted to those that had endured for five years, or on which not less than six yearly premiums had been paid (a very common period in mutual offices), the Bonus to be declared would give a

reversionary addition of £2 2s. per cent. per annum on the original sums assured, equivalent to 46 [Qy. 36] per cent. on the Premiums paid. By the new regulations, all policies that have endured for one complete year or more now participate, and the total number so entitled is 6,553; but if participation was restricted to policies that had endured for five years, the number would be reduced to 2,950. Policies of one year's standing, instead of five years' as formerly, being admitted to participation in Profits has affected the rate of bonus to be declared on the present occasion to the extent of 12s. per cent.; 3,603 policies are now entitled to participate, which would not have received their share of profits under the former By-laws, until they had attained a duration of five complete years.

The policies in force on the 29th February, 1864, were 3,725, assuring with bonus additions £1,981,900, and producing an annual revenue of £66,900 in premiums. The total number of policies then entitled to participate was 925, covering assurances with bonus additions amounting to £514,449 19s. 8d. The policies in force on the 28th February last, numbered 7,677, assuring with bonus additions £3,796,315, and producing an annual revenue of £128,172 in premiums. The total number of policies now entitled to participate is 6,553 for assurances, including bonus additions, amounting to £3,322,315 17s. 2d. The average amount of each policy at the Investigation of 1864, was £525; now the average is £482.

The average amount per annum of increase during the past Quinquennium, has been 1,072 policies assuring £489,914, yielding new annual income of £17,864 14s. 9d.

The total Premiums received from the commencement of the Society to the 29th February, 1864, a period of fifteen years, was £288,459 6s. 2d. During the last five years the premiums received have amounted to £475,745 18s. 10d., exceeding by 60 per cent. the total premiums of the previous fifteen years.

The amount at credit of the Assurance Fund in 1864, was £218,461 1s. 7d. The amount now is £536,901 0s. 2d., exceeding the previous fifteen years' accumulations by £318,439 18s. 7d. The realised assets of the Fund amount to 70 per cent. of the entire Assurance Premiums received by the Society.

The share of each member in the divisible Surplus, is determined by the Laws of the Society, in proportion to the absolute value of his policy, *including previous additions.*

The Reserve which I have recommended to be made for Prospective Bonuses, to policies that may become claims by death during the next five years, is ample to provide for additions, at the rate of £1 10s. per cent. per annum on the original sum assured, for each complete year of the existence of the policy after this investigation. The rate at which the prospective bonuses are to be declared, is for the board to determine annually; but I recommend that it should be at a fixed rate for at least two years; and not less than £1 10s. per cent. per annum.

The sum of £50,000, which I have recommended to be retained out of the ascertained surplus for a Reserve Fund, is not more than I consider necessary for a Mutual Life Assurance Society to possess, taken in connection with the magnitude of the business, and the fact that the Liabilities are valued by the Carlisle Table of Mortality, in combination with 4 per cent. Interest. The average rate of Interest earned on the

Funds during the last five years has been £5 14s. 1d. per cent.; during the previous five years it was at the rate of £5 15s. 10d. per cent. The percentage of Interest obtained on the Investments, no doubt warrants 4 per cent to be used in the calculations, especially as the Society's Tables of Premiums are calculated on a Carlisle 4 per cent. basis. The Interest which has been realised is quite one per cent. above that of any first-class English Assurance Office; and fully more than £1 10s. 0d. per cent. above the rate obtained by the majority of the largest and most successful offices. We are therefore, as regards Interest, on a par with offices carrying out a 3 per cent. Carlisle Valuation. The element of Interest in the calculations would not of itself call for any special Reserve, although it must be evident that the rate of Interest hereafter to be realised on Funds rapidly accumulating, will certainly not increase.

The necessity for a Reserve Fund becomes more apparent when the probable future rate of Mortality is considered. On that point the question arises, whether the Table in use by the Society most correctly represents the Mortality that has been found to prevail among assured lives. It is now a well-established fact, that the Mortality deduced from the observations upon which the Carlisle Table was founded, shows the duration of life to be more favourable than Assurance Offices have experienced for persons above the age of 40. This has been proved by the published experience of the oldest Mutual and Proprietary Institutions, as well as by the combined experience of Offices now twice ascertained. If the Table of Mortality known as the "Combined Experience," obtained from data furnished by seventeen offices, had been used in valuing the business, a much larger Reserve for the Liabilities would require to have been made. Or, in other terms, if the "Experience" Tables had been applied to this Valuation, more than half the proposed Reserve Fund would disappear.

To prevent misapprehension on this subject, there is another point to which I would here refer. It has been the practice of the Society for some years to report annually, a comparison of the expected claims by the Carlisle Table, with the actual claims paid. From these returns it appears, that the number of claims which has emerged in the successive years of the past Quinquennium, has been on an average, 70 per cent. of what might have been expected according to the Carlisle Table. A lower rate of Mortality than that furnished by the Tables, has been found in all carefully conducted Offices to prevail for several years after lives are assured, and this, it is well known, results from the selection of the lives assured. Assurance Offices in general find (and this Society has also found) it to be a source of large profit; its permanence however, depends on a continual influx of new lives in large numbers. It is in after years, when the Society will have a great number of policies of long standing, that a Reserve Fund will be found needful; at a time in short, when the Mortality experienced may exceed that shown by the Table on which the Liabilities are valued. Practically, the Society had a Reserve Fund, at last Investigation, of within £18,000 of the amount now proposed to be reserved. A Reserve Fund has therefore been gradually forming, and the result is, that the whole profits of future Investigations will be available for division among the members; the present Reserve Fund reappearing specifically stated in the accounts at subsequent periods of division, although it may have to be augmented with the increase of the business. I therefore unhesitatingly recommend, that the sum of £50,000 should be

now reserved. Not only will it give stability to the Society, and provide for increased Mortality, (if that should happen) but it will also render the Society independent of fluctuations in the amount of new business, while the accruing Interest on the Fund will materially augment the rate of future Bonuses.

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#### ENDOWMENT AND ANNUITY BRANCHES.

The practice of the Society, in undertaking to divide profit among members of the Endowment and Annuity Branches, is novel, and I believe peculiar to this Society. But if, for the benefits contracted for, an adequate addition has been charged in the single or annual Premiums, to provide for Bonuses, there can be no objection to return a surplus, from whatever source obtained, on an equitable basis.

To ascertain whether a profit has been realised on the transactions of either Branch at a given date, it is essential that the Funds of each Branch should be separately accounted for; that they should be improved respectively at the average rate of interest, annually obtained by the Society in each year on its total Funds; and that the total expenses of the Society should be rateably apportioned between the Branches. The difference between the amount at credit of the Fund, and the value of the Liabilities will determine the Profit or Loss at any given time. The first of these conditions is made a requirement in the Society's Act of Incorporation, and is still more explicitly laid down in the By-laws. I have been thus particular in detailing the obvious business conditions that must be adhered to, before a true state of either of the Branches can be ascertained; because I have been unable to trace in the books of the Society or from the Quinquennial Reports, presented to the Members in 1859 and 1864, any evidence that the realised Assets belonging to the respective Branches, have been dealt with in conjunction with the Policy Liabilities as determined at these two Investigations. The division of the Total Surplus between the Branches was clearly set out, but there was no evidence to show whether a profit had been earned or a loss sustained on either Branch.

The existing laws of the Society, in regard to this Investigation, require that "a careful valuation of the total assets of the Society, and of the liabilities and risks of each Branch, separately, shall be made, &c." In conducting this Investigation, and with that demand to be met, it became necessary to ascertain the amount of realised assets properly belonging to each Fund; and, consequently, the amount at the commencement of the Quinquennium. But, having no data to show this, except by taking the sum represented by the value of the Liabilities of the respective Funds, augmented by the additions respectively made to them, out of the gross surplus at the last division of Profits, I proceeded to trace the accumulation of the Funds, in the three Branches, from the commencement of the Society, and at each Investigation. Looking to the amount of Bonus, that had been successively declared to the holders of Endowment and Annuity Policies, on the one hand, and to the consideration received by the Society, for the original benefits contracted for on the other, I was anxious to discover what were the sources of profit that produced such results, and to know whether the Funds of the respective Branches were being improved at a uniform rate of Interest, and the Expenses equitably charged to the Funds. With that end in view, the following statements have been compiled.

**FIRST.**—A summary of the Total Receipts and Disbursements from the commencement of the Society, to 29th February, 1869, and of each Branch separately.

**SECOND.**—A summary of the Assurance, Endowment, and Annuity Funds, and the corresponding amount of Assets, for every year, and at each Quinquennial Investigation.

**THIRD.**—A Table showing the average yearly rate of Interest received on the Total Funds of the Society.

**FOURTH.**—A Table showing the total amount of Expenses at each Quinquennium, and the share chargeable and charged to each Fund.

**FIFTH.**—Statements of the Liabilities and Assets of the Assurance, Endowment, and Annuity Branches, at 1859 and 1864, first with, and second without, the adjustments for Interest and Expenses to be now explained, The exact figures reported on, as the value of the Liabilities are incorporated, and the result shewn by both methods.

The following Table shows the uniform rate of Interest obtained on the Endowment and Annuity Funds; and also at what rate they would require to have been accumulated to possess the amount of Interest appearing respectively at their credit during the ten years, 1854-1864.

*TABLE shewing the actual and assumed rate of Interest on the Endowment and Annuity Funds.*

Year.	ENDOWMENT BRANCH.		ANNUITY BRANCH.	
	Uniform Rate of Interest on the Funds.	Required rate per cent. to obtain amount of Interest carried to Credit of the Funds.	Uniform Rate of Interest on the Funds.	Required rate per cent. to obtain amount of Interest carried to Credit of the Funds.
1855	3.990	18.433	3.990	12.685
1856	5.926	24.312	5.926	16.731
1857	6.779	26.970	6.779	18.561
1858	5.098	20.306	5.098	13.974
1859	5.338	29.337	5.338	20.190
1860	4.627	6.215	4.627	10.225
1861	5.787	7.421	5.787	12.209
1862	4.573	5.626	4.573	9.256
1863	5.877	7.187	5.877	11.826
1864	8.092	11.971	8.092	19.693

Referring now to the Expenses—

At the end of each Quinquennial period, the total expenses of the Society for the previous five years have to be distributed among the Branches. The principle adopted has been, to divide the expenses in proportion to the ascertained value of the Liabilities of the respective Branches. Keeping in view the method on which the Profits of the Society are divided, this is the correct and equitable plan, and it has been followed on this occasion.

I have been thus particular in explaining the method by which Interest and Expenses have been dealt with, and their adjustment as between Branches, inasmuch as it will be found that the Endowment and Annuity Funds are not in the satisfactory condition in which they were on former occasions reported to be.

I have shewn that there has been an inequitable apportionment of Interest and Expenses, as between the Branches, to the extent of £9,315 11s. 2d. This sum is composed of £5,852 9s. 4d. for Interest, and £3,463 1s. 10d. for Expenses. The Assurance Fund has been diminished by that sum in two Instalments: first, in 1859, by £2,593 4s. 10d., and again, in 1864, by £6,722 6s. 4d. The Endowment Fund has been benefited by the sum in question to the extent of £3,885 6s. 7d., and the Annuity Fund to the extent of £5,430 4s. 7d. The purposes to which these respective amounts have been applied are as under:—

**ENDOWMENT BRANCH.**

1859.				
February 28, Deficiency in the Fund	..	..	£102	7 3
Profit Divided	..	..	618	1 7
Reserve Fund	..	..	154	19 1
1864.				
February 29, Deficiency in the Fund	..	..	1,375	14 3
Profit Divided	..	..	1,631	9 4
Reserve Fund	..	..	2	15 1
			<u>£3,885</u>	<u>6 7</u>

**ANNUITY BRANCH.**

1859.				
February 28, Deficiency in the Fund	..	..	£68	6 6
Profit Divided	..	..	1,318	17 5
Reserve Fund	..	..	330	13 0
1864.				
February 29, Profit to be Divided	..	..	2,760	16 7
Reserve Fund	..	..	951	11 1
			<u>£5,430</u>	<u>4 7</u>

An Abstract of Actual and Expected Claims appears in the following

**TABLE.**

Year.	Expected Deaths.	Actual Deaths.	Expected Amount of Claims by the Carlisle Table.	Actual Amount of Claims.
28th February, 1865.	46	27	£ 30,000	£ 18,542
" 1866.	62	52	43,000	36,381
" 1867.	61	34	37,800	26,260
" 1868.	66	54	40,950	32,397
" 1869.	85	58	50,702	30,705
Total . . . .	320	225	202,452	144,285

The business of the Society has been conducted on a scale of Expenditure, which has been gradually decreasing, while the amount of new business transacted has been annually increasing, as shewn in the following

TABLE.

ASSURANCE, ENDOWMENT, AND ANNUITY BRANCHES.			
Year ending 28th February.	Subsisting Assurances.	Gross Annual Revenue from Premiums and Interest.	Rate per cent. of Expenses on Gross Annual Revenue.
	£	£	
1850	5,200	417	41.4
1851	10,300	869	21.8
1852	21,050	1,610	25.0
1853	37,600	2,530	20.2
1854	52,800	3,861	16.0
1855	86,900	5,326	24.3
1856	148,910	9,613	20.8
1857	214,460	13,000	23.5
1858	293,735	17,399	21.3
1859	398,985	22,897	19.4
1860	529,415	29,386	13.9
1861	710,090	38,984	14.6
1862	919,640	48,032	14.5
1863	1,202,260	62,459	13.5
1864	1,551,538	80,985	12.4
1865	2,085,450	103,246	18.6
1866	2,416,477	116,804	14.0
1867	2,950,632	142,085	11.0
1868	3,381,602	157,288	10.3
1869	3,865,667	179,026	10.6

The miscellaneous sources of profit have been inconsiderable. The very liberal amount hitherto allowed for the Surrender of Policies has precluded almost any profit being derived from that source.

The future elements of profit are now considerable, but they have not been taken into account or anticipated in any way whatever.

\* \* \* \* \*

*To the Directors of the Australian Mutual Provident Society.*

Gentlemen,—I have examined the Actuary's Report on the Investigation of this year, dated July 6th, 1869, and have the honor to report that I concur in his recommendations, and certify that the proposed distribution of £97,338 among the Members of the Assurance Branch, being 35 per cent. on the values of their Policies, is correct, safe, and equitable to all the Members, in terms of the By-laws.

In joining with the Actuary in recommending that so large a sum as £50,000 shall be retained as a Reserve Fund, I must beg to draw your attention to the following circumstances. At the time of the last Investigation in 1864, it was generally supposed, from the experience of the Society, and from the published Returns of the Registrar-General, that the rates of mortality prevailing in New South Wales were more favourable than those indicated by the Carlisle Tables, upon which all our calculations are founded. It has been since however proved, by an analysis of the results of two Census Returns, and of the Annual Reports of the Registrar-General of New South Wales, that the rates of mortality in this Colony are greater than was supposed, and that at ages above 45, they are considerably greater than the Carlisle rates. We have no right therefore to assume that the low rate of mortality which has hitherto prevailed among the Members of the Society will continue as their ages advance, but should rather anticipate that it will be greater than what has been assumed as the basis of our calculations.

I have myself checked the greater part of the Policy Valuations under Table A, and all of those under Table B. The calculations regarding the remaining Tables under all the Branches have been checked or calculated by the Actuary, and I have satisfied myself that in all cases, the right formulæ have been made use of, and correctly applied, and that all the requisite adjustments have been made.

I have the honour to be, Gentlemen,  
Your obedient Servant,

M. B. PELL,

University, July 8th, 1869.

Consulting Actuary.

[An unusual number of elaborate Tables are given in an Appendix, from which we have taken only those that are most important and most likely to interest the English reader.—ED. J. I. A.]

*Balance Sheet of the Australian Mutual Provident Society,  
on 28th February, 1869.*

LIABILITIES.	
I. ASSURANCE BRANCH:—	
Value of Liabilities under Assurance Policies, per Summary	
Valuation Statement.....	£369,433 15 0
Claims by Death, admitted, but not due till after 28th February	14,029 14 6
II. ENDOWMENT BRANCH:—	
Value of Liabilities under Endowment Policies, per Summary	
Valuation Statement.....	48,410 12 7
III. ANNUITY BRANCH:—	
Value of Liabilities under Annuity Policies, per Summary	
Valuation Statement.....	19,743 12 6
Annuities due, but not paid till after 28th February.....	300 2 10
IV. COMMISSION due to Agents on Premiums in course of collection, and all other outstanding accounts at 28th February ....	
	1,082 15 0
BALANCE.....	149,646 0 2
	£602,646 12 7
ASSETS.	
New South Wales Debentures .....	£178,761 15 0
Victoria do. ....	10,100 0 0
Queensland do. ....	39,163 15 0
Queensland Treasury Bills .....	23,684 0 0
Sydney Corporation Debentures.....	4,000 0 0
Loans on Mortgages .....	169,610 18 1
Loans to Members on Security of their Policies .....	50,385 13 7
Value of Freehold Estate .....	9,932 9 1
Value of Society's House, Pitt-street, Sydney .....	12,734 8 7
Cash on Deposit at Interest with National Bank, Melbourne .....	21,000 0 0
Do. do. Oriental Bank .....	50,000 0 0
Premiums due at Head Office, and in Agencies, the time for payment of which has not expired .....	19,828 9 0
Value of Office Furniture .....	1,075 0 0
Interest on Investments accrued since last payment .....	3,359 17 4
Cash at Bankers' (Current Accounts) .....	9,010 6 11
	£602,646 12 7

Separate Balance Sheets are also given of the three Funds. The Assurance Fund shows a surplus of £153,437 10s. 8d.; and the Endowment and Annuity Funds show deficiencies of £2,769 14s. and £1,021 16s. 6d. respectively.

## Summary and Valuation of the Policies of the Australian Mutual Provident Society, as at 28th of February, 1869.

Description of Transactions.	No. of Policies in Force.	Under Table.	PARTICULARS OF THE POLICIES FOR VALUATION.										VALUATION.					
			Sums Assured.	Office Yearly Premiums.	Nett Yearly Premiums.	Margin, being Provision for Future Expenses and Bonuses.	Value by the Graduated Table of Mortality.						Value of Sums Assured and Bonuses.	Value of Office Yearly Premiums.	Value of Net Yearly Premiums.	Value of Margin.	Interest 4 per Cent.	
							Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.						Col. 7.
<b>I. ASSURANCE BRANCH.</b>																		
For the whole Term of Life	5,354	A	2,714,569	87,983	64,992	22,991	1,136,824	1,278,500	929,409	349,091	207,415							
Existing Bonuses	..	..	75,591	..	..	..	38,540	..	..	..	38,540							
For the whole Term of Life: Premiums by a limited number of payments	688	B	398,464	15,662	13,052	2,610	151,419	116,862	95,484	21,378	55,935							
Existing Bonuses	..	..	13,776	..	..	..	6,002	..	..	..	6,002							
For the whole Term of Life: Joint Lives	37	G	16,100	772	644	128	8,797	8,737	7,203	1,534	1,594							
Existing Bonuses	..	..	464	..	..	..	267	..	..	..	267							
For the whole Term of Life: Increasing Premiums	2	N	1,000	19	15	4	288	254	211	43	77							
Survivorship Policies	7	H	410	148	124	24	1,581	1,658	1,373	285	208							
Existing Bonuses	..	..	4	..	..	..	17	..	..	..	17							
Endowment Assurances	1,550	J	546,950	22,295	18,579	3,716	263,409	258,754	213,634	45,120	49,775							
Existing Bonuses	..	..	7,263	..	..	..	4,277	..	..	..	4,277							
Short Term Policies	39	A short	21,725	509	324	185	1,283	1,846	1,045	801	238							
Additional Reserve for Climate and other Special Risks	..	..	..	784	..	..	..	..	..	..	2,496							
Adjustment for Claims payable three months after Death	..	..	..	..	..	..	..	..	..	..	2,592							
	7,677		3,796,316	128,172	97,730	29,658	1,612,704	1,666,611	1,248,359	418,252	369,433							
<b>II. ENDOWMENT BRANCH.</b>																		
With Return of Premiums in event of Death	1,100	D	185,656	9,847	9,118	729	103,912	64,351	58,278	6,073	45,634							
Existing Bonuses	..	..	3,379	..	..	..	2,622	..	..	..	2,622							
Without Return of Premiums in event of Death	5	E	840	48	44	4	467	..	343	30	154							
	1,105		169,875	9,895	9,162	733	107,001	64,694	58,591	6,103	48,411							
<b>III. ANNUITY BRANCH.</b>																		
Immediate and Deferred	57	K & C	2,569	717	598	120	19,457	6,385	5,094	1,291	14,863							
Existing Bonuses	..	..	311	..	..	..	2,479	..	..	..	2,479							
Temporary Annuities	49	K	477	..	..	..	2,902	..	..	..	2,902							
	106		3,357	717	598	120	24,838	6,385	5,094	1,291	19,744							
Total of the Results	9,988		3,966,191	138,764	107,490	30,511	1,744,548	1,787,960	1,312,044	426,646	437,596							

NOTE.—Column 9 subtracted from Column 7 gives the Net Value of the Liabilities in Column 11. Column 9 subtracted from Column 8 gives the Value of the Margin in Column 10.

TABLE of Policies Granted, Void, and in Force.

ASSURANCE BRANCH.				
	No. of Policies.	Annual Revenue.		Amount.
		£	s.	d.
Granted . . .	9,831	170,913	12	9
Void . . .	2,154	43,525	15	8
In Force . . .	7,677	127,387	17	1
		3,699,217	0	0
ENDOWMENT BRANCH.				
	No. of Policies.	Annual Revenue.		Amount.
		£	s.	d.
Granted . . .	1,495	14,801	19	3
Void . . .	390	4,407	0	2
In Force . . .	1,105	9,894	19	1
		166,496	0	0
ANNUITY BRANCH.				
	No. of Policies.	Annual Revenue.		Amount.
		£	s.	d.
Granted . . .	157	2,599	4	7
Void . . .	51	1,881	19	11
In Force . . .	106	717	4	8
Purchase Money for Present Annuities		10,702	0	10
		3,045	14	4
				Per Annum.

TABLE A, FOR THE WHOLE TERM OF LIFE.

*Examples showing the Amounts of the Reversionary and Cash Bonuses on Policies of £1,000 each, on which the Reversionary Bonus has been left as an Addition to the Policy.*

Date of Policy.	Age at Entry.	No. of Premiums Paid.	Annual Premium.	Former Bonus Additions.	Reversionary Bonus in respect of last 5 Years.	Rate per cent. per Annum.	Total Bonus Additions at 31st February, 1869.	Average Rate per cent. per Annum.	Amount of Premiums Paid.	Percentage of Total Cash Bonuses on Premiums Paid.
<i>Policies that have had Bonuses allotted at the 1st, 2nd, 3rd, and 4th Investigations.</i>										
1849	35	20	£ s. d. 27 13 4	£ s. d. 747 16 10	£ s. d. 442 7 8	8.84	£ s. d. 1,190 4 6	5.95	£ s. d. 553 6 8	100.14
1851	40	18	32 4 2	614 12 7	398 12 9	7.96	1,013 5 4	5.63	579 15 0	88.60
<i>Policies that have had Bonuses allotted at the 2nd, 3rd, and 4th Investigations.</i>										
1853	54	16	53 9 2	390 17 0	322 6 0	6.44	713 3 0	4.45	855 6 8	54.21
1853	40	16	32 4 2	333 8 7	275 3 3	5.50	608 11 10	3.80	515 6 8	57.83
1853	30	16	24 1 8	266 13 3	223 2 4	4.46	489 15 7	3.06	385 6 8	50.12
<i>Policies that have had Bonuses allotted at the 3rd and 4th Investigations.</i>										
1859	55	11	56 1 8	118 15 1	174 4 2	3.48	292 19 3	2.66	616 18 4	29.49
1859	40	11	32 4 2	93 18 10	135 15 1	2.71	229 13 11	2.09	354 5 10	29.00
1859	20	11	18 14 2	77 17 2	114 6 10	2.28	192 4 0	1.75	205 15 10	28.11
<i>Policies that will have Bonuses allotted at the 4th Investigation.</i>										
1864	61	6	73 17 6		71 10 5	1.43			443 5 0	10.54
1864	40	6	32 4 2		50 6 11	1.01			193 5 0	10.91
1864	20	6	18 14 2		45 16 6	.92			112 5 0	11.55

## LA ROYALE BELGE,

COMPAGNIE ANONYME D'ASSURANCES A FORFAIT SUR LA VIE, ET CONTRE LES ACCIDENTS.

We have been favoured by M. Adan, Corresponding Member of the Institute at Brussels, with the Bonus Report of the above Company, describing its progress during the three years ending on the 31st December, 1868. It is of considerable length, and contains many more details than are usually given in English Reports. It is divided into two "chapters," of which the former is subdivided into two "sections." In § 1 of the former of these sections, we first have particulars given of the Assurances in force on the 31st December, 1865, under the different heads—(1) Assurances on one life without profits, (2) ditto with profits, (3) Joint life assurances, (4) Survivorship assurances, (5) Term assurances. The number of policies was 2414, and the sum assured £531,793, the insurances with profits being for less than £23,000.

The number of new policies effected in the three years was 2273, insuring £453,794.

Making a total of 4687 policies, insuring . . . £985,587

On the other hand, there were

Claims . . . . .	197 policies for	£39,485
Lapsed, surrendered, expired, &c.	998 „	182,452
Reassured . . . . .		149,598

<u>1,195</u>	<u>£371,535</u>
--------------	-----------------

So that there were existing on { 3,492 policies }  
the 31st December, 1868 . { insuring } £614,052

The distribution of these assurances into various classes is as follows:—

Non-participating assurances—

On one life . . . . .	2,955 policies for	£478,978
On two joint lives . . . . .	80 „	7,958
One life against another . . . . .	10 „	2,378
Temporary (excluding tontine counter-assurances) . . . . .	297 „	93,904
Various . . . . .	26 „	8,945
Do. deferred . . . . .	3 „	543

Participating assurances—

On one life . . . . .	121 „	21,346
<u>3,492</u>		<u>£614,052</u>

Of the above policies

680 were on the lives of 678 females, assuring	£41,601
And 2,812 on the lives of 2,725 males „	<u>572,451</u>

The average sum assured on one life in 1865 was £229 and in 1868 £181, showing a diminution in the interval of £48. The average age of the assured is 42½ years; in 1865 it was 44 years.

Then is given a table, showing the distribution of the policies, both as respects the age of the lives, and the magnitude of the sums assured; the ages being grouped in quinquennial periods, and the sums assured by steps of 1000 francs (£40) up to £480, and the few policies of larger amount being disposed in 5 groups. Next are given several items of information as to the ages of the youngest and oldest of the lives assured in 1865 and 1868 respectively, and as to the average duration of the insurances. Then follows a still more elaborate table, showing for each year of the existence of the Company, the number and amount (1) of the proposals received, and (2) of the policies completed; (3) the number of lives and the total sum assured at risk during the year; (4) the deductions for expiry, lapse, surrender, death, and reassurance; (5) the net increase in number of lives and sums assured; (6) the number of lives assured and the amount of the assurances at the end of each year; and lastly, the average amount insured on each life, and the average age of the lives.

§ 2 gives the particulars in 1865 and 1868 of the survivorship annuities, which, however, are few in number, and small in amount.

§ 3 is devoted to “tontine counter-assurances” (*contre-assurances*), which is apparently a very important branch of the business.

On the 31st December, 1865, the amount of the engagements of the Company, to reimburse to the subscribers to tontine associations the amount of their payments in the event of the death of the assured before the time of division,

Under . . . . .	7,147 policies, was	£443,767
Add—		
New policies effected in the period 1866–8 . . . .	403 „	22,171
Total . . . . .	7,550	£465,938
Deduct—Run off in the same period—		
By death . . . . .	126	£6,923
Expiry of risk . . . .	1,309	54,558
Annulled . . . . .	205	16,307
	1,640	77,788
In force on 31st Dec., 1868 .	5,910	for £388,150

The amount of the claims paid in this branch of the business was £4172.

A Table is given showing the distribution of the “counter-assurances” (1) according to the “date of liquidation,” and (2) according to the age of the nominees.

§ 4 treats of the annuities, immediate and deferred.

The immediate annuities amounted on 31st December, 1868, to £6412 per annum, the number of the lives being 281, and the amount of purchase money received thereon having been £68,211.

There were also 29 deferred annuities in force on 25 lives, the purchase money having amounted to £589.

§ 5 gives particulars of the endowments, arranged under three heads, the last of which includes apparently the assurance of a sum to purchase a substitute in case the nominee is drawn in the conscription.

§ 6 gives particulars of the insurances against accidents.

This description of business was commenced by this Company in 1867.

The amounts assured are—

Against death by accident . . . . .	£276,318
Against accidents entailing permanent inability to work . . . .	3,830
Against accidents entailing temporary inability . . . . .	323
Total . . . . .	£280,471

The number of lives assured in this branch is 4270, and the claims paid have been £179.

§ 7 relates to insurances of sums certain.

It is stated that a single premium of £3,003 has been received under this head.

§ 8 states that the premiums received in the three years amount to £80,674.

§ 9 contains a table showing the numbers and amounts of the insurances in the several classes on December 31st, 1865, and December 31st, 1868, and the difference between the two, whether increase or decrease. On the whole, there is a diminution in the number of Policies, but an increase in the sum assured.

The amounts here shown, are with great satisfaction compared with those of three of the principal French Life Insurance Companies at the corresponding period of their existence. As regards the prospects of the Company for the future, a calculation of Professor Karup is quoted, that in

Great Britain and Ireland, containing 30 millions of inhabitants, there is on the average £15 insured on each person; while in France, with 38 millions of inhabitants, there is only £1. 13s. assured per person; in Germany, with 50 millions, only £1. 1s.; and in the rest of Europe, with 172 millions, only 2½*d.* The opinion is expressed that these figures indicate a wide field for the extension of Life Insurance.

The second section of the first chapter treats of the mortality experienced by the Company in the sixteen years of its existence. It appears that, excluding from consideration the tontine counter-assurances, 337 deaths have taken place, the expected number being 317. A table is given, showing how these deaths are distributed over 55 named causes of death and over each decennial period of life. Another table compares the actual and the anticipated mortality, both as regards lives and as regards sums assured, during each year of the Company's existence. It is not stated however by what table of mortality the anticipated deaths are computed.

Among the lives on which tontine counter-assurances have been granted the number of deaths has been 602, whilst the expected mortality was 1002; and in the annuity branch the actual mortality has been 148 deaths, against an expectation of 195.

Chapter II. is devoted to the financial position of the Company on the 31st December, 1868. The following is the Balance Sheet.

*Balance Sheet, 31st December, 1868.*

ASSETS.		£
Unpaid Capital . . . . .		102,000
Public Funds of Prussia and Wurtemberg . . . . .		3,412
Preference Shares and Obligations of Prussian and Baden Railways . . . . .		5,953
Preference Shares and Obligations of Belgian Railways . . . . .		46,422
Shares in the National Bank, Shares in the <i>Société Générale</i> , and Perpetual Annuities of the city of Brussels . . . . .		3,585
Loans on security of the Obligations of various Belgian Railways . . . . .		9,600
Bills in hand . . . . .		8,409
Real Property situated in Belgium, and Mortgages on the same . . . . .		44,760
Sundry Accounts . . . . .		25,845
Balance with the <i>Société Générale</i> . . . . .		11,193
Cash in Office . . . . .		207
		<hr/>
		£261,386
LIABILITIES.		£
Subscribed Capital . . . . .		120,000
Statutory Reserve . . . . .		6,897
Premium Reserve . . . . .		126,806
Claims unpaid . . . . .		4,079
Sundry Accounts . . . . .		449
Deposits at Compound Interest . . . . .		3,123
Profit and Loss . . . . .		32
		<hr/>
		£261,386

The Report proceeds:—

The extreme smallness of the balance precludes the declaration of a dividend. This result is attributable to various causes, which may be termed accidental; for during the period under review, first war and then cholera prevailed, the latter of which caused 21 claims for the sum of £4,644. But one important cause has been the obligation laid on the Company by the Prussian Government since 1866, of writing off at once and in full the commuted commissions paid, instead of writing them off in

a term of years, which would be fairer towards the shareholders. When it is borne in mind that in addition to this immediate writing off of the commission on the first year's premium, the first year's reserve on the policy has to be provided, it will be seen that it may easily happen that the very period in which the Company has effected the greatest number of new assurances may be that which apparently has produced the least favourable results.

Further, a comparison with the processes employed by other Offices has led us to suppose that our Company pursues one of the most rigorous systems. There is one system which values the sums assured at one rate of interest and the future premiums at another rate, and this admits of negative reserves for recent assurances on young lives, whereas by the method we follow the reserve increases yearly, being always equal to the single premium for that part of the sum assured which, in consequence of the increased age of the life assured, is not covered by the annual premium.

We shall pursue these comparisons further, but we have no intention of altering the system at present in use, as our personal experience has not yet been sufficiently long to enable us to speak positively on the point. But we think it well to note these differences, if only to point out that the smaller the reserve which a Company makes for its liabilities, the more easily can it declare a bonus.

It must not be thought that we accept proposals too easily; for out of 7433 proposals for insuring £1,835,713 we declined 1099 for £376,322, being about the seventh part of the whole in number, and the fifth in amount.

#### *Assurances with Profits.*

These assurances receive one half of the profits realized in their branch; and on the present occasion this amounts to 11·42 percent of the premiums paid during the triennial period.

We take the opportunity of avowing that we have great hesitation in recommending this kind of assurance, for the reason that it presents the contract of life assurance as one *de lucro captando*, as a good investment, or a speculation likely to enrich the assured, rather than as a contract *de damno vitando*. We believe that the legitimate object of assurances with profits is to enable the premiums to be reduced, when it shall appear from the mortality experienced that this can be done with propriety.

The Report concludes with a refutation of some false statements that have been circulated in reference to the Company.

It has been asserted that the Company pays its claims, not in cash, but in Belgian 2½ percent stock taken at par, so that the assured suffer a loss of 40 or 45 percent. This assertion is absurd on its very face, and is a simple untruth.

It has been further asserted that the assets of the Company are invested in the shares of the *Credit foncier international* and of the *Credit foncier et industriel*. This is wholly without foundation. The investments of the Company, as can be seen from the Balance Sheet, have always been made in strict conformity with the statutes, which do not allow of the funds being invested in securities of the above-named description.

From the circumstance that the office of the Company is situated in the same building as other *établissements de crédit*, it has been asserted that it is only a fraction of the group formed by them. But this is quite inaccurate, as it is wholly independent of them, and economy in house rent is the only reason for its being located in the same premises with them.

Lastly, it is pretended that the small amount of insurance retained by the Office at its own risk on a single life is an indication of its weakness; but this is a great mistake. It may, indeed, be stated that, as large assurances are always few in number, it is difficult if not impossible to make an average of them. It is necessary therefore to seek a remedy for the undue risk arising from large assurances, unless we overlook one of the most elementary principles, that of the division of risks. The Royal Belge has found this remedy in a wise limitation of its risks by means of reinsurance; and every competent man will see in this limitation a pledge of security, and not an index of weakness. We might, if necessary, fortify our opinions on this point by the arguments set forth by Mr. Sprague, a member of the Institute of Actuaries of London, in his paper *On the Limitation of Risks*. (See *Journal of the Institute of Actuaries*, vol. 13, page 20.)

### STAR LIFE ASSURANCE SOCIETY.

*Established 1843.*

#### *REPORT of the ASSETS and LIABILITIES of the Society to December 31st, 1868.*

The 31st of December concluded the Fifth Quinquennial period of the Society's history, and, according to the provisions of the Deed of Settlement, the exact position of the Company is to be ascertained, and the surplus, if any, is to be divided amongst the Proprietors and Policy-holders who are entitled to participate. The Secretary has been entrusted with this duty.

In this, as in previous valuations, the Carlisle rate of mortality and 3 per cent. interest are the data upon which the valuation is made.

The following is a—

### GENERAL SUMMARY.

#### Policies in force on December 31st, 1868.

No. of Policies.	Description.	Sums Assured.			Reversionary Bonus.			Annual Premium.		
		£	s.	d.	£	s.	d.	£	s.	d.
<b>ASSURANCES.</b>										
11,253	Whole of Life . . . . .	3,994,351	0	0	165,815	17	0	132,062	11	0
380	Joint Lives . . . . .	77,855	0	0	4,100	12	0	3,792	14	8
47	Last of two Lives . . . . .	20,850	0	0	562	16	0	347	11	11
9	"    three    "    . . . . .	6,400	0	0				124	16	4
17	Survivorship Assurance . . . . .	24,500	0	0	137	0	0	631	17	14
1	At 25 or death . . . . .	100	0	0				2	19	1
90	"    50    "    . . . . .	14,425	0	0	155	5	0	709	6	5
61	"    55    "    . . . . .	11,650	0	0	90	0	0	573	6	4
118	"    60    "    . . . . .	20,300	0	0	145	10	0	850	1	8
48	"    65    "    . . . . .	6,400	0	0				261	7	2
10	Limited No. of Payments . . . . .	7,750	0	0	167	0	0	410	15	3
11	Ascending Scale . . . . .	4,350	0	0	59	10	0	96	5	10
30	Temporary Assurances . . . . .	16,200	0	0				382	10	0
70	Endowments . . . . .	8,750	0	0				438	13	6
12,145		4,213,881	0	0	171,233	10	0	140,684	17	4
<b>ANNUITIES.</b>										
6	Survivorship . . . . .	130	0	0	per Annum			36	18	8
5	Deferred . . . . .	90	0	0	"			40	8	3
23	Immediate . . . . .	624	15	10	"					

The following is an analysis of the Policies issued for the whole of life, with the corresponding Annual Premiums at each decade of age, together with the Sum assured:—

Age.	No. of Policies.	Sum Assured.		Reversionary Bonus.		Annual Premium.	
		£	s. d.	£	s. d.	£	s. d.
80 and upwards ..	11	3,775	0 0	525	5 0	254	8 7
70 to 79 inclusive	236	93,680	0 0	10,501	15 0	6,489	7 1
60 to 69 "	1,045	393,189	10 0	35,150	5 0	19,242	17 8
50 to 49 "	2,697	957,181	10 0	52,720	4 0	38,319	8 3
40 to 39 "	3,237	1,199,410	0 0	52,561	5 0	36,025	1 6
30 to 29 "	2,707	912,020	0 0	12,901	13 0	22,808	6 6
20 to 19 "	1,253	409,845	0 0	1,443	5 0	8,489	12 3
Under 20 .....	85	25,250	0 0	12	5 0	433	9 2
	11,253	3,994,351	0 0	165,815	17 0	132,062	11 0

The following are the results of the valuation:—

**BALANCE SHEET of the STAR LIFE ASSURANCE SOCIETY,  
December 31st, 1868.**

<i>Dr.</i>		£	s.	d.
To present value of £4,184,581, assured for the whole of life and other periods .....		2,206,698	3	8
„ present value of £171,174, Bonus additions on the aforesaid ....		101,993	15	11
„ Amount to be reserved for Assurances effected on the ascending scale ..		126	5	10
„ Amount to be reserved for Temporary Assurances .....		382	10	0
„ Amount to be reserved for Endowments .....		1,871	7	11
„ Amount to be reserved for Deferred and Survivorship Annuities ..		419	5	4
„ Amount to be reserved for Immediate Annuities .....		5,242	0	5
		2,316,733	9	1
Less present value of Sums Re-assured, and additions thereto ..		100,446	18	9
		2,216,286	10	4
„ Capital Stock .....		5,000	0	0
„ Claims unpaid and Bonuses thereon .....		14,327	7	5
„ Balances—				
Reserve for future expenses and Profits on existing Policies ..		418,374	9	10
Surplus .....		191,505	12	6
		£2,845,494	0	1
<i>Cr.</i>		£	s.	d.
By present value of £139,767 8s. 0d. per annum, being the annual Premium on amounts per contra .....		2,091,872	9	0
Less value of Premiums for re-assurance .....		103,514	2	6
		1,988,358	6	6
These have been taken at the market price of the day, viz., 31st Dec. 1868. { Value of £34,000 Consols .....		31,407	10	0
	„ £4,000 New 3 per cents. ....	3,700	0	0
	„ £6,780 (67,800 Rs.) India 4 per cent. ....	6,186	15	0
	„ £2,500 New Brunswick 6 per cent. ....	2,562	10	C
	„ £20,618 11s. 4d. Canada Dominion Stock ....	20,618	11	4
	„ £7,000 Madras Railway Company Bonds ....	7,315	0	0
	„ £10,500 Great Indian Peninsular Railway Shares ..	10,946	5	0
	„ £20,000 (Delhi) Scinde Railway Stock .....	20,200	0	0
	„ £4,000 Great Southern of India Railway Stock ..	4,140	0	0
	„ £2,000 Oude and Rohilkund Railway Stock ..	2,070	0	0
	„ £10,000 Punjab Railway Stock .....	10,100	0	0
Carried forward .....		£2,107,604	17	10

	Cr.—(continued).	£	s.	d.
Brought forward . . . . .		2,107,604	17	10
By Loans on Mortgage, &c., (including advances on Wesleyan Methodist Chapels) . . . . .		662,078	16	3
„ Home Department Premiums due . . . . .		3,582	0	3
„ Unpaid Half-premiums . . . . .		21,212	2	4
„ Aliquot Interest on Loans . . . . .		13,170	4	6
„ Balance at Bankers, in hands of Agents, &c. . . . .		37,845	18	11
		<u>£2,845,494</u>	<u>0</u>	<u>1</u>

The surplus of **£191,505. 12. 6.** is now available for division among the persons who, in accordance with the Deed of Settlement, are entitled thereto. The bonus to be apportioned to the Policy-holders will afford a reduction of from 25 to 30 per cent. on the Premiums paid during the last five years, or an addition to the Policy exceeding that of any previous distribution.

The Directors wish only to add that the application of the bonus will not take effect until the 1st of August next, and will continue for the following five years, according, however, to the mode of appropriation selected by each Policy-holder.

\* \* \* \* \*

The STAR has now been established for a quarter of a century. It has paid in claims **£612,081. 4. 10.**; distributed in bonuses **£239,804. 11. 4.**, and has a Reserve or Insurance Fund of more than Three-quarters of a Million.

## CORRESPONDENCE.

### “ON THE LIQUIDATION OF AN INSOLVENT LIFE OFFICE.”

*To the Editor of the Assurance Magazine.*

SIR,—Will you allow me to use your *Journal* as the channel of a few remarks on the scheme for the liquidation of Life Insurance Companies, recently propounded by Mr. Bunyon.

The following is a brief outline of Mr. Bunyon's proposal as I understand it. The business of the Company in liquidation is to be carried on by a committee of joint liquidators; the policyholders who do not choose to withdraw are to continue to pay their full premiums; the creditors of the Company whose claims have matured are to be paid (in a certain order of priority) out of the realized assets a proportion of their debts, which proportion is to be ascertained from an estimate of the values of the assets and liabilities at the time of payment; the matured debts which cannot at once be paid are to bear a low rate of interest; and finally, when the estate has become so small that it will not bear the cost of management, the assets and a proper proportion of liabilities are to be taken over by a solvent Insurance Company.

Mr. Bunyon's scheme is clear and comprehensive; it could, I believe, be easily worked at less cost than the present mode of winding up such Companies. It has, however, the disadvantage that the operation of it would produce a certain amount of inequality among the persons interested

in the assets. It could not probably be carried out without legislative enactment. It would, I think, be more expensive, and would certainly be more dilatory than the modification of it I am about to propose.

Probably every one would admit that inequalities between those among whom the assets of an insolvent Company are to be divided should not for a moment be weighed against a substantial benefit to the whole body, such as the preventing the estates being eaten up by costs; at any rate, provided no person, however much relatively worse off than others he is made, is absolutely a loser by the scheme that works those inequalities. On the other hand, notwithstanding some glaring exceptions, the principle of proportionate distribution of assets of insolvent estates has always been the guide, both of the Court of Chancery and the legislature, in the winding up of insolvent estates, whether of individuals or associations; and a scheme of uneven distribution would be universally condemned unless in other respects it offered great advantages. Now Mr. Bunyon's scheme favours those persons whose claims have already matured, and in a less degree those persons who are, by reason of their holding old policies, paying light premiums, over the holders of more recently issued policies, inasmuch as he contemplates their continuing to pay the full premiums, though they will ultimately receive only a proportion of the sum insured. Now it is clear that it would be worth while for a number of the assured to sacrifice their policies (which in most instances ought to be worth something), and take new policies in other Offices, rather than go on paying premiums, a part of which must go to the aggrandizement of their co-assured.

I think I can best expound my view of the course of liquidation that ought to be pursued by illustration. Let us take the simplest case possible, that an Insurance Company is to be wound up, whose liabilities entirely consist of those upon policies originally issued by itself, and has no liabilities on annuities or otherwise; and we will also suppose that nothing is paid for commission on the premiums, and that the Office is small enough to be taken up by some other Office. There must be some price at which a solvent Company would undertake the whole of the liabilities of the first Company, the policyholders paying the same premiums to the new Office that they did to the old Office; and if half, or some other proportion of that price, only were paid, the new Office might undertake to pay half or the other proportion of the sums insured, and receive only half or the other proportion of the premiums.

On the winding up of an insolvent Insurance Company, I would propose that after deducting the necessary expenses of liquidation, or so much, if any, of them as are payable out of the funds applicable to payment of the policy claims, the whole of that fund should be handed over to another Office, which would issue fresh policies, of amount proportionably smaller with premiums smaller in the same proportion than those paid on the original policies. Each policyholder might also be allowed an option of retiring, and receiving a dividend on what I may call a minimum value of his policy, instead of receiving a new policy.

Of course it is clear that the allowing every policyholder to prove for the value of his policy, taking into consideration the state of health and circumstances relating to the person whose life is insured, would lead to endless expense and fraud; but the value could be fixed, on the supposition

that the person whose life is insured is in perfect health, and the policyholder's receiving a dividend on such minimum value would work no injustice to the other claimants.

If the business of the Company to be liquidated were too large to be absorbed by one Office, it could be distributed and taken over by more than one Office.

There would be less difficulty still in handing over liabilities on annuities and otherwise to be proportionably paid by the Office taking over such partial liabilities.

I have suggested the barest outline of a scheme, which could be better developed by persons more practically familiar with the details of insurance business than I am. But I believe such a scheme could be carried out under the 159th section of "The Companies Act, 1862." I do not say that the Court of Chancery would have no power to sanction under that section Mr. Bunyon's scheme, but I do not think, in the exercise of its discretion, it would, inasmuch as it offends the spirit of the Winding-up Acts and principles of equity.

It is almost needless to say that any liquidation that can be effected without having recourse to fresh legislative enactment is, on that ground alone, to be preferred. Piecemeal legislation is at all times to be deprecated, and at this time there are so many changes in contemplation, and the public mind is so filled with other matters, that it would be more than ordinarily difficult to pass a satisfactory measure on this subject.

The matter of costs can be best estimated by experience. Past experience tends to show that the cost of carrying on a business in liquidation in Chancery far exceeds that of carrying on the same business in private hands.

Mr. Bunyon's scheme is to end in amalgamation. It seems to me that an amalgamation brought about within a moderate time would be preferable to one brought about after long delay. One of the objects of winding up Companies is, that the shareholders should be able to ascertain what their liabilities are, and discharge them at once. Now, it is to be observed that the liabilities of shareholders in Insurance Companies, as generally constituted, is not really limited to the nominal value of their shares, for the shareholders are bound to discharge in full all the costs of liquidation and all liabilities on contracts not in terms limited to the funds of the Company, and by keeping matters open, an amount of uncertainty is introduced—some shareholders may become insolvent in the meanwhile, and greater expense be consequently thrown on the others. It may, however, be admitted that the giving of time to some few contributories may enable them to pay up a larger part of their liabilities than they otherwise could; but ample power is given by "The Companies Act, 1862," to make any arrangement with separate contributories.

Mr. Bunyon's able and lucid pamphlet suggests many interesting questions; for instance, in what cases a Company is to be deemed so insolvent that it ought to be wound up, and whether the powers of policyholders to petition might not beneficially be extended, and again to what extent the doctrine of marshalling will be applied as between different classes of creditors, that is to say, how far creditors, who have more than one fund to go upon, are to be paid out of the fund in which other creditors have no interest. It would be trespassing on your space too much, and would be beyond

the object of my letter, to pursue those questions further; but I ought perhaps to notice two passages in Mr. Bunyon's pamphlet which seem to me likely to mislead. The first (p. 13) relates to the priority of annuities over claims on policies. It is stated that annuities in many instances have priority "by virtue of the deeds of settlement." Now I do not doubt that Companies which have deeds of settlement providing for such priority, issue policies and grant annuities on terms that provide for that priority. But the priority would not exist by virtue of the deed of settlement alone. The other passage (p. 22) relates to the allowance of interest on claims in winding up proceedings. It seems to me a hasty deduction to say that, because interest on debts is to cease as between the creditors, therefore creditors, the dividends on whose claims are deferred to suit the convenience of all, are not to be allowed interest on such deferred dividends; and in the passage referred to that deduction has been drawn.

I am, Sir,

Your obedient servant,

*Lincoln's Inn, 16th Feb., 1870.*

DAVID PITCAIRN.

\* \* \* The suggestion to reduce the premiums payable by the assured in the same proportion as the sums assured, appears to us to be novel, and has much to recommend it on the ground of simplicity; but it appears to us to overlook the circumstance that a considerable part of the sum assured under a life policy is provided for by the future premiums. To take an extreme case, suppose that the whole of the funds of the Office have been spent, and that there are no shareholders to fall back upon. Here Mr. Pitcairn's scheme would virtually say—there is nothing left to wind up; there is an end of the whole concern. But a scheme of liquidation, to be complete, should take into account the possibility of the lives assured, or some of them, being willing to constitute themselves a Mutual Insurance Society, paying the old premiums, but with reduced sums assured. In this case, it is a problem of some difficulty to adjust equitably the rights of the assured in different classes, having regard to their several ages and standing. But the problem can be satisfactorily solved; and we shall probably return to the question at no distant date, unless we find our ideas anticipated by others.—ED. J. I. A.

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## ERRATUM.

P. 173, first diagram,

For  $\Delta\lambda a_x$  read Ar. Co.  $\Delta\lambda a_x$

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JOURNAL  
OF THE  
INSTITUTE OF ACTUARIES  
AND  
ASSURANCE MAGAZINE.

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*Explanation of a New Method of Adjusting Mortality Tables;  
with some observations upon Mr. Makeham's modification of  
Gompertz's Theory. By W. S. B. WOOLHOUSE, F.R.A.S.*

[Read before the Institute, 31st January, 1870.]

ALL tabular numbers drawn directly from observations, especially those appertaining to statistics, are more or less affected by unavoidable imperfections arising from defective information, insufficiency in the number of observations recorded by the experience, and other known or unknown incidental causes. When a consecutive series of tabular numbers is taken and differenced up to a certain order, the existence of the imperfections or errors alluded to, is at once revealed by the differences, which exhibit a conspicuous disturbance of their law of progression. To eradicate the errors and effect a proper adjustment of the original numbers is, however, an operation of peculiar and somewhat novel difficulty, and a critical examination of the empirical expedients, usually resorted to for such purpose, will show that instead of meeting the indispensable requirements of the problem, they only evade its most essential conditions, and thereby impregnate the results with a new series of theoretical errors, which too readily evade detection in consequence of the systematic uniformity of their law.

In compliance with the expressed desire of the Experience Committee I have recently adjusted the graduation of the New Experience Tables of Mortality for healthy lives. The required operation and the conditions attending it, not merely possessing the interest excited by an unusual speciality, which I have discussed on former occasions, but being one of very great practical importance, inasmuch as the adjusted tables are eventually to form the basis of an extensive superstructure of other tables, I could not refrain, at the outset, from giving to the subject a renewed and independent consideration. The grand desideratum is that of obtaining a continuous curve, or properly graduated table, which shall, as closely as possible, represent the actual facts derived from the observations. After thoroughly testing various schemes for accomplishing this object, I have at length succeeded in devising a method that is both satisfactory in principle and efficient in practice. As the method finally determined upon and adopted may be applied with the greatest facility to all analogous cases of adjustment, and promises to be one of very general utility, it has been considered advisable to present a detailed explanation of the process to the members of the Institute.

As suggested in my paper\* "*On the Construction of Tables of Mortality*," the number-living at each age in the mortality table is the most manageable element for final adjustment; at ages beyond the limiting age of the table it is at once conveniently, as well as accurately, put down as zero, a practical facility that cannot be over-estimated; and it has also this essential advantage, that precisely the same aggregate tabular mortality or decrement must necessarily be finally retained between all points of actual coincidence, in whatever way the intermediate numbers may be modified. As a consequence of this last-mentioned principle, the number of such coincidences with experience, in the curve of the number-living, may be regarded as one test of close adjustment and substantial exhibition of the actual mortality.

The method I have adopted may be briefly stated, and the rationale of the process and its fortunate adaptation to what is chiefly required will be at once apparent. The data, for the reasons already stated, are the numbers-living at successive years of age as deduced, without any adjustment, from the original facts. If we begin at the first age in the table and extract the

\* *Journal*, vol. xiii., p. 95.

numbers-living at quinquennial intervals, that is, according to the usual notation,  $l_{10}, l_{15}, l_{20}, l_{25} \dots$  we can, by the formula for interpolation, determine all the intermediate values at the other ages, and so obtain a complete series of values that shall be continuous. Geometrically speaking we shall thus pass a continuous curve-line through the indicated quinquennial points. Against the adoption of such curve-line as the basis of the final table there is manifestly this tangible objection, that the numbers at the ages 10, 15, 20, 25,  $\dots$  are made use of exclusively, and that the original numbers between those ages are wholly ignored as data. This rather material objection, which is inherent in other methods of adjustment, is entirely removed by varying the epoch of the adopted quinquennial data, that is, by taking the five distinct series hereunder stated, viz.:—

$$\begin{array}{ccccccc} l_{10}, & l_{15}, & l_{20}, & l_{25}, & . & . & . \\ l_{11}, & l_{16}, & l_{21}, & l_{26}, & . & . & . \\ l_{12}, & l_{17}, & l_{22}, & l_{27}, & . & . & . \\ l_{13}, & l_{18}, & l_{23}, & l_{28}, & . & . & . \\ l_{14}, & l_{19}, & l_{24}, & l_{29}, & . & . & . \end{array}$$

Then by separately interpolating the intermediate values for each of these series, and by finally taking the arithmetical average, or mean value, of the five completed sets of results. The logical premises that virtually guides us to this last deduction is the recognised principle, that the probabilities of positive and negative errors are equal.

Reverting again to a graphical illustration, all the points of the original data are thus occupied by five distinct curves, assimilating to the experience and to one another, and forming in combination a sort of network; and at every age the resulting ordinate of the adjusted curve is the arithmetical mean of the five corresponding ordinates, and the five curves are as it were mutually drawn in towards a central course. That such central curve must exhibit a correct average of the original observations, without giving undue weight to any one of them, it is unnecessary to explain, and it will be instantly perceived that every element of the data is equally employed in its determination.

It is not requisite however to compute these five curves separately, a labour which would be unnecessarily circuitous. For the purpose of actual calculation we proceed mathematically to reduce the preceding system of operations to a direct process.

For any given age let  $l$  denote an interpolated value of the

number-living, and let  $l_z$ , which denotes the original number for an age  $z$  years older, be the nearest quinquennial point of the corresponding curve. Then the series of values, from which  $l$  is found, are  $l_{z-5}$ ,  $l_z$ ,  $l_{z+5}$ , &c., and by interpolating with central values and stopping after second differences, we shall have

$$\begin{aligned} l &= l_z - \left(\frac{z}{5}\right)a + \left(\frac{z}{5}\right)^2 \frac{b}{2} \\ &= l_z - \left(\frac{z}{5}\right) \frac{l_{z+5} - l_{z-5}}{2} + \left(\frac{z}{5}\right)^2 \frac{l_{z+5} - 2l_z + l_{z-5}}{2} \\ &= \frac{z(5+z)}{50} l_{z-5} + \frac{25-z^2}{25} l_z - \frac{z(5-z)}{50} l_{z+5}. \end{aligned}$$

For the several values of  $l$ , as deduced from the five respective curves, we must make  $z$  separately equal to  $-2$ ,  $-1$ ,  $0$ ,  $1$ ,  $2$ . The five values of  $l$  are therefore,

$$\begin{aligned} l &= -.12l_{-7} + .84l_{-3} + .28l_{+3} \\ l &= -.08l_{-6} + .96l_{-1} + .12l_{+4} \\ l &= 0 \quad \quad \quad + l \quad \quad + 0 \\ l &= +.12l_{-4} + .96l_{+1} - .08l_{+6} \\ l &= +.28l_{-3} + .84l_{+2} - .12l_{+7} \end{aligned}$$

Hence, if we put

$$\begin{aligned} \gamma_1 &= l_{-1} + l_{+1}, & \gamma_2 &= l_{-2} + l_{+2}, & \gamma_3 &= l_{-3} + l_{+3} \\ \gamma_4 &= l_{-4} + l_{+4}, & \gamma_6 &= l_{-6} + l_{+6}, & \gamma_7 &= l_{-7} + l_{+7} \end{aligned} \quad (A)$$

$$\begin{aligned} f &= \gamma_1 - \gamma_3, & g &= \gamma_2 - \gamma_3 \\ h &= \gamma_6 - \gamma_3, & k &= \gamma_7 - \gamma_4 \end{aligned} \quad (B)$$

and if ( $l$ ) denote the required average number-living at the given age, we shall have by adding together the five values of  $l$ ,

$$\begin{aligned} 5(l) &= l + .96\gamma_1 + .84\gamma_2 + .28\gamma_3 + .12\gamma_4 - .08\gamma_6 - .12\gamma_7 \\ &= l + \gamma_1 + \gamma_2 - .04\{(\gamma_1 - \gamma_3) + 4(\gamma_2 - \gamma_3) + 2(\gamma_6 - \gamma_3) + 3(\gamma_7 - \gamma_4)\} \\ &= l + \gamma_1 + \gamma_2 - .04(f + 4g + 2h + 3k) \quad (C) \end{aligned}$$

By means of these last formulæ (A), (B), (C), the required adjusted values of the numbers-living are readily computed.

*Example.*—In the Table H<sup>MF</sup>, Healthy Lives—Male and Female, it is required to find the adjusted value of  $l$  for age 25.

The data, contained in the table at page 396, are taken from Table H<sup>MF</sup>, "The Mortality Experience," vol. 1, page 277, and the calculation, *in extenso*, is given hereunder.

Age 25	1	2	3	4	6	7
$l$ 9250	$l_{-1}$ 9300	$l_{-2}$ 9374	$l_{-3}$ 9432	$l_{-4}$ 9496	$l_{-6}$ 9631	$l_{-7}$ 9688
$\gamma_1$ 18479	$l_{+1}$ 9179	$l_{+2}$ 9112	$l_{+3}$ 9048	$l_{+4}$ 8975	$l_{+6}$ 8829	$l_{+7}$ 8761
$\gamma_2$ 18486						
46195	$\gamma_1$ 18479	$\gamma_2$ 18486	$\gamma_3$ 18480	$\gamma_4$ 18471	$\gamma_6$ 18460	$\gamma_7$ 18449
Corr <sup>a</sup> + 3 <sup>32</sup>	$\gamma_3$ 18480	$\gamma_3$ 18480			$\gamma_3$ 18480	$\gamma_4$ 18471
			$2k$ - 40			
	$f$ - 1	$g$ + 6	$3k$ - 66		$k$ - 20	$k$ - 22
$(5)46198\cdot32$	$4g$ + 24					
			-106			
$(d) \dots 92397$	+ 23	. . . . . + 23				
			- 83	$\times -\cdot 04 = + 3\cdot 32 \text{ Corr}^a$		

In filling in the data in the six ruled columns, the computer will bear in mind that a chasm precedes the last two columns, and that  $l_{-5}$  and  $l_{+5}$  are there passed over unheeded. This is further indicated by the numerals placed at the top of the respective columns. It will be further observed that the final result is here put down and retained to an extra place of figures.

Since the expression (C) is linear with respect to the several values of  $l$ , it is evident that precisely the same formulæ may be applied to adjust the yearly decrements; and these being much smaller numbers, the calculation of the table may be thus considerably abbreviated and expedited, the numbers-living being then deduced by successive subtraction of the adjusted decrements. The adjustment of the decrements undoubtedly offers the greatest possible facility. As an example, taking the same table as before, the following is the calculation of the adjusted decrement from age 25 to age 26.

25 to 26	1	2	3	4	6	7
$d$ 51	$d_{-1}$ 70	$d_{-2}$ 74	$d_{-3}$ 58	$d_{-4}$ 64	$d_{-6}$ 77	$d_{-7}$ 57
137	$d_{+1}$ 67	$d_{+2}$ 64	$d_{+3}$ 73	$d_{+4}$ 71	$d_{+6}$ 68	$d_{+7}$ 77
138						
326	137	138	131	135	145	134
- 2 <sup>36</sup>	131	131			131	135
			$2k$ + 28			
	$f$ + 6	$g$ + 7	$3k$ - 3		$k$ + 14	$k$ - 1
$(5)323\cdot64$	$4g$ + 28					
			+ 25			
$(d) \dots 647$	+ 34	. . . . . + 34				
92397 . . . . . (l) age 25			+ 59	$\times -\cdot 04 = - 2\cdot 36$		
91750 . . . . . (l) " 26						

The resulting adjusted decrement 647 being here subtracted from 92397, the adjusted number-living at age 25, we obtain 91750 for the adjusted number-living at age 26.

In the actual construction of the New Experience Tables I have first made independent computations of the adjusted numbers-living for every fifth year of age; and afterwards calculated the values throughout for every age, by adjusting the decrements, thence deducing by successive subtraction the numbers-living, and making use of the former calculations at the end of each quinquennial interval, as a periodical check on the accuracy of the work. The final results, on being differenced to second differences, are generally found to be remarkable for their orderly progression, though at exceptional places there may yet exist some slight traces of irregularity, but they are quite isolated and so trivial as to be readily amended by inspection.\* Judging by my own experience, I have no doubt that the simplicity and efficiency of the manipulation will be favourably appreciated by those who may hereafter have occasion to put it in practice.

Another matter incidental to the final completion of the table, it will be requisite to explain. As the data for each separate calculation must extend over an interval of seven years preceding and following the given age, the formulæ will obviously not apply to the first seven years of the table, and the numbers for those years, viz., ages 10 to 16 will therefore be wanting. To effect a continuous juncture at age 17, I have considered it most expedient to supply the required numbers by means of constant third differences.

At age 10 the radix of the table is  $l_{10}=100000$ . If  $\Delta_1$ ,  $\Delta_2$ ,  $\Delta_3$  be the differences immediately following age 17, and  $n=7$ , we shall have

$$l_{10}=l_{17}-n\Delta_1+\frac{n(n+1)}{2}\Delta_2-\frac{n(n+1)(n+2)}{2.3}\Delta_3$$

$$=l_{17}-7\Delta_1+28\Delta_2-84\Delta_3$$

from which

$$\Delta_3=-\frac{l_{10}-l_{17}}{84}-\frac{\Delta_1}{12}+\frac{\Delta_2}{3}.$$

When the series of numbers is put down in a retrograde order, the differences that are of an odd order will change sign. In such case therefore we shall have to begin with  $l_{17}$  and apply the three orders of differences, after having reversed the signs of the first and third. The third difference should be calculated to one, or perhaps two, additional places of figures, and then the continued summation of the differences will sufficiently check the accuracy of the computation.

\* Those who are not practically familiar with progressions of differences, and the disturbances caused by isolated errors, need only to have recourse to the elementary Rule given by me in a former paper, *Journal*, vol. xii., page 140.

The first calculated numbers, of the  $H^{MF}$  Table, beginning at age 17, with the accompanying differences, are

	( <i>l</i> )		
Age 17	97189		
„ 18	96720	-469	
„ 19	96195	-525	-56

&c.

Here we have  $\Delta_1 = -469$ ,  $\Delta_2 = -56$ , and

$$\begin{aligned}\Delta_3 &= -\frac{l_{10}-l_{17}}{84} - \frac{\Delta_1}{12} + \frac{\Delta_2}{3} \\ &= -\frac{2811}{84} + \frac{469}{12} - \frac{56}{3} = -13.05.\end{aligned}$$

Hence changing the signs of the odd orders the three commencing differences are +469.00, -56.00, +13.05, and the retrograde calculation is as follows:—

	( <i>l</i> )		+13.05	= $\Delta_3$
Age 17	97189.00	+469.00	-56.00	
„ 16	97615.05	426.05	-42.95	
„ 15	98011.20	396.15	-29.90	
„ 14	98390.50	379.30	-16.85	
„ 13	98766.00	375.50	- 3.80	
„ 12	99150.75	384.75	+ 9.25	
„ 11	99557.80	407.05	+22.30	
„ 10	100000.20	+442.40	+35.35	

The column of second differences is formed by the repeated addition of the constant third difference placed above it, and the other columns are hence obtained by continued addition, or the inverse operation to that of differencing.

In the foregoing method of adjustment every individual element of the data supplied by the experience has its proper influence in determining the several results, and there does not exist anything of an arbitrary nature in the process. No extraneous condition or restriction is placed upon the quantities, which are freely permitted to manifest and assert their own law. Hence also, if there should be any particular phases in the absolute law of mortality, or any special peculiarities at certain periods of life, the same will be brought out with greater clearness in the final table, after the casual irregularities have been eliminated.

The adjustments have been separately made on the  $H^M$ ,  $H^F$  and  $H^{MF}$  Tables, and also on an  $H^M$  Table, excluding the first five years of assurance, this last table being designed for the general

purposes of valuations. For the  $H^{MF}$  mortality, the unadjusted numbers-living and corresponding expectations of life, together with the results after adjustment of the former, are shown in the following table. In the course of the table there are upwards of forty coincidences with the experience.

*Table  $H^{MF}$ , Healthy Lives—Male and Female.*

Age	UNADJUSTED			ADJUSTED		
	Number-living	Decr.	Expectation	Number-living	Decr.	Expectation
10	10000	97	49·97	100000	442	49·9814
11	9903	0	49·46	99558	407	49·2010
12	9903	25	48·46	99151	385	48·4010
13	9878	21	47·58	98766	376	47·5877
14	9857	63	46·68	98390	379	46·7676
15	9794	52	45·98	98011	396	45·9465
16	9742	19	45·22	97615	426	45·1309
17	9723	35	44·31	97189	469	44·3265
18	9688	57	43·47	96720	525	43·5391
19	9631	77	42·72	96195	581	42·7739
20	9554	58	42·06	95614	621	42·0308
21	9496	64	41·31	94993	645	41·3023
22	9432	58	40·59	94348	653	40·5813
23	9374	74	39·84	93695	651	39·8606
24	9300	70	39·15	93044	647	39·1360
25	9230	51	38·44	92397	647	38·4065
26	9179	67	37·65	91750	651	37·6739
27	9112	64	36·93	91099	668	36·9395
28	9048	73	36·18	90431	686	36·2087
29	8975	71	35·47	89745	703	35·4816
30	8904	75	34·75	89042	718	34·7578
31	8829	68	34·04	88324	726	34·0363
32	8761	77	33·30	87598	733	33·3142
33	8684	72	32·59	86865	743	32·5911
34	8612	77	31·86	86122	754	31·8680
35	8535	73	31·15	85368	768	31·1451
36	8462	78	30·41	84600	789	30·4233
37	8384	81	29·69	83811	811	29·7049
38	8303	86	28·97	83000	830	28·9903
39	8217	89	28·27	82170	844	28·2781
40	8128	82	27·57	81326	854	27·5664
41	8046	87	26·85	80472	860	26·8536
42	7959	86	26·14	79612	869	26·1383
43	7873	86	25·42	78743	888	25·4212
44	7787	91	24·69	77855	913	24·7055
45	7696	95	23·98	76942	948	23·9927
46	7601	99	23·27	75994	989	23·2858
47	7502	106	22·57	75005	1029	22·5862
48	7396	104	21·89	73976	1067	21·8934
49	7292	109	21·20	72909	1102	21·2065
50	7183	119	20·51	71807	1133	20·5243

Table HMF—(continued).

Age	UNADJUSTED			ADJUSTED		
	Number-living	Decr.	Expectation	Number-living	Decr.	Expectation
51	7064	118	19·84	70674	1167	19·8453
52	6946	121	19·17	69507	1204	19·1701
53	6825	114	18·50	68303	1251	18·4992
54	6711	125	17·81	67052	1304	17·8350
55	6586	149	17·14	65748	1358	17·1788
56	6437	146	16·53	64390	1414	16·5306
57	6291	144	15·90	62976	1471	15·8905
58	6147	154	15·26	61505	1531	15·2586
59	5993	146	14·64	59974	1601	14·6354
60	5847	176	13·99	58373	1677	14·0231
61	5671	174	13·42	56696	1760	13·4231
62	5497	187	12·83	54936	1849	12·8371
63	5310	197	12·26	53087	1936	12·2668
64	5113	199	11·72	51151	2014	11·7122
65	4914	210	11·17	49137	2080	11·1717
66	4704	210	10·65	47057	2138	10·6434
67	4494	216	10·12	44919	2186	10·1262
68	4278	232	9·61	42733	2224	9·6186
69	4046	241	9·13	40509	2268	9·1193
70	3805	212	8·68	38241	2331	8·6305
71	3593	218	8·16	35910	2401	8·1582
72	3375	256	7·65	33509	2469	7·7070
73	3119	253	7·24	31040	2531	7·2802
74	2866	295	6·83	28509	2567	6·8822
75	2571	236	6·56	25942	2542	6·5137
76	2335	248	6·17	23400	2476	6·1670
77	2087	222	5·85	20924	2369	5·8376
78	1865	240	5·48	18555	2247	5·5190
79	1625	214	5·22	16308	2110	5·2106
80	1411	186	4·93	14198	1969	4·9106
81	1225	188	4·61	12229	1823	4·6208
82	1037	156	4·36	10406	1672	4·3427
83	881	163	4·04	8734	1522	4·0783
84	718	131	3·84	7212	1360	3·8335
85	587	128	3·58	5852	1186	3·6082
86	459	100	3·44	4666	1014	3·3982
87	359	79	3·26	3652	849	3·2029
88	280	73	3·05	2803	689	3·0216
89	207	48	2·94	2114	548	2·8434
90	159	42	2·68	1566	435	2·6635
91	117	33	2·46	1131	336	2·4956
92	84	33	2·25	795	247	2·3390
93	51	23	2·34	548	181	2·1679
94	28	0	2·90	367	131	1·9905
95	28	3	1·90	236	86	1·8178
96	25	16	1·06	150	56	1·5733
97	9	4	1·00	94	44	1·2128
98	5	5	0·50	50	33	0·8400
99	0	0	·00	17	17	0·5000
100	..	..	..	0	0	·0000

If it were required to construct a table according to any proposed theory, it is evident that a previous adjustment of the data must supply the best possible preliminary for a satisfactory determination of the values of the theoretical constants, since the curve is in such case at once made to pass through the respective centres of the observations which appertain to the vicinity of each selected point. An interesting and useful example will be the formation of a table on the formula propounded by Mr. Makeham, as a modification of the well-known formula of Gompertz, respecting which I shall, however, first avail myself of the opportunity of making a few summary observations, bringing under review their relations, distinctive properties, and important bearing on the progress of our general knowledge of the theory of assurance.

Gompertz's law of mortality, which first appeared in the *Philosophical Transactions* in 1825, expresses the number-living at each age of the mortality table by the exponential formula

$$l_x = k(g)^{q^x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

By duly assigning the numerical values of the constants  $kgq$  involved in this function it is capable of representing numerically, with tolerable accuracy, the results of observation for a series of consecutive ages, with a range of at least thirty years. This last-mentioned limitation however imposes the necessity of introducing new sets of constants at certain periods of life, in order to complete the table of mortality, and this alone has been a bar to the general adoption of the formula as a general basis of investigation.

The logarithm of the number-living is

$$\log l_x = \log k + (\log g) \cdot q^x$$

which differentiated with respect to  $x$ , and the algebraic sign reversed, gives the force of mortality, viz.:—

$$\mu_x = -(\log g \log q) \cdot q^x$$

or

$$\mu_x = Bq^x \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

putting  $B = -\log g \log q$ , which is a positive coefficient, since  $\log g$  is negative.

The values of the forces of mortality at successive ages therefore form a geometrical series, of which the constant  $q$  is the common ratio. It was upon this elementary law physiologically stated with great perspicuity by Mr. Gompertz that he originally

established the formula (a) that has deservedly obtained so much celebrity.

In the *Philosophical Magazine* for November, 1839, Professor De Morgan first communicated his discovery that the well-known rule given by Thomas Simpson for finding the value of an annuity on three lives would be strictly correct if Gompertz's theory were true with one set of constants throughout the whole remainder of life. This conclusion is derived from what Professor De Morgan appropriately calls "the law of uniform seniority," which may be briefly stated as follows:—

If an age  $s$  be determined from the ages  $x, y, z$ , &c., by the relation

$$*q^s = q^x + q^y + q^z \dots$$

then, multiplying by  $q^t$ ,

$$q^{s+t} = q^{x+t} + q^{y+t} + q^{z+t} \dots$$

Hence, as the chosen relation subsists in the general case of uniform seniority, it must evidently continue to hold during the whole of life. From this it is readily shown that the value of an annuity on a life aged  $s$  is equal to that of a joint annuity on the lives aged  $x, y, z$ , &c., since the values of any two corresponding payments of the collated annuities are identical.

Professor De Morgan's explanation of this interesting property of Gompertz's law is given in the *Journal*, vol. viii. page 181, and his earlier paper on the subject was reprinted in vol. x. page 27. In the same volume, page 121, I have shown that the expedient resorted to in Simpson's rule, according to which a status of equivalent annuity value is substituted in the calculation of a joint annuity on several lives, the accuracy of which Professor De Morgan had proved to follow the assumption of Gompertz's law, cannot be true on any other hypothesis. If Gompertz's formula with one set of constants could have been sustained to the end of life, Professor De Morgan's discovery of this remarkable congruence would have been of immense value, since in such case monetary contingencies involving any number of lives would have been reducible in the simplest manner to those of corresponding single lives, and the unwieldy construction of joint life tables would have become entirely superseded.

The modification of Gompertz's formula proposed by Mr. Makeham is

\* This relation might otherwise be simply stated  $\mu_s = \mu_{xy\&c.}$

$$l_x = \frac{k}{a^x} (g)^{q^x} \quad . . . . . (c)^*$$

and may be regarded as a tolerably correct exponent of the numerical law of mortality for at least seventy years, or, we might say, the whole period of adult life. It gives

$$\begin{aligned} \log l_x &= \log k - (\log a) \cdot x + (\log g) \cdot q^x \\ \mu_x &= \log a - (\log g \log q) \cdot q^x \end{aligned}$$

or 
$$\mu_x = A + Bq^x \quad . . . . . (d)$$

The force of mortality according to Mr. Makeham's formula is therefore made up of two constituent portions, or partial forces, of which one is constant for all ages, and the other increases from year to year in a geometrical progression. This combination in the force of mortality is much in accordance with Mr. Gompertz's admirable philosophical reasoning upon the subject. It would indeed appear that Mr. Gompertz had, on his first discovery and subsequent investigation of the general principles of the law, definitely contemplated the existence of such conditions. Perhaps no better explanation of the specific rationale of Mr. Makeham's formula could be given than Mr. Gompertz's lucid and satisfactory statement in the paper referred to, that "it is possible that death may be the consequence of two generally coexisting causes; the one, chance, without previous disposition to death or deterioration; the other a deterioration, or an increased inability to withstand destruction."

Mr. Makeham (*Journal*, vol. ix. page 361) states that in the formula for the probabilities of living the additional constant ( $a$ ) is introduced "in such way that it should combine with the constant representing the interest of money in the corresponding formula for the *values of sums* depending upon those probabilities; and by this means preserve an important property of Mr. Gompertz's formula, first observed by Professor De Morgan,—viz., the power of substituting an equivalent single age, easily determined, for any combination of joint ages; with this difference however, that, by the introduction of the additional constant referred to, the substitution consists of an *equal number of lives*, of a certain common age, in lieu of a *single life*."

\* If stated in the form

$$l_x = \frac{k}{a^x (g)^{h^x}}, \text{ or perhaps } \frac{k}{l} = e^{ax + \beta h^x},$$

it might be more convenient to deal with numerically, as the logarithms of the constants in either case would then all be positive.

The property referred to depends as before on a specific law of uniform seniority, and it is at once derived by a simple substitution analogous to that made by Professor De Morgan. It is in that way deduced as follows:—

Put  $G = q^{t-1} \dots (e)$ ;  
then

$$p_{x,t} = \frac{G^x}{a^t}, p_{y,t} = \frac{G^y}{a^t}, p_{z,t} = \frac{G^z}{a^t}, \&c.$$

and, if  $n$  denote the number of lives,

$$\therefore p_{xyz\dots,t} = \frac{G^{x+y+z+\dots}}{a^{nt}}$$

The same probability for  $n$  equal lives, each aged  $s$ , is

$$p_{us\dots,t} = \frac{G^{ns}}{a^{nt}}.$$

Hence if  $s$  be determined from  $x, y, z, \dots$  by the relation

$$q^x + q^y + q^z \dots = nq^s \dots (f)$$

then  $a_{xyz\dots} = \Sigma p_{xyz\dots,t} v^t$   
 $= \Sigma p_{us\dots,t} v^t = a_{us\dots}$

That is the required value of the joint annuity on the  $n$  lives aged  $x, y, z, \dots$  is equal to that of the joint annuity on the  $n$  equal lives, each aged  $s$ , the same indeed being identical as to the separate values of each payment. The property has been similarly proved by Mr. Sprague, *Journal*, vol. xiii., page 355.

I would here suggest that the sought annuity may be otherwise determined in the form of an equivalent annuity on a *single life*, by making

$$\left. \begin{aligned} q^x + q^y + q^z \dots &= q^{s_1} \\ \frac{v}{a^{n-1}} &= v_1 \end{aligned} \right\} \dots (g)$$

For then,

$$\begin{aligned} p_{xyz\dots,t} &= \frac{G^{x+y+z+\dots}}{a^{nt}} \\ &= \frac{G^{s_1}}{a^{nt}} = \frac{p_{s_1,t}}{a^{(n-1)t}} \end{aligned}$$

$$\therefore a_{xyz\dots} = \Sigma p_{xyz\dots,t} v^t = \Sigma p_{s_1,t} v_1^t.*$$

The required value of the annuity on the joint lives  $x, y, z, \dots$

\* For continuous annuities  $\Sigma$  is replaced by  $\int dt$  and the values are also identical throughout.

is therefore identical with that on the *single life* aged  $s_1$ , provided only that interest of money is taken such that the present value of £1 due a year hence is  $v_1$ .

For the purpose of determining the values of annuities involving several lives there are thus two alternatives available, viz., to construct joint life tables with equal ages, say to the extent of five lives; or to tabulate annuities on single lives only, but with a more minute subdivision of the rates of interest. Instead of prematurely signifying any preference I am disposed for the present to content myself with the observation that the rival advantages of these alternatives may be worth some consideration previously to the actual formation of a system of tables. It will be perceived that the latter method of tabulation I have suggested, which might easily be carried out by means of lateral interpolation, would equally facilitate the working of inverse problems, in which it might be required to determine the rate of interest from given data appertaining to annuities or assurances.

It may further be stated that a rigid analytical proof might be given that Mr. Makeham's formula, which includes that of Gompertz, is the most general form of function possible to which a law of uniform seniority can in any way be applicable; and that, as it is vitally important to retain the extraordinary facilities which such a law presents, we cannot hope for any further generalization. On the assumption of any other theory of mortality the advantages alluded to must be wholly surrendered.

As the constant Mr. Makeham has introduced may always be combined with the factor depending on interest of money, the expressions which arise in any investigation must preserve the same functional forms as those of Gompertz. But they are all alike intractable: none of them admit of independent summation or integration excepting by infinite series; not even in the ordinary case of determining the value of a life annuity or of the expectation of life. On the other hand the formula, assuming it to be practically true, gives the force of mortality without series. Also, when the values of joint annuities are known, it serves to elicit the value of a survivorship assurance. For such assurance Mr. Makeham (*Journal*, vol. ix. page 361) has given the correct formula, the same in substance being equivalent to the following,

$$\begin{aligned}\bar{A}_{xy} &= \frac{q^x}{q^x + q^y} \bar{A}_{xy} - \frac{q^x - q^y}{q^x + q^y} \log a \cdot \bar{a}_{xy} \\ &= \frac{q^x}{q^x + q^y} (1 - \delta \bar{a}_{xy}) - \frac{q^x - q^y}{q^x + q^y} \log a \cdot \bar{a}_{xy}.\end{aligned}$$

It may be briefly deduced thus. Since

$$\log p_{x,t} = q^x \log G - t \log a$$

$$\log p_{xy,t} = (q^x + q^y) \log G - 2t \log a$$

$$\therefore \log p_{x,t} = \frac{q^x}{q^x + q^y} \log p_{xy,t} + \frac{q^x - q^y}{q^x + q^y} \log a \cdot t,$$

which differentiated, supposing  $t$  to vary, gives

$$\frac{dp_{x,t}}{p_{x,t}} = \frac{q^x}{q^x + q^y} \frac{dp_{xy,t}}{p_{xy,t}} + \frac{q^x - q^y}{q^x + q^y} \log a \cdot dt$$

Multiply by  $p_{xy,t}$ ; then

$$dp_{x,t} \cdot p_{xy,t} = \frac{q^x}{q^x + q^y} dp_{xy,t} + \frac{q^x - q^y}{q^x + q^y} \log a \cdot p_{xy,t} dt$$

Lastly, multiplying by  $v^t$ , changing signs and integrating,

$$\bar{A}_{xy}^1 = \frac{q^x}{q^x + q^y} \bar{A}_{xy} - \frac{q^x - q^y}{q^x + q^y} \log a \cdot \bar{a}_{xy}.$$

Mr. Makeham states the formula for the more general case of an assurance on the joint existence of  $m$  lives  $xx'x'' \dots$  against the joint existence of  $n$  lives  $yy'y'' \dots$ , the same being equivalent to

$$\frac{S_x}{S_x + S_y} \bar{A}_{xx' \dots y'y'' \dots} - \frac{nS_x - mS_y}{S_x + S_y} \log a \cdot \bar{a}_{xx' \dots y'y'' \dots}$$

where

$$S_x = q^x + q^{x'} + q^{x''} \dots$$

$$S_y = q^y + q^{y'} + q^{y''} \dots$$

It is somewhat similar in form to the other, which is a particular case, and may be deduced in like manner. These formulæ, of course, assume a theoretical table with one set of constants.

Gompertz's formula contains three independent constants ( $k, g, q$ ), and can therefore be made to accurately represent the numbers-living at any three given ages. Mr. Makeham having introduced a fourth arbitrary constant ( $a$ ), his formula can be made to represent the exact numbers-living at any four given ages; and it is convenient to fix upon these at equal intervals embracing the chief portion of life. To determine the constants of Mr. Makeham's formula we have, for a given age  $x$ ,

$$\log l = \log k - x \log a + q^x \log g \quad . \quad . \quad . \quad (1)$$

and, taking uniform intervals of  $t$  years,

$$\Delta \log l = -t \log a + q^x (q^t - 1) \log g \quad . \quad . \quad (2)$$

$$\Delta_2 \log l = q^x (q^t - 1)^2 \log g \quad . \quad . \quad . \quad (3)$$

$$\therefore t \log q = \Delta \log (\Delta_2 \log l)$$

which last gives the value of  $\log q$ , and hence  $\log g$ ,  $\log a$  and  $\log k$  are found from (3), (2), (1) respectively.

For calculation let the four ages be 20, 40, 60, 80; then from the adjusted numbers—living on page 396 we get the following:—

Age	$\log l$	$\Delta$	$\Delta_2$	
20	4.9805215	— .0702921	— .0737253	$\log = 8.8676166$
40	4.9102294	— .1440174	— .4699674	„ = 9.6720666
60	4.7662120	— .6189848		
80	4.1522272			
				$t=20) \cdot 8044500$

$$\log q = \cdot 0402225$$

$$q^t - 1 = 5.374559$$

$$\log(q^t - 1) = \cdot 7303429$$

And we hence find,

$$\log k = 5.0396484$$

$$\log a = \cdot 0028287$$

$$\log g = - \cdot 0004004$$

Again, for a second determination, beginning with ages midway between the former, viz., taking 30, 50, 70, 90, we obtain

Age	$\log l$	$\Delta$	$\Delta_2$	
30	4.9495949	— .0984281	— .1802095	$\log = 9.2557777$
50	4.8561668	— .2786376	— 1.1140998	„ = 0.0469241
70	4.5825292	— 1.3877374		
90	3.1947918			
				$t=20) \cdot 7911464$

$$\log q = \cdot 0395573$$

$$q^t - 1 = 5.182247$$

$$\log(q^t - 1) = \cdot 7145182$$

Hence also,

$$\log k = 5.0442858$$

$$\log a = \cdot 0029327$$

$$\log g = - \cdot 0004365$$

To ascertain briefly the degree of approximation presented by these separate determinations, begin with  $x=10$  and make the interval  $t=10$ ; then with the two sets of constants we find, for age 10,

	1st Curve.	2nd Curve.
$\log l =$	5.0103502	5.0138735
$\Delta \log l =$	— .0298287	— .0309403
$\Delta_2 \log l =$	— .0023503	— .0023981
$10 \log q =$	4.022250	3.955732

The values of  $l$ , calculated from these, are inserted in the following table, and are compared with the original adjusted values.

Age	Adjusted $l$	1st Curve	2nd Curve	DEVIATIONS FROM ADJUSTED		
				$c_1$	$c_2$	$\frac{2c_1 + c_2}{3}$
20	95614	95614	96146	0	+ 532	+ 177
30	89042	88786	89042	- 256	0	- 171
40	81326	81326	81338	0	+ 12	+ 4
50	71807	71967	71807	+ 160	0	+ 107
60	58373	58373	58234	0	- 139	- 46
70	38241	38000	38241	- 241	0	- 161
80	14198	14198	14859	0	+ 661	+ 220
90	1566	1306	1566	- 260	0	- 173
100	0	3	6	+ 3	+ 6	+ 4

The last column shows an estimate of the modified deviations for an assumed intermediate curve favourably chosen for nearer approximation; and for this curve the various elements are found thus :

1st curve,	$2 \log q = .0804450$	} Similarly we get, for the interval $t=10$ ,
2nd ,,	$\log q = .0395573$	
	$3) .1200023$	
Intermediate curve	$\log q = .0400008$	
		$\log l_{10} = 5.0115246$
		$\Delta \log l_{10} = - .0301992$
		$\Delta_2 \log l_{10} = - .0023662$

The set of constants finally obtained from these are

$$\left. \begin{array}{l} \log k = 5.0411939 \\ \log \alpha = .0028634 \\ \log g = - .0004121 \\ \log q = .04 \end{array} \right\} \dots \dots (h)^*$$

the value of  $\log q$  being slightly curtailed to facilitate computations relating to the law of seniority.

Hence when  $t=1$  we find, for the direct formation of a complete table, the following values,

$$\begin{aligned} \log l_{10} &= 5.0115246 \\ \Delta \log l_{10} &= - .0029633 \\ \left\{ \begin{array}{l} \Delta_2 \log l_{10} = - .0000096 \\ \log = (-) 4.9838727 \end{array} \right. \end{aligned}$$

\* In a similar calculation for the former Experience Table (17 Offices) I have chosen the curve midway between those determined by ages 20, 40, 60, 80, and 30, 50, 70, 90, and the resulting constants are,

$\log k = 5.0291738$	} Also, $\log l_{10} = 4.9992772$
$\log \alpha = .0028744$	
$\log g = - .0004635$	
$\log q = .03956$	
	$\Delta \log l_{10} = - .0029843$
	$\Delta_2 \log l_{10} = - .0000105$
	$\log = (-) 5.02045$

from which a table may readily be computed. It will be understood that  $\log k$  is not a significant element of the law of mortality, as it merely determines the radix of the table.

The values of  $\Delta_2 \log l$ , from year to year, are got by adding successively  $\log q = .04$  to  $\log(\Delta_2 \log l)$  and taking out the several natural numbers; and then the values of  $\Delta \log l$  and  $\log l$  are obtained by continued addition of the differences. The results are given in the following table, and they are preceded by corresponding numbers according to the experience, for the purpose of exhibiting how far the formula may be relied upon as a near approximation. Throughout a considerable portion of the table the deviations are small, and may be considered to be within the limits of errors of observation.

Age	BY EXPERIENCE		BY MR. MAKEHAM'S FORMULA		
	$l$	Expectation $e$	$\log l$	$l$	Expectation $e$
18	9688	43.47	4.9874896	97160	43.3270
19	9631	42.72	4.9844175	96476	42.6311
20	9554	42.06	4.9813253	95791	41.9321
21	9496	41.31	4.9782110	95107	41.2303
22	9432	40.59	4.9750725	94422	40.5257
23	9374	39.84	4.9719075	93736	39.8185
24	9300	39.15	4.9687134	93049	39.1087
25	9230	38.44	4.9654874	92361	38.3966
26	9179	37.65	4.9622264	91670	37.6822
27	9112	36.93	4.9589270	90976	36.9657
28	9048	36.18	4.9555855	90279	36.2474
29	8975	35.47	4.9521970	89577	35.5273
30	8904	34.75	4.9487597	88871	34.8057
31	8829	34.04	4.9452661	88159	34.0828
32	8761	33.30	4.9417117	87440	33.3588
33	8684	32.59	4.9380906	86714	32.6339
34	8612	31.86	4.9343964	85980	31.9084
35	8535	31.15	4.9306221	85236	31.1826
36	8462	30.41	4.9267599	84481	30.4566
37	8384	29.69	4.9228013	83715	29.7309
38	8303	28.97	4.9187371	82935	29.0058
39	8217	28.27	4.9145571	82140	28.2815
40	8128	27.57	4.9102501	81330	27.5583
41	8046	26.85	4.9058038	80501	26.8368
42	7959	26.14	4.9012048	79653	26.1172
43	7873	25.42	4.8964384	78784	25.3999
44	7787	24.69	4.8914884	77891	24.6853
45	7696	23.98	4.8863371	76973	23.9739
46	7601	23.27	4.8809651	76027	23.2660
47	7502	22.57	4.8753511	75050	22.5622
48	7396	21.89	4.8694717	74041	21.8630
49	7292	21.20	4.8633013	72996	21.1686
50	7183	20.51	4.8568118	71914	20.4798

Age	By Experience		By Mr. MAKEHAM'S FORMULA		
	$l$	Expectation $e$	$\log l$	$l$	Expectation $e$
51	7064	19.84	4.8499725	70790	19.7969
52	6946	19.17	4.8427496	69622	19.1206
53	6825	18.50	4.8351061	68408	18.4512
54	6711	17.81	4.8270014	67143	17.7893
55	6586	17.14	4.8183910	65825	17.1355
56	6437	16.53	4.8092261	64450	16.4903
57	6291	15.90	4.7994532	63016	15.8542
58	6147	15.26	4.7890137	61520	15.2278
59	5993	14.64	4.7778433	59957	14.6115
60	5847	13.99	4.7658715	58327	14.0059
61	5671	13.42	4.7530209	56627	13.4115
62	5497	12.83	4.7392068	54854	12.8288
63	5310	12.26	4.7243362	53007	12.2583
64	5113	11.72	4.7083072	51087	11.7004
65	4914	11.17	4.6910080	49092	11.1555
66	4704	10.65	4.6723161	47024	10.6241
67	4494	10.12	4.6520971	44885	10.1066
68	4278	9.61	4.6302037	42678	9.6033
69	4046	9.13	4.6064743	40409	9.1145
70	3805	8.68	4.5807318	38083	8.6406
71	3593	8.16	4.5527819	35709	8.1817
72	3375	7.65	4.5224117	33298	7.7381
73	3119	7.24	4.4893877	30859	7.3100
74	2866	6.83	4.4534538	28409	6.8974
75	2571	6.56	4.4143293	25961	6.5005
76	2335	6.17	4.3717064	23535	6.1193
77	2087	5.85	4.3252476	21147	5.7537
78	1865	5.48	4.2745828	18818	5.4038
79	1625	5.22	4.2193062	16569	5.0694
80	1411	4.93	4.1589728	14420	4.7504
81	1225	4.61	4.0930948	12391	4.4466
82	1037	4.36	4.0211372	10499	4.1578
83	881	4.04	3.9425135	8760	3.8838
84	718	3.84	3.8565806	7188	3.6242
85	587	3.58	3.7626333	5789	3.3786
86	459	3.44	3.6598984	4570	3.1469
87	359	3.26	3.5475280	3528	2.9285
88	280	3.05	3.4245925	2658	2.7231
89	207	2.94	3.2900726	1950	2.5303
90	159	2.68	3.1428507	1389	2.3496
91	117	2.46	2.9817013	959	2.1805
92	84	2.25	2.8052807	639	2.0227
93	51	2.34	2.6121156	409	1.8756
94	28	2.90	2.4005905	252	1.7389
95	28	1.90	2.1689341	148	1.6119
96	25	1.06	1.9152041	82	1.4944
97	9	1.00	1.6372709	43	1.3858
98	5	.50	1.3327994	22	1.2857
99	0	.00	0.9992293	10	1.1937
100	0	..	0.6337532	4	1.1094

The values of life annuities would come out with still less discrepancy than the expectations, since amongst the components of such values the extreme and less certain portions towards the end of life become principally affected by monetary discount.

According to (d) the theoretical Force of Mortality, involved in Mr. Makeham's formula, is

$$\mu = A + Bq^x$$

where  $A = \log_e a$ ,  $B = -\log_e g \log_e q$ , in which alone the logarithms must necessarily be Napierian or hyperbolic. In the present case the numerical constants are therefore

$$A = \cdot 0065932$$

$$B = \cdot 0000874$$

$$\log q = \cdot 04$$

and these might be regarded as the chief constituents of the theoretical table.

The corresponding values obtained for the new  $H^M$  and the former Experience Table (17 Offices) are

	New $H^M$ .	17 Offices.
$A =$	$\cdot 0061470$	$\cdot 0066186$
$B =$	$\cdot 0000988$	$\cdot 0000972$
$\log q =$	$\cdot 0394660$	$\cdot 0395613$
$\log k =$	$5\cdot 0400600$	$5\cdot 0291738$
$\log a =$	$\cdot 0026696$	$\cdot 0028744$
$\log g =$	$-\cdot 0004724$	$-\cdot 0004635$

It would be useless to give constants for the new  $H^F$  Table and for the  $H^M$  table, excluding the first five years of assurance, as these tables are less compatible with Mr. Makeham's formula. This incongruity might be partly accounted for by the more limited experience data upon which the computations are made.

The subject of these remarks would be incomplete were I not to notice that Gompertz, in his supplementary memoir given in the *Philosophical Transactions* for 1862, extends his researches on the science connected with Human Mortality. By introducing three additional terms his function is made practically complete as a representation of the law of mortality from birth to the limit of life, but it thereby loses its simplicity and elegance and, in a corresponding degree, becomes cumbrous and unmanageable as an instrument of investigation.

My observations have already occupied more space than was anticipated, and must now be brought to a termination. In

conclusion I have only furthermore to state my general impression that the important and reliable aid to be derived from Mr. Makeham's formula will be found to be not so much perhaps in the construction of mortality and annuity tables as in the useful application of the general properties evolved, of which the most valuable is the law of uniform seniority, to tables independently and more accurately deduced from experience. It will at least entirely obviate in all cases the necessity of forming joint annuity tables on more than two lives, excepting for equal ages, and it is by no means impossible that future investigations founded on Mr. Makeham's hypothesis may reveal other important relations at present unknown.

In intimate connexion with this subject I may here reproduce, and announce as an extraordinary general theorem, the formula for an assurance established in my paper "On an Improved Theory of Annuities and Assurances," viz.,

$$\bar{A} = \left( \mu - \frac{d}{dx} \right) \bar{a} \dots$$

which I have found to be **UNIVERSAL**. It represents, with rigid mathematical accuracy, the value of an assurance of £1 payable on the failure of the life  $x$  under any compounded conditions of survivorship, or otherwise, amongst any number of lives, the symbol  $\bar{a} \dots$  denoting a continuous annuity of £1 per annum on the life  $x$ , and payable under precisely the same conditions. The complete theory of assurances is thus, at once, reduced to the exclusive computation of life annuities.

It may be further interesting to announce that I have recently succeeded in establishing another general theorem, viz.,

$$\frac{\bar{A}}{\delta} - \frac{A}{d} = \frac{\mu}{12}.$$

This remarkable theorem is also **UNIVERSAL**; it is true whether the assurance be absolute or contingent, and however complicated the contingency by which it may be affected, the one-twelfth of the force of mortality stated on the right hand being in every case that only of the life or lives, on the failure of which the assurance is determinable. The equation fixes a very simple and commodious bond of relationship between an assurance of £1 payable at the instant of death and a similar assurance payable at the end of the year of decease, whatever may be the nature of the assurance. In the second term the assurance payable at the end of the year of decease is always the true value, and, in cases of survivorship, is not subject to any objectionable supposition of uniform decrement.

The theorem might be thus enunciated:—

*The present value of a continuous reversionary perpetuity of £1 per annum, whether absolute or contingent, commencing from the day of decease, exceeds that of a corresponding annual reversionary perpetuity, first payment at the end of the year of decease, by one-twelfth of the force of mortality at the given age or ages of the life or lives in possession.*

An equivalent relation, in respect of ordinary annuities and assurances, was similarly enunciated in my paper before referred to (see *Journal*, vol. xv., page 112). But I had not then made the discovery that it was capable of unlimited generalization.

It is important to note generally that the marvellously comprehensive theorems here announced do not depend on any assumed law of mortality, but are founded on absolute principles, not only applicable to all tables, but, in connexion therewith, to every possible condition that may come under consideration.

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\*\*\* In connection with the above paper it may interest our readers to state that Mr. Makeham has handed to us two letters he has received from Mr. L. W. Meech, Actuary of the Charter Oak Life Insurance Company, Hartford, Connecticut. In the former of these, dated 10th Dec., 1868, Mr. Meech states that in graduating the American Life Table (Males), published in the Thirteenth Massachusetts Insurance Report, he found Mr. Makeham's formula would not apply below the age of 20. Above that age, taking  $l_x = ds^x g^a$ , he found

$$\lambda s = \bar{1} \cdot 9961371'93$$

$$\lambda g = -0 \cdot 0003619'30$$

$$\lambda q = 0 \cdot 0396800'44$$

In a postscript he adds, "Let  $\frac{s^2}{1 \cdot 04} = \frac{s}{1 \cdot 05}$ ; then  $s = \frac{1 \cdot 04}{1 \cdot 05} = .990476$ , which is very near the true value ( $s$  above  $= .991145$ ). "With it 4 per cent joint lives could be found in the table for 5 "per cent single life." This follows at once from Mr. Woolhouse's formula ( $g$ ), putting  $n=2$ . This change in the value of  $s$  would, as Mr. Meech remarks, reduce the extent of tables.

In the second letter, dated 2nd July, 1869, Mr. Meech gives the summary of an elaborate research into a method for finding the approximate value of a life annuity by means of series. We hope that at some future time we shall be enabled to give this and other researches of Mr. Meech in full.—ED. J. I. A.

*On the Proper Method of Estimating the Liability of a Life Insurance Company under its Policies.* By THOMAS B. SPRAGUE, M.A., Vice-President of the Institute of Actuaries.

[Read before the Institute, 25th April, 1870.]

SOME years ago I read before the Institute a paper bearing on this subject (see *Journal*, vol. xi., p. 90) in which I examined at considerable length the method of valuation which has been on various occasions so strongly advocated by Mr. Tucker,\* and which is denominated by him the "reinsurance method." This term, however, appeared to me unsuitable, and I termed the method the "hypothetical method" of valuation; because the sums assured and the premiums are valued by means of a hypothetical table of the values of reversions and annuities, deduced by an inverse process from the premiums actually charged by the office. In that paper I compared Mr. Tucker's method with the net-premium method of valuation, and gave my reasons for believing the latter to be not only greatly superior to the former, but the method that should be generally employed. Subsequent consideration has however satisfied me that the net-premium method of valuation is also open to very serious objections, and that it is not applicable in all cases; and I propose on the present occasion, first to point out some of these objections, and then to consider briefly what method of valuation should be followed when the net-premium method is inapplicable. The subject however is a very wide one; and the present paper must be considered rather as a contribution to the discussion of a question that is all-important to those who have the responsibility of advising as to the liabilities of Life Insurance Companies, than as claiming to be itself a complete essay on the subject.

Such further discussion appears to be the more desirable at the present time, on account of the intolerant manner in which some of the advocates of the net-premium method of valuation insist upon its being the one method that should be employed in all cases. A recent writer, who may probably be considered as representing the most extreme views on this subject, thus expresses himself:—"The first and most urgent of the principles affecting the solvency of Life Offices is THAT THE WHOLE OF THE FUTURE LOADING AT EACH INVESTIGATION BE RESERVED INTACT." "If, in estimating its liability under its policies, a Company debit

\* See *Journal*, vol. x., p. 312; and the recent pamphlet "*On the Causes which lead to the Insolvency of Life Insurance Companies.*" Laytons. 1870.

" itself only with the sum that would be actually requisite to pay  
 " the amount assured at death, and credit itself with the *gross*  
 " future payments under its contracts, instead of the *nett* premiums  
 " merely, it will have anticipated the total worth of this loading ;  
 " and there will be no fund to provide future working expenses,  
 " or to be a source of profit ; and, so far as this item is concerned,  
 " the Company is insolvent. If a part of this loading only be  
 " discounted, and the remainder be sufficient to pay the future  
 " charges of management, the Company will be solvent in the  
 " sense that it will be able to pay its policies."\* It is scarcely  
 necessary to point out that if a Company is able to meet its  
 engagements, it is solvent—not in this or that sense, but abso-  
 lutely, solvent—and that anything more is a question, not of  
 solvency, but of surplus. Writings such as the above, although  
 well meant, appear to me to have an extremely mischievous  
 tendency, and to be calculated to propagate wholly erroneous  
 views on life insurance finance.

One of the principal objections to Mr. Tucker's method of  
 valuation is that it does not exhibit the actual financial position of  
 the Company to which it is applied. As stated above, the sums  
 assured and the premiums are multiplied by certain numbers  
 which have no real significance in themselves, but which we know  
 beforehand will bring out a value of the policy that may be fairly  
 acted upon in the case of a prosperous Insurance Company doing  
 a steady business ; such value being, as I have shown in the paper  
 already referred to, either greater than the value as found by the  
 net-premium method, equal to it, or less than it, according to the  
 method in which the premiums are loaded. To make use of a  
 comparison that has been often applied to the solution of problems  
 by algebraical processes, the figures are put into a sort of actuarial  
 mill, the handle is turned, and out comes the result we want. In  
 other words, the process does not exhibit to us the true value,  
 either of the sums assured, or of the premiums payable. It is  
 not my purpose on the present occasion to discuss any further the  
 merits of Mr. Tucker's method of proceeding, although there are  
 some points in his recent pamphlet which appear to invite  
 criticism. At the same time I must not omit to add that in some  
 respects I think better of that method than I did formerly.

The above mentioned objection applies with equal force to the  
 net-premium method of valuation, when the value of the premiums

\* "*The Principles affecting the Solvency of a Life Assurance Company.*" By James  
 R. Macfadyen. Glasgow, 1870.

actually payable is either not ascertained or not stated in the valuation balance sheet. The object of every valuation should be to exhibit in the fullest possible manner the exact financial position of the Company; and this is not done unless the value of the premiums actually payable is stated. As Mr. Jellicoe pointed out long ago (see *Journal*, vol. iii., p. 188)—“If the gross premiums are not valued, a Society charging very low premiums would show *cæteris paribus* the same assets as another charging very high ones, supposing both to value by the same table.” In other words, the two Offices would appear to be on an equality, whereas there can be no doubt that the position of the Office charging high premiums is, *cæteris paribus*, much more stable and much better secured than that of the Office charging low premiums; and its prospects as regards future profits, greatly preferable.

No further argument is required to show that the actual premiums payable ought to be valued by a true table of mortality, and brought into account. At the same time, it is undeniable that altho' it is right to exhibit the true value of the gross premiums, it is not correct to take credit for them in the same way as for the realized assets. If this is done, then, as Mr. Tucker reminds us that David Jones has said at p. 1094 of his book, the Office, having valued the gross premiums by a true table of mortality, places itself in point of security in precisely the same situation as if it had originally charged only the net premiums. “If the rate of interest and mortality be assumed as what will actually obtain, and the whole surplus be divided, the sum reserved by the Company together with the future premiums to be received on the policies will just enable the Office to discharge the various claims that will be made on it as the lives assured die off. It therefore follows that all the profits are anticipated and that no future bonus can be declared but at the expense of new assurers.” It might have been added that there will be no fund, except the profits on new assurances, out of which the expenses of management can be paid. It thus appears that altho' the gross premiums ought to be valued and brought into account, yet it is equally undeniable that credit must not be taken for their full amount in estimating the liabilities of the Company; but that a deduction must in all cases be made from the value of the gross premiums, to provide for future expenses, profits, and contingencies. The question then arises, what is the proper deduction to make from the Office premiums for these purposes? To this question

the advocate of the net-premium method of valuation have a ready answer. They say that the exact loading of the premiums must be reserved, neither more nor less,—that the addition which was originally made to the net premiums for the above specified purposes, is the proper deduction to make ever afterwards. This answer may be considered as to a certain extent satisfactory in the ordinary case where a Company is well established, is doing a steady business, and is economically managed; and it may then be said of this method, as we have already said of Mr. Tucker's, that it gives a good working reserve, the results of the application of the two methods being, indeed, in general not very different.

But there are certain cases in which this method is found to be quite inapplicable; and this being the case, I contend that the method has no claim to complete and scientific accuracy, and is not entitled to the exaggerated praise bestowed upon it by the more ardent of its admirers. First, take the case of the transfer of the business of one Company to another. Suppose, for the sake of definiteness, the transfer of the business of a small Company to another much larger: and, for simplicity, assume in the first place that the policies are all without profits. Then the net-premium method of valuation would bring out the value of the transferred policies, or the sum to be paid to the Office undertaking the risk, quite independent of the premiums actually payable. But this is manifestly unjust. If the premiums are high, it is undeniable that a smaller amount of realized assets will suffice to meet the liability under them; and conversely, if the premiums are very low, the amount of realized assets should be larger. It is conceivable that the premiums charged may be actually less than the net premiums as given by the tables used in the valuation; and in this case, even the strongest advocates of the net-premium method could not defend its use. If the policies of the transferred Office are to share in the future profits, the net-premium method of valuation is equally, if not still more, inapplicable; for whether the policyholders of the transferred Company are to share equally with the original policyholders in the future profits of the joint Company, or whether they are to have any other scale of profits, it is of vital importance to know the amount of their contribution towards the profits, which cannot be done unless the exact premiums they pay are brought into account. Whether, therefore, the policies are to share in the profits, or not, in both cases the actual premiums payable must be valued; and the value of the policy will depend upon the magnitude of those

premiums, being greater or less according as the scale of premiums paid is low or high. In this case, then, the net-premium method does not give the proper deduction to be made from the gross premium; and I will now proceed to show what deductions should be made from the premiums in the two cases.

First, supposing the policies to be without profits, and neglecting for the present the deterioration which the lives on the average experience with the lapse of time, it is obviously of no consequence to the purchasing Company at what ages the several policies were taken out. The data required for valuation are the sum assured, the premium actually payable, and the present age; and the deduction to be made from the gross premium may be determined by comparison with the premiums charged by the purchasing Office for new insurances. In other words, the value of the contributions of a policyholder in the transferred Company of a given present age must be the same as that of the contributions of a new insurer of the same age. Thus, then, the policyholder is to be credited with the premium he pays, and to be charged with that he would pay if he effected a new insurance, and the value of his policy will therefore be the present value of the difference between the premium he actually pays and that which he would pay according to the rates of the Office. Suppose that the policy was taken out at the age  $x$ , and has been  $n$  years in force, the sum assured being 1, the premium charged by the purchasing Office at that age being  $P_x$ , and the premium actually payable,  $P'_x$ ; further, that  $\varpi_{x+n}$  is the net premium at the present age,  $P_{x+n}$  the premium charged by the purchasing Office for that age, and  $\phi_{x+n} = P_{x+n} - \varpi_{x+n}$ , the loading. Then the value of the policy will be

$$\begin{aligned} A_{x+n} - (1 + a_{x+n})(P'_x - \phi_{x+n}) &= (1 + a_{x+n})(\varpi_{x+n} + \phi_{x+n} - P'_x) \\ &= (1 + a_{x+n})(P_{x+n} - P'_x). \end{aligned}$$

If we allow for the commission of 5 percent on the premium, we must deduct that percentage, both from the original premium, and from that now payable, so that the value of the policy will be

$$\frac{19}{20}(1 + a_{x+n})(P_{x+n} - P'_x).$$

It is of course obvious that the purchasing Office may be willing to take over the selling Office on easier terms. Having satisfied itself that the business of the selling Office is of a good class, or that the connection is likely to prove valuable, it may reason that

there is no necessity to charge the same sum for the management of a ready-made business, as for one that has to be obtained by advertising, canvassing, &c.; and this argument is perfectly legitimate. These views will be carried into effect by substituting for  $P_{x+n}$  a smaller premium; care being always taken to avoid the negative values which would occur if  $P_{x+n}$  were less than  $P'_x$ . In order to give full effect to the preceding views, it is essential that the *true* rate of interest should be used in the calculations, *i.e.* the highest rate which can be fairly reckoned on as likely to be maintained in the future—say four percent.

If the policies are to participate in the profits, the question is not so simple, and much may depend upon the method according to which the profits are divided. If, for example, it is agreed that the policyholders of the transferred Company shall share in the future profits just as if their policies had been originally effected with the purchasing Company, it is clear that in the valuation they should be charged with the same contributions to profits, &c., as are payable by the policyholders of the same standing in the purchasing Company; that is to say, they should be credited with the premiums they actually pay, and from these should be deducted the loading on the premiums charged by the Office for their several ages at entry, the item of commission being separately dealt with if thought desirable. The value of the policy in this case will therefore be

$$\begin{aligned} A_{x+n} - (1 + a_{x+n})(P'_x - \phi_x) &= A_{x+n} - (1 + a_{x+n})\{P'_x - (P_x - \varpi_x)\} \\ &= (1 + a_{x+n})\{\varpi_{x+n} - \varpi_x - (P'_x - P_x)\} \end{aligned}$$

If we allow for the commission of 5 percent we must put  $\frac{19}{20}(P'_x - P_x)$  instead of  $P'_x - P_x$ .

The above assumes that the purchasing Company values its own policies by the net-premium method; and since the value of a policy by that method is

$$(1 + a_{x+n})(\varpi_{x+n} - \varpi_x)$$

the above expression becomes

$$V_{x|n} - (1 + a_{x+n})(P'_x - P_x).$$

This indicates that the value of any policy in the transferred Company is to be deduced from that of the policy similarly circumstanced in the purchasing Company, by adding or subtracting the present value of the difference between the premiums payable on the two policies, according as the premiums of the

purchasing or of the transferred Company are the higher. In this form, the expression is perfectly general, provided we understand by  $V_{x|n}$  the value of a policy taken out at age  $x$ , and  $n$  years in force, according to the method of valuation employed by the Company. If the policies are to have an agreed rate of bonus, the value of this bonus is to be added to that of the sum assured, and the policy then treated as a nonparticipating policy.

The arguments applied in the case of the non-participating policies will not now hold. Indeed, the conditions which affect the two classes of policyholders are widely different. The participating policyholders are to be treated precisely as those of the purchasing Company, the same reserve must be made for future expenses and profits in the two cases, and of course the liabilities must be valued at the same rate of interest and by the same table of mortality in the two cases.

The values of both participating and non-participating policies found by the above described methods will in many cases be widely different from those found by the net-premium method, which is thus proved, as already stated, to be quite inapplicable; and it is not difficult to see that it will always be inapplicable in estimating the terms on which the business of one Office may be transferred to another.

In America the net-premium method of valuation is proving to be inapplicable for a different reason. The members of the Institute are well aware that by the laws of the State of Massachusetts a valuation is made yearly by the Insurance Commissioner of the liabilities of all life insurance companies doing business in the State. The basis of this valuation has hitherto been the "Experience" Table of Mortality with 4 percent interest; but with the increasing competition and the tendency of newly formed Companies to reduce as far as possible the premiums, it is found that the premiums charged by some Companies very little exceed, or even fall short of, the net premium used in the valuation. In this latter case, it is a manifest absurdity to give a Company credit in the valuation for a net premium actually greater than that it will receive; and in the former case, it is little less absurd to give it credit for a net premium which shows a margin quite insufficient to defray the necessary expenses of management.

Mr. Barnes, Superintendent of the New York Insurance Department, thus expresses himself in his Sixth Annual Report, published in 1865, p. xcvi:—"The tendency of some companies to reduce

"the rates of premium will probably sooner or later render this method (the net-premium method) ineffectual and unscientific for the reason that it is based on an *assumed* net premium, which is in effect credited alike to all companies, although their actual rates may be above or below this standard, and may differ ten, fifteen, twenty or any other percentage. \* \* \* Valuations for Governmental purposes will however ultimately have to be made upon the *actual premiums* receivable by the different companies, deducting an allowance for loading, as may be provided by law." How these sound and moderate views are to be reconciled with the claptrap statement on the next page, that "it may now be considered as an established and axiomatic truth of the science of Life insurance that the loading on premium should never be credited or allowed as realized assets," I will not undertake to explain.

A third case in which the net-premium method of valuation appears inapplicable is the unfortunately too common case in which a Company has been at great expense to procure new business, and has spent much more than the loading of the premiums. The expenditure may have been wise and justifiable; and the actuaries who most strongly insist upon the net-premium method of valuation being always followed, may be under the necessity of admitting, when pressed upon the point, that the Company is not only perfectly solvent and able to meet its liabilities, but that it possesses a valuable business, which may be expected to produce a considerable profit in the future. But the net-premium method of valuation takes no cognizance of these circumstances. It gives the Company no credit for the wise outlay it has made, nor for the valuable business it has actually obtained; but crediting it with the net premium only, brings out a deficiency. No further argument is required to prove that in this case, too, the net-premium method of valuation is wholly inapplicable.

The question as to the proper reserve to be made in this case is one of the greatest possible practical importance. The expenses of life insurance companies appear to be decidedly on the increase, and it must be a rare circumstance for a new Company starting at the present time to be able to limit its expenditure to the margin of the premiums received during the first years of its existence. It appears to me, therefore, essential to devise some rule that shall lead to more reasonable results in this case. If actuaries of standing and repute insist upon a rigid adherence in all cases to a

rule which is only applicable to certain cases, the consequence must be that the managers of the Company thus unfairly treated will have recourse to the services of pretenders, and incompetent persons, who, from ignorance or worse motives, will bring out any result that is desired. It appears to me that true wisdom consists in facing the question fairly and investigating a rule for determining what expenses a Company may incur without becoming insolvent. This will enable us to decide as to the proper answers to the further questions, When is a life insurance company to be considered insolvent? and how are its affairs to be administered in that case? The really scientific actuary must be prepared to deal with each case according to its own merits, and not insist, like a quack doctor, upon applying the same specific in all cases indiscriminately.

In Germany this point of the inapplicability of the ordinary net-premium method of valuation appears to have attracted considerable attention. It is there a very common practice to pay the agent a high rate of commission upon the completion of a policy, say 1 percent upon the sum assured, the premiums in future years being free from commission, or only subject to a very small deduction to defray the bare cost of collection and remittance. A very little calculation is sufficient to show that after allowing for this commuted commission, and the other expenses attendant upon the issue of the policy, also for the current risk of the Office during the first year of the insurance, there will be very little of the first year's premium remaining at the younger ages, and at all ages considerably less than the policy-values required by the net-premium method of valuation.

The same remarks apply when the ordinary 5 percent commission is commuted for a single payment of, say, 50 or 60 percent of the first year's premium, as, I believe, is the case in America. The net-premium method of valuation takes no account of the payments that may have been made; and thus lands us in the absurdity of requiring that the same reserve should be made for policies on which the commission has been commuted, as for those which are still subject to the payment of 5 percent commission. An attempt has been made to meet this difficulty by entering in the accounts as an asset the sum paid for the commutation of future commissions. This, under proper restrictions, appears to be quite allowable, although open to some objection; but the attempt on the part of some of the American Companies to adopt this course, has been discouraged by the

Insurance Commissioners of Massachusetts and New York, who unhesitatingly strike out all such items from the list of assets. In following this course, and yet requiring the full net-premium reserve as given by the Tables prescribed by law, it appears to me that they commit a great injustice towards the Companies. The Massachusetts Commissioner thus states (13th Report, pp. lxxx., lxxxi.) his reasons for rejecting the item "commuted commissions" as an asset against the net-premium reserve:—"Commissions form an important part of the expenses which the margin or loading was intended to provide for, and whether paid in advance, or from year to year, the money paid on account of them is an expenditure, and not an investment. \* \* \* \* It may be very prudent for a Company which is under such a liability to get rid of it. But discharging a liability is not an asset. We find no difficulty, therefore, in concurrence with the able and efficient Superintendent of Insurance in New York in rejecting 'commuted commissions' as unrealized assets. We go still further, and pronounce them *unreal* assets. The question should be, What assets has the Company *now* to satisfy the required reserve?—and not what it may have some years hence, provided it should receive these margins discharged of the commission, and should not spend them for something else."

To return to Germany. Some of the points here considered have been discussed at great length by Dr. Zillmer, the President of the German Life Insurance Institute, in a very able work entitled *Beiträge zur Theorie der Prämien-Reserve bei Lebens-Versicherungs-Anstalten*, Stettin, 1863. He proposes a method of looking at the question, according to which it may be said that the net premium only is valued, although proper allowance is made for the effect of the payment of the commuted commission. He argues that it is not essential that the net premium should be constant throughout the duration of the policy, but it may be so fixed as to have a minimum value the first year and a correspondingly larger value for the remainder of life. Thus, supposing the commuted commission to be at the rate of 1 for each 100 assured, let the net premium of the first year for an insurance of 1 be less by .01 than the net premium for the remainder of life, which we suppose to be equal to P; then we have for the determination of P the following equation

$$P - .01 + Pa_x = (1 + a_x)\varpi_x$$

or

$$P(1 + a_x) = .01 + (1 + a_x)\varpi_x$$

whence

$$\begin{aligned}
 P &= \varpi_x + \frac{\cdot 01}{1 + a_x} \\
 &= \varpi_x + \cdot 01(\varpi_x + d) \\
 &= 1\cdot 01\varpi_x + \cdot 01d.
 \end{aligned}$$

The reserve to be made for the policy after  $n$  years will then be  $A_{x+n} - (1 + a_{x+n})P$ . Dr. Zillmer points out that the reserve thus made, altho' less than that given by the ordinary method, fulfils the same conditions:—*Firstly*, that it is formed from the modified net premiums actually received, by deducting the claims that will occur according to the rate of mortality assumed, and adding interest at the rate assumed in the calculation of the premiums; *Secondly*, that, together with the net premiums to be paid in the future, allowing interest at the tabular rate, it will exactly suffice to meet the engagements of the Company. He further points out in the course of his work the importance of avoiding the bringing out of a negative value for any policy, and the precautions that must be taken to prevent this.

In the cases here considered, where a liability has been extinguished, the propriety of making an allowance for the outlay can scarcely be questioned; but it appears to me that the case is the same in principle when the heavy outlay in the first year has been incurred for other purposes, and that in all cases when a large outlay is made in the first year, allowance should be made for it in estimating the values of the policies.

In valuing by the net-premium method, the supposition is tacitly made that the expenses chargeable to a policy are equally spread over its existence. But this is very far indeed from the truth, as the expenses incurred in the first year of the policy's existence far exceed those of subsequent years. Indeed, by far the greatest part of the expenses of every Life Insurance Company are chargeable to the new business—in particular, all the advertising, inspection of agencies, and medical fees, and a considerable proportion of the postage, directors' fees, salaries, and office expenses. For if a Company decided, as some have done, on taking no new business, and closed its doors with the intention of working out the existing business as economically as possible, it is obvious that the expenses, exclusive of the commission, might be reduced to a very trifling sum. This reduced sum, or at all events such a sum as an established Company would undertake to conduct the business for, is all that should be charged to the old business.

If we keep these considerations steadily in view, we shall see

that the only case in which the net-premium method of valuation can be expected to lead to satisfactory results is when, from the magnitude of the business transacted or other causes, the total expense of conducting the business, irrespective of commission, amounts only to a small percentage of the premiums. In all other cases, we must seek for some other rule. In other words, in most cases the deduction of the original loading from the gross premium does not give satisfactory results, and we have to investigate a rule for determining what deduction ought to be made.

As a first step towards this, we may say without hesitation that any commission payable should be deducted, so that we credit the Office, not with the gross premium, but with the actual premium receivable. Then, what further deduction should be made? In considering this point we very soon arrive at one inflexible rule, perhaps the only inflexible rule that is applicable to the subject. In the case of recent policies, if no deduction beyond the commission, or only a small one, is made, we shall find on valuing by a true table of mortality that the value of the future premiums is greater than that of the sum assured, so that in technical language the value of the policy is negative. If the figures thus resulting are entered in the account, then the policy counts as an asset instead of a liability. The effect is to increase the assets of the Company by a quantity that is purely fictitious, or at all events, delusive. It is no doubt quite true that in certain cases the present value of the future premiums is greater than that of the sum assured. But there is no certainty that those premiums will ever be received, for the policyholder is under no obligation to keep his policy in force; and if credit is taken for the policy as an asset, then the nominal assets of the Company are liable to be reduced by the dropping of the policy. The error of taking credit for policies of this description is of exactly the same kind as if a trader should in taking stock value all his stock-in-trade at the retail selling price instead of at the cost price. It appears then that in no case is it permissible so to value the liabilities of a Life Insurance Company as to lead to negative policy-values. Any method of proceeding which has this effect, or exhibits any policies as assets instead of liabilities, is quite inadmissible and cannot be too strongly condemned.

Provided that this cardinal error is avoided, it appears to me that the question of the amount to be deducted from the gross premium is one that must be answered according to the special

circumstances of each case. Thus, if a Company has from any cause incurred a large expenditure for its new business, it ought not to be required to have so large an amount in hand as another Company which has acquired its business at less expense; and conversely, a Company which has acquired its new business at a comparatively very trifling cost cannot be held justified in making so small a reserve for its liabilities as may fairly suffice for its neighbour that has been put to greater expense to obtain business. It has indeed been argued by some authorities, that an actuary should state the value of the liability of a Life Office without any reference, either to the amount of its expenses, or its assets; but this course appears to me both unscientific and unpractical. It is neither reasonable nor just to lay down one and the same law for regulating the reserve of all Life Insurance Companies, the circumstances of which differ very widely from each other.

The object of the actuary in making a valuation should be to equalize the profits of the Company; in other words, to spread the profit derivable from each policy or class of policies as far as possible uniformly over their continuance. The larger the expense at which the new business is obtained, the less will be the profit that is to be looked for during the continuance of the policies; but the actuary should aim at dividing this smaller profit uniformly over the currency of the policies. When the expenses of obtaining business have exceeded the loading on the premiums, the effect of valuing by the net-premium method is to represent each new policy effected as causing a loss to the Office, which, as already pointed out, is an absurd result; and to postpone for several years the period when any profit is considered as derived from the policy. This is on the supposition that the policy is continued in force. If, however, the policy should lapse after a year or two, then the net-premium method leads to the further absurdity of representing that a profit has accrued to the Office by the loss of business that was obtained at great expense, and which it would have been far more to its interest to retain.

The consistent advocates of the net-premium method of valuation are compelled to urge that under no circumstances should the expenses exceed the loadings of the premiums received; but their arguments appear to me quite inconclusive. They argue expressly, or by implication, that the full net premiums are required for the payment of the sum assured, and that they should be scrupulously accumulated and reserved for this sole purpose. But it is obviously unnecessary that the sum annually

invested out of the year's premium and set apart for the purpose of forming a fund for payment of the sums assured, should be the same every year; and it is enough that the present value of all the sums so set apart should be equal to the present value of the net premiums. The Office premium being constant, while the expenses are greatest in the first year, the proper plan obviously is to set apart a small sum out of the first year's premium, and larger sums afterwards, which together fulfil the above condition. We thus see that it is by no means necessary that the current expenses should be less than the current loadings.

When the expenses have exceeded the current loadings, it has been a very common course to enter in the accounts a portion of the heavy outlay incurred, under some such name as "preliminary expenses," "extension of agencies," "purchase of business," "commuted commissions," etc. etc.; and this appears to be the proper place to consider in what light such items ought to be regarded. When such an item is inserted in the account, the tacit supposition is made that the sum in question has not been spent and lost to the Company, but that a certain portion, either of the Proprietors', or the Assurance Fund, has been invested in the purchase of business of at all events an equal value. Now when we are considering the value of an asset entered under one of the above headings in the balance sheet, it is not enough to be satisfied that so much money has actually been spent, in the hope of obtaining business that will repay the outlay. We must rather look at the business that has been actually obtained. The money may have been spent, but it may not have produced any adequate return; and in considering how far such an asset may properly be reckoned good, we must not make allowance for business that may possibly be obtained in the future, but must look exclusively to the results already obtained. This being premised, we have next to consider whether the excess of the value of the future premiums over that of the sums assured is greater than the amount entered in the balance sheet as preliminary expenses, etc., and if this is the case, it may at first sight appear that this amount can always be considered as a good asset. But there is another consideration to be borne in mind; for it is clear that such an asset can only be considered good so far as it is well secured; or in other words, the fund formed by the excess of the value of the future premiums over that of the sums assured, or to put it rather differently, by the future loadings, can only be considered available as against present outlay so far as it is well secured. How far then is this the case?

The policyholder being under no obligation to keep up his policy, we can only look to that he has already paid; but in the case of an ordinary whole life policy, the policyholder pays a much larger sum than is required to meet the current risk, and the excess may be considered as standing to his credit in the Company's books, and liable to be forfeited if he discontinue his policy. This sum in deposit, then, virtually forms a security for the payment of the future loadings; and to the extent of this sum, we may consider the future loading well secured. Provided, therefore, the sum expended on the credit of the future loadings does not exceed this sum in deposit, we may fairly set the one off against the other. We thus arrive at the conclusion that items among the assets of the above-mentioned kinds are justifiable only so far as their amount falls short of the present value of the future loadings on the one hand, and of the value of the policies on the other hand. This then is the proper way in which all such assets should be regarded. As far as they are justifiable they represent the present value of future loadings on existing policies; and although it is true those loadings may never be received, yet they may be considered as a good asset so far as secured by the sums in deposit to the credit of the policyholders. It follows from these considerations that when the balance-sheet contains such an item as one of those above specified, it will be wrong to enter in it the value of the gross premiums receivable; for in that way credit would be virtually taken twice over for the value of the future loadings. It is further easy to see that in all cases where such an item does appear among the assets, it is essential that there should yearly be written off from it a certain amount, as the loading, the value of which it represents, is actually received on the one hand, or lost to the Company on the other hand by the policy being dropt or surrendered. When such an item is thus carefully limited in amount and reduced properly year by year, its insertion in the balance-sheet appears perfectly defensible. The account should then be arranged as follows:—

ASSETS.		LIABILITIES.	
Mortgages, &c. ....	A	Claims admitted, &c. ....	F
Preliminary expenses ....	B	Value of sums assured ....	G
Value of future premiums ....	C	Balance .....	H
Less deduction for expenses ..	D		
	— E		
	<u>T</u>		<u>T</u>

where  $E = C - D$ , and  $T = A + B + E = F + G + H$ . Then a comparison of

D with C shows what proportion of the Office premiums is reserved as loading, and a comparison of B with D shows what part of that loading has been already expended.

But it is both simpler and less open to objection, as we shall presently see, to arrange the account differently—to omit altogether from the account on the one hand any such item as “preliminary expenses,” and on the other hand to diminish D, the deduction from the Office premiums, to the same extent.

Bearing in mind now the principle established above, that the values of policies should never be negative, let us examine within what limits the deduction from the gross premium should fall. Let  $P_x$  be the actual premium receivable after deduction of any commission payable; and suppose it be just due. Then the difference between the value of the sum assured and that of the reduced Office premiums is

$$A_{x+n} - (1 + a_{x+n})P_x = (1 + a_{x+n})(\varpi_{x+n} - P_x).$$

If  $\varpi_{x+n}$  is less than  $P_x$ , this value is negative, which shows that the deduction from the gross premium must be certainly not less than  $P_x - \varpi_{x+n}$ , in addition to the commission. If we have a table of the values of  $\varpi_x$ , simple inspection informs us when  $x$  is given for what values of  $n$ ,  $\varpi_{x+n}$  is less than  $P_x$ . If  $\varpi_{x+n}$  is greater than  $P_x$ , then the above expression is always positive, and it is not necessary, so far as avoiding negative values is concerned, to make any deduction from  $P_x$ , although of course it is necessary to do so in order to provide for the future working expenses.

If the expenditure in obtaining new business has been such that the current claims and the expenses have together absorbed the whole of the first year's premiums, then a simple method of valuation, which in my opinion is perfectly justifiable, will be to reserve for policies in their first year only sufficient to meet the unexpired current risk, and to consider that all other policies have been effected at the next higher age. The value will thus be

$$A_{x+n} - (1 + a_{x+n})\varpi_{x+1} = (1 + a_{x+n})(\varpi_{x+n} - \varpi_{x+1}).$$

In this case the deduction from the gross premium will be (commission +  $P_x - \varpi_{x+1}$ );  $P_x$  being as before the premium after deduction of commission. If the expenses attendant on procuring new business have been still heavier, so that together with the current risk they have absorbed the whole of the premiums for the two first years, then for policies of less than two years' standing a reserve may be made for the unexpired current risk, and for

policies of more than two years' standing a reserve equal to

$$A_{x+n} - (1 + a_{x+n})\varpi_{x+2} = (1 + a_{x+n})(\varpi_{x+n} - \varpi_{x+2}).$$

In this case, the deduction from the Office premium is (comm. +  $P_x - \varpi_{x+2}$ ).

The same course is applicable in the extreme case where the expenses and current risk absorb the premiums for the first  $t$  years; the reserve for policies of less than  $t$  years' standing being simply the unexpired current risk, and that for policies of greater standing

$$A_{x+n} - (1 + a_{x+n})\varpi_{x+t} = (1 + a_{x+n})(\varpi_{x+n} - \varpi_{x+t}).$$

Here the deduction from the gross premium is (comm. +  $P_x - \varpi_{x+t}$ ), which shows that the process is only possible so long as  $t$  is so small that  $\varpi_{x+t} < P_x$ . This is determined by simple inspection of a table of the values of  $\varpi_x$ . But  $t$  cannot in practice have so large a value; for some reserve, however small, must be made for the future expenses of conducting the business, even if the greatest part of the expenses is considered chargeable to the new business. In all these cases it will be noted the policies never have a negative value.

When this course is pursued, we are at once reminded that upon all subsequent occasions the policy must reckon as if taken out at the age  $x+t$ . Thus, after the lapse of  $t+n$  years, the policy must rank for the purpose, either of surrender, or of distribution of bonus, as if it had been taken out at the age  $x+t$ , and  $n$  premiums only had been paid upon it. As enabling us to bear this in mind constantly, as well as for the reason pointed out previously, the above method is greatly superior to the common one of entering in the assets of the Office a lump sum as preliminary expenses, &c., &c. An asset of that description may possibly represent nothing but an injudicious expenditure for which no return of any value can reasonably be anticipated. An outlay made by a Life Insurance Company for the purpose of obtaining business ought to produce an early return; and if it does not within at most two or three years produce an amount of business sufficient to reimburse the Office, it cannot be considered money well spent.

Of course, if the Society is under any contract, express or implied, to allow a surrender value of a certain magnitude for its policies, it is essential that it should reserve at least this amount to meet its liabilities under the policies: and this, again, renders it necessary that the expenses of the early years should not exceed a certain amount.

While on this topic, it will not be out of place to refer to the

inconsistent course followed by many Offices that issue policies on the half-credit scale of premiums. They virtually allow for the surrender of such policies half the amount of the premiums paid at any time during the first five years of the existence of the policy: whereas if the holder of an ordinary policy applies for the surrender value, he is allowed a very much smaller value, perhaps 30 percent of the premiums paid, instead of 50 percent; and that only if the policy has been in force, for perhaps five years. As regards consistency, it is a far preferable plan to allow one third of the premiums to remain on credit; and in that case, there seems very little objection to extending the credit to the whole of life-as is done by some Offices.

The preceding remarks refer principally to non-participating policies; and in valuing policies which participate in the profits we must further bear in mind that in general a special reserve must be made to provide for the bonuses to be declared in the future. In other words, a deduction must be made from the gross premium beyond what is necessary for the future expenses; and it seems fair to fix the minimum amount of this deduction as equal to the sum that may be considered to have been added specially to provide for the bonuses, viz. the difference between the participating and non-participating premiums, so that we shall not, in consequence of the participating premiums being greater than the non-participating, make a less reserve for a participating policy of a given standing on a life of a given age, than for a similarly circumstanced non-participating policy; but shall make at all events as large a reserve for the former as for the latter.

There is, however, a fundamental distinction to be borne in mind in making a valuation of participating policies for the purpose of a division of profits, according to the mode in which the profits of the Office are to be declared. If it is the practice of the Office to declare an unconditional cash bonus, the greatest care must be taken that no profit is divided that has not actually been realized out of the profits of the past; but this is not the case if the bonus declared is only conditional on the policy being kept in force for a certain time. For example, if the bonus is given by way of reduction of the premium next falling due, the assured will not have the benefit of it unless he pays another premium. Again, if the bonus be applied to extinguish the premiums after a certain age, or to make the sum assured payable on the life attaining a certain age, then all the holders of policies in which the lives drop previously to those ages will obtain no

actual benefit from the bonus. In these and similar cases it is by no means essential that the bonus declared should be provided for out of the realized profits of the past. Indeed, justice requires that the profits of the future should also be taken into account. Even in the case of the bonus being given by way of reversionary addition to the sum assured, the same will hold good provided that the assured has no advantage from the bonus except in the case of the policy becoming a claim, so that, for example, in case the policy is surrendered no allowance is made in the surrender value for the bonus.

Let us next consider the case of a well established and prosperous Office, which spends annually only a small fraction of its loadings in obtaining new business, so that there is no inducement to anticipate any part of its future profits: what deduction should it make from its gross premiums in valuing its policies? In order to fix our ideas, let us suppose that at the time of valuation there are several persons of the same present age paying the same annual premium to the Office, but insured at different ages, and consequently for different sums. Suppose the common present age to be 60, and that persons insured at the ages 20, 25, 30, 35, 40, 45, 50, 55, all pay to the Office the same premium, viz. £100, their several sums assured being those shown in the following table.

Age at Entry.	Sums Assured.	Office Premium.	EXPERIENCE.		CARLISLE.	
			Net Premium.	Loading.	Net Premium.	Loading.
20	£ 5,634	£ 100	82·99	17·01	84·17	15·83
25	4,980	100	82·92	17·08	84·81	15·19
30	4,388	100	83·59	16·41	85·65	14·35
35	3,827	100	84·57	15·43	85·46	14·54
40	3,306	100	86·12	13·88	85·92	14·08
45	2,817	100	88·26	11·74	85·02	14·98
50	2,339	100	89·70	10·30	84·72	15·28
55	1,884	100	89·81	10·19	85·63	14·37
	29,175	800	687·96	112·04	681·38	118·62
60	1,485	100	89·47	10·53	85·99	14·01

The premiums used in the calculations are the “without-profit” rates of a well established proprietary Office.

Then if we value by the net-premium method, using the Experience Table, the effect is that from the £100 premium paid by the person who insured at the age of 20, we throw off £17·01

as loading; while from the £100 paid by the person who insured at the age of 55 we throw off only £10·19. These several deductions are regulated by the age at entry; but this does not appear to be altogether satisfactory. Supposing the present age known, together with the sum assured and the premium actually payable, it is immaterial what was the age at entry. In the above instance, the sum assured being £3,827, the premium £100, and the present age 60, the same reserve should surely be made for the policy whether the age at entry was 35, as supposed; or whether on the other hand, either a much higher scale of premiums being charged the age at entry had been, say, 30; or a much lower scale of premiums being charged, the age at entry was, say, 40. This argument is obviously unanswerable when the case is that of the transfer of non-participating policies from one Office to another. When however the valuation is required for the purpose of division of profits in the Company which originally granted the policies, the only valid objection that can be urged against it is that by the net-premium method of valuation the portion of loading applicable for profits yearly is a constant quantity, whereas the adoption of another method might lead to irregularities in that respect. But it may very easily happen that other considerations may overrule this objection.

When there are given, as above, the present age, the sum assured, and the Office premium, there are two methods of fixing the deduction to be made which immediately present themselves. We may fix our attention principally on the premium paid, and consider what percentage of it should be treated as margin. Supposing we are valuing by the Carlisle table, we may say that if a person of the age 60 now insures his life at the tabular rate, out of every £100 he pays we consider £14·01 as margin, and consequently we will deduct 14·01 percent from the premiums paid by all persons of the present age 60. But on examination it will be found that this plan is not to be recommended. It appropriates a certain percentage of the premiums—whether high or low—for the expenses; but all the expenses, except commission, chargeable to a policy of a given amount are the same whether the premiums are high or low, and it thus appears that the policy on which a low premium is paid, is not charged with a sufficient contribution to the expenses under this plan. Taking the extreme case of a policy on which the premiums have been fully paid up, the above plan will make no reserve for the expenses attaching to such a policy.

The other plan which suggests itself is that on every policy of

a given amount on a life now 60, there shall be made the same actual charge for expenses of management as is made on a policy of the same amount now taken out on a life of 60. In the above illustration, we assume that a person of the age of 60 now assuring his life would have to pay £6·73 per annum for every £100 assured. From this may properly be deducted the commission of 5 percent, leaving £6·39, of which, if we value by the Carlisle table, we consider £5·66 to be the net premium and ·73 the loading. This same sum then, we may say, should be charged on each £100 assured on a life now aged 60, whatever the age at entry. In our illustration, this amounts to  $£29,175 \times \cdot 73 = £213$ , and adding the commission of 5 percent, the total deduction from the annual Office premiums of £800 is £253, being more than double what is made by the net premium method, whether the Carlisle or Experience table is used. This result is caused partly by the high premium charged at 60, and partly by the sums assured at the lower ages being so much larger than at the middle and higher ages, a condition of things that will not prevail in practice. In this particular case the above result clearly shows that the method of valuation now suggested could not be employed. If this method could be adopted, the process of valuation would become very much simpler, inasmuch as it would be unnecessary to tabulate the net premium, or the loading, for each separate insurance. All policies relating to lives of the same present age would be grouped together, as is commonly done at present; the grouping exhibiting the sum assured, the bonus, if any, and the Office premium; and in the valuation there would be deducted from this Office premium, (1) the commission of 5 percent and (2) a sum proportional to the sum assured, determined for each age by the premium charged for that age.

This method of valuation may be properly described by the term "reinsurance method" which Mr. Tucker has applied to his method, as it appears to me with less propriety. The reserve made for each policy would be the single premium for which a person of a given present age might be admitted to pay the premium for any assigned lower age. This is exactly what Mr. Tucker claims for his method; but it is obvious that the single premium assigned by his method, being the product of the difference of the premiums at the two ages by an artificial annuity, which is less than the real annuity, is not the real equivalent of that difference. The sum given by the method now proposed however is the exact equivalent.

Assuming the premiums to be formed by adding a fixed sum to the net premium at each age according to the table used in the valuation, and  $\frac{1}{10}$ th for commission, it is obvious that the method now proposed will give precisely the same reserve as the net-premium method. This method of loading the premiums has been recommended and adopted by Mr. Jellicoe, and it appears to me to be in some respects greatly preferable to the more usual one of adding a percentage to the net-premiums.

The method now proposed would in other respects simplify and improve the calculations for a bonus valuation. In the case of premiums which have been fully paid up, either in a single payment or in a limited number of payments, a little consideration shows that whereas the loading intended to provide for the expenses and profits through the whole currency of the policy has been received, a certain portion of the sum so received must be set aside to provide for the future expenses and bonuses. The method now proposed exactly provides for this case. The actual premium payable having been valued whether payable for life or for a term of years, a further reserve is made of the value of the loading corresponding to the present age, which may either be considered as a deduction from the premiums payable or as an addition to the value of the sum assured. This principle will meet the case of insurances of every class. It will also meet the case of premiums payable half-yearly or quarterly. In this case, the value of the policy is less than when the premiums are paid yearly, and in order to get the correct value we have only to value the premiums actually payable, and to reserve the same sum for the future expenses, &c., as in the case of annual premiums.

The same principle applies to the case of policies effected with a limited number of payments, when the stipulated number of premiums has not been fully paid up. In this, as in the former cases, the premium actually payable is to be valued and the deduction above mentioned to be made for the future expenses.

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*A Table for determining the Amounts, &c., of Continuous Annuities Certain. By WILLIAM MATTHEW MAKEHAM, Fellow of the Institute of Actuaries.*

THE annexed Table contains the values of the function  $\log \frac{e^x - 1}{x}$  for all values of  $x$  differing by  $\cdot 01$ , from 0 to 10.4.

*Mode of using the Table.*

To find the value of the function for any intermediate value of  $x$ , enter the table with the value of  $x$  nearest to the given value, whether greater or less, and take out the quantities  $\Delta$  and  $\Delta'$  opposite the interval to which the given value belongs. Then,  $x'$  denoting the given value and  $x$  the nearest tabular value, the difference to be added to the value of the function corresponding to  $x$ , in order to give the value of the function corresponding to  $x'$ , will be

$$100(x' - x)(\Delta \pm \Delta')$$

the positive sign to be taken when  $x' - x$  is positive and the negative sign when negative.

[*Note.*—To avoid ciphers  $\Delta$  and  $\Delta'$  are given as whole numbers, —hence, in the result, the decimal place must be removed seven places to the left.]

*Example 1.*—Required the value of  $\log \frac{\epsilon^x - 1}{x}$  when  $x = .603$ .

Here the tabular value of  $x$  nearest to  $.603$  is  $.60$ , and  $100(x' - x) = .3$  a positive quantity

$$\Delta + \Delta' = 23882 \quad \text{and multiplying by } .3$$

$$\hline + 7165$$

which added to  $\cdot 1367833$  the value corresponding  $x = .60$ .

gives  $\cdot 1374998$  the result required.

*Example 2.*—Required the value of  $\log \frac{\epsilon^x - 1}{x}$  when  $x = 2.686$ .

In this case  $x' - x = 2.686 - 2.69 = -.004$  a negative quantity. Hence

$$\Delta - \Delta' = 30441 \quad \text{and multiplying by } -.4$$

$$\hline - 12176$$

which added to  $\cdot 7079713$  the value corresponding to  $x = 2.69$

gives  $\cdot 7067537$  the result required.

To find the value of the argument ( $x'$ ) for any intermediate value of the function, ascertain by inspection the argument corresponding to the tabular value of the function nearest to the given value, whether greater or less, and take out  $\Delta$  and  $\Delta'$  corresponding to the interval to which the given value of  $x$  belongs. Then the quantity to be added to the argument corresponding to the nearest

tabular value of the function, in order to give the argument required, is

$$\frac{\text{Given value of function—nearest tabular value}}{100(\Delta \pm \Delta')}$$

the positive sign to be taken when the numerator is positive and the negative sign when negative.

*Example 1.*—Required the value of  $x$  corresponding to  $\log \frac{\epsilon^x - 1}{x} = 1.8970000$ .

The nearest tabular value of the function is 1.8957011, corresponding to  $x=6.19$ . Hence we have

$$\frac{8970000 - 8957011}{36508 - 3} = \frac{12989}{36505} = .3558$$

and dividing by 100 we have  $6.19 + .003558 = 6.193558$  for the argument required.

*Example 2.*—Required the value of  $x$  in the equation  $\log \frac{\epsilon^x - 1}{x} = .7280000$ .

$$\frac{7280000 - 7293480}{30616 + 6} = -\frac{13480}{30622} = -.4402$$

Consequently the required argument is  $2.76 - .004402 = 2.755598$ .

[*Note.*—The numerator of the fraction should never exceed one-half of the denominator; should it do so in any case (which may possibly happen if the interval is nearly bisected) the tabular value of the fraction on the other side of the given value must be taken.]

#### *Application of the Table.*

To find the logarithm of the amount of a continuous annuity certain for  $t$  years, at a force of interest of  $r$  per £1.

The amount of the annuity in question is  $\frac{\epsilon^{rt} - 1}{r}$ . (See *Baily "On Interest and Annuities,"* p. 47.) Dividing and multiplying by  $t$  and taking the logarithms we have

$$\log \frac{\epsilon^{rt} - 1}{rt} + \log t$$

*Example.*—It is required to determine the amount of a continuous annuity of £70. 13s. 6d. for 10 years, at a force of interest of  $4\frac{1}{2}$  per cent.

$$\begin{aligned}
 \log \frac{e^{.45} - 1}{.45} &= .1013745 \\
 \log 10 &= 1. \\
 \log 70.675 &= 1.8492658 \\
 \hline
 &2.9506403
 \end{aligned}$$

Hence it appears that the required answer is £892.566.

As an example of the inverse use of the Table, let it be required to determine the force of interest in a continuous annuity of £1, the amount of which in 101 years is  $\log^{-1} 3.4809745$ .

$$\frac{3.4809745 - 3.4799936}{39237 - 1} = \frac{9809}{39236} = .25$$

Hence  $rt = 10.3525$  and the force of interest is  $\frac{10.3525}{101} = .1025$ , or  $10\frac{1}{4}$  per cent.

In the absence of a Table of the present values of continuous annuities  $\left(\frac{1 - e^{-rt}}{r}\right)$  the accompanying Table may be used for determining such values, for  $\frac{1 - e^{-rt}}{r} = \frac{1 - e^{-rt}}{rt} \times t = \frac{e^{rt} - 1}{rt} \times t \times e^{-rt}$ , and therefore the logarithm of the present value of a continuous annuity certain is

$$\log \frac{e^{rt} - 1}{rt} + \log t - \log e \times rt \quad (\log e = .43429448).$$

By means of the Table we may also determine the amounts of annuities payable at any given interval of time. Thus the amount of an annuity for  $t$  years payable at intervals of  $n$  years  $\left(\frac{t}{n}\right.$  being a whole number) is  $\frac{e^{rt} - 1}{e^n - 1} \times n = \frac{e^{rt} - 1}{rt} + \frac{e^{rn} - 1}{rn} \times t$ . Hence the logarithm of the amount of the annuity described is

$$\log \frac{e^{rt} - 1}{rt} - \log \frac{e^{rn} - 1}{rn} + \log t.$$

*Example.*—Required the amount of an annuity of £9. 3s. 6d. (payable half-yearly) for 25 years, at a force of interest of 10 per cent.

$$\log \frac{e^{2.50} - 1}{2.50} = .6505987$$

$$\begin{aligned}
 -\log \frac{\epsilon^{.05} - 1}{.05} &= 1.9890974 \\
 \log t &= 1.3979400 \\
 \log 9.175 &= .9626061 \\
 \hline
 &3.0002422
 \end{aligned}$$

Whence it appears that the amount of the given annuity is £1000.558.

### Concluding Observations.

It thus appears that a single Table of the values of the function  $\log \frac{\epsilon^x - 1}{x}$  for all values of  $x$  positive and negative will afford the means of obtaining directly the logarithms of the amounts and present values of annuities certain for any given term, whole or fractional, for any given force of interest, and payable at any given intervals of time. This comprehensive property is analogous to that attaching to the formula which I have suggested for the construction of Life Annuity Tables. For by the reduction of the integral  $\int b^x \cdot s^x \cdot dx$  to  $\int \epsilon^{-v} \cdot v^{n-1} \cdot dv$ , given in my paper "On the Law of Mortality," (see vol. xiii., p. 349), it appears that a single Table of the integral  $\int_x^\infty \epsilon^{-v} \cdot v^{n-1} \cdot dv$  for successive values of  $n$  and  $x$  will afford the means of obtaining directly the value of an annuity on any life or combination of lives at any required rate of interest, and by *any Table of Mortality whatever* that can be represented in the form  $L_x = ab^x s^x$ —thus obviating the necessity of constructing special Tables of Annuities for different rates of interest and mortality.

The method given for interpolating intermediate values of the tabular function is derived from the formula  $y_n = y_0 + n \left( \Delta y_0 - \frac{\Delta^2 y_0}{4} \right)$ , the value of  $n$  being confined between the limits  $-\frac{1}{2}$  and  $+\frac{1}{2}$ . The usual formula is  $y_n = y_0 + n \left( \Delta y_0 - \frac{1-n}{2} \Delta^2 y_0 \right)$  which coincides with the preceding when  $n = \frac{1}{2}$ . For other cases the difference between the two values is  $n \cdot \frac{n-\frac{1}{2}}{2} \cdot \Delta^2 y_0$ , the greatest possible numerical value of which is  $\frac{\Delta^2 y_0}{32}$ . As the second difference of the tabulated

function in the highest case amounts to 36, we have  $\frac{\Delta^2 y_0}{32} = 1.125$ ;

hence the results can always be depended upon within an unit or two in the last decimal place. By the ordinary interpolation with first differences the limit of the error would be  $\frac{36}{8}$ , or just four times as great.

$x$	$\log \frac{x^2-1}{x}$	$\Delta$	$\Delta'$	$x$	$\log \frac{x^2-1}{x}$	$\Delta$	$\Delta'$
.00	0.0000000	+21733	-9	.46	0.1037100	+23392	-9
.01	0.0021733	+21769	-9	.47	0.1060492	+23427	-9
.02	0.0043502	+21805	-9	.48	0.1083919	+23463	-9
.03	0.0065307	+21841	-9	.49	0.1107382	+23499	-9
.04	0.0087148	+21878	-9	.50	0.1130881	+23535	-9
.05	0.0109026	+21914	-9	.51	0.1154416	+23570	-9
.06	0.0130940	+21950	-9	.52	0.1177986	+23606	-9
.07	0.0152890	+21986	-9	.53	0.1201592	+23642	-9
.08	0.0174876	+22022	-9	.54	0.1225234	+23677	-9
.09	0.0196898	+22058	-9	.55	0.1248912	+23713	-9
.10	0.0218957	+22095	-9	.56	0.1272625	+23749	-9
.11	0.0241051	+22131	-9	.57	0.1296374	+23784	-9
.12	0.0263182	+22167	-9	.58	0.1320158	+23820	-9
.13	0.0285349	+22203	-9	.59	0.1343978	+23856	-9
.14	0.0307552	+22239	-9	.60	0.1367833	+23891	-9
.15	0.0329792	+22275	-9	.61	0.1391724	+23927	-9
.16	0.0352067	+22312	-9	.62	0.1415651	+23962	-9
.17	0.0374379	+22348	-9	.63	0.1439613	+23998	-9
.18	0.0396726	+22384	-9	.64	0.1463611	+24033	-9
.19	0.0419110	+22420	-9	.65	0.1487644	+24068	-9
.20	0.0441530	+22456	-9	.66	0.1511712	+24104	-9
.21	0.0463986	+22492	-9	.67	0.1535816	+24139	-9
.22	0.0486479	+22528	-9	.68	0.1559955	+24175	-9
.23	0.0509007	+22564	-9	.69	0.1584130	+24210	-9
.24	0.0531571	+22601	-9	.70	0.1608340	+24245	-9
.25	0.0554172	+22637	-9	.71	0.1632585	+24281	-9
.26	0.0576809	+22673	-9	.72	0.1656866	+24316	-9
.27	0.0599481	+22709	-9	.73	0.1681182	+24351	-9
.28	0.0622190	+22745	-9	.74	0.1705533	+24386	-9
.29	0.0644935	+22781	-9	.75	0.1729919	+24422	-9
.30	0.0667716	+22817	-9	.76	0.1754341	+24457	-9
.31	0.0690532	+22853	-9	.77	0.1778797	+24492	-9
.32	0.0713385	+22889	-9	.78	0.1803289	+24527	-9
.33	0.0736274	+22925	-9	.79	0.1827816	+24562	-9
.34	0.0759199	+22961	-9	.80	0.1852378	+24597	-9
.35	0.0782160	+22997	-9	.81	0.1876975	+24632	-9
.36	0.0805157	+23033	-9	.82	0.1901608	+24667	-9
.37	0.0828189	+23069	-9	.83	0.1926275	+24702	-9
.38	0.0851258	+23105	-9	.84	0.1950977	+24737	-9
.39	0.0874363	+23141	-9	.85	0.1975714	+24772	-9
.40	0.0897503	+23176	-9	.86	0.2000486	+24807	-9
.41	0.0920680	+23212	-9	.87	0.2025293	+24842	-9
.42	0.0943892	+23248	-9	.88	0.2050135	+24877	-9
.43	0.0967141	+23284	-9	.89	0.2075011	+24911	-9
.44	0.0990425	+23320	-9	.90	0.2099923	+24946	-9
.45	0.1013745	+23356	-9	.91	0.2124869	+24981	-9

$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$	$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$
.92	0.2149850			1.50	0.3656987	+26966	-8
.93	0.2174865	+25016	-9	1.51	0.3683954	+26999	-8
.94	0.2199916	+25050	-9	1.52	0.3710952	+27031	-8
.95	0.2225001	+25085	-9	1.53	0.3737984	+27063	-8
.96	0.2250120	+25120	-9	1.54	0.3765047	+27096	-8
.97	0.2275274	+25154	-9	1.55	0.3792143	+27128	-8
.98	0.2300463	+25189	-9	1.56	0.3819271	+27160	-8
.99	0.2325686	+25223	-9	1.57	0.3846431	+27192	-8
1.00	0.2350944	+25258	-9	1.58	0.3873623	+27224	-8
1.01	0.2376236	+25292	-9	1.59	0.3900848	+27256	-8
1.02	0.2401563	+25327	-9	1.60	0.3928104	+27288	-8
1.03	0.2426924	+25361	-9	1.61	0.3955392	+27320	-8
1.04	0.2452319	+25395	-9	1.62	0.3982713	+27352	-8
1.05	0.2477749	+25430	-9	1.63	0.4010065	+27384	-8
1.06	0.2503212	+25464	-9	1.64	0.4037449	+27416	-8
1.07	0.2528711	+25498	-9	1.65	0.4064865	+27448	-8
1.08	0.2554243	+25532	-9	1.66	0.4092312	+27479	-8
1.09	0.2579809	+25567	-9	1.67	0.4119792	+27511	-8
1.10	0.2605410	+25601	-9	1.68	0.4147303	+27543	-8
1.11	0.2631045	+25635	-9	1.69	0.4174845	+27574	-8
1.12	0.2656714	+25669	-8	1.70	0.4202420	+27606	-8
1.13	0.2682417	+25703	-8	1.71	0.4230025	+27637	-8
1.14	0.2708153	+25737	-8	1.72	0.4257662	+27669	-8
1.15	0.2733924	+25771	-8	1.73	0.4285331	+27700	-8
1.16	0.2759729	+25805	-8	1.74	0.4313031	+27731	-8
1.17	0.2785568	+25839	-8	1.75	0.4340762	+27762	-8
1.18	0.2811440	+25872	-8	1.76	0.4368524	+27794	-8
1.19	0.2837346	+25906	-8	1.77	0.4396318	+27825	-8
1.20	0.2863286	+25940	-8	1.78	0.4424143	+27856	-8
1.21	0.2889260	+25974	-8	1.79	0.4451999	+27887	-8
1.22	0.2915267	+26007	-8	1.80	0.4479886	+27918	-8
1.23	0.2941309	+26041	-8	1.81	0.4507804	+27949	-8
1.24	0.2967383	+26075	-8	1.82	0.4535752	+27980	-8
1.25	0.2993491	+26108	-8	1.83	0.4563732	+28011	-8
1.26	0.3019633	+26142	-8	1.84	0.4591743	+28041	-8
1.27	0.3045809	+26175	-8	1.85	0.4619784	+28072	-8
1.28	0.3072017	+26209	-8	1.86	0.4647857	+28103	-8
1.29	0.3098259	+26242	-8	1.87	0.4675960	+28134	-8
1.30	0.3124535	+26276	-8	1.88	0.4704093	+28164	-8
1.31	0.3150844	+26309	-8	1.89	0.4732257	+28195	-8
1.32	0.3177186	+26342	-8	1.90	0.4760452	+28225	-8
1.33	0.3203561	+26375	-8	1.91	0.4788677	+28256	-8
1.34	0.3229970	+26409	-8	1.92	0.4816933	+28286	-8
1.35	0.3256412	+26442	-8	1.93	0.4845219	+28316	-8
1.36	0.3282886	+26475	-8	1.94	0.4873535	+28347	-8
1.37	0.3309394	+26508	-8	1.95	0.4901882	+28377	-8
1.38	0.3335935	+26541	-8	1.96	0.4930259	+28407	-8
1.39	0.3362509	+26574	-8	1.97	0.4958666	+28437	-8
1.40	0.3389116	+26607	-8	1.98	0.4987103	+28467	-7
1.41	0.3415756	+26640	-8	1.99	0.5015570	+28497	-7
1.42	0.3442428	+26673	-8	2.00	0.5044067	+28527	-7
1.43	0.3469134	+26705	-8	2.01	0.5072594	+28557	-7
1.44	0.3495872	+26738	-8	2.02	0.5101151	+28587	-7
1.45	0.3522643	+26771	-8	2.03	0.5129738	+28617	-7
1.46	0.3549447	+26804	-8	2.04	0.5158355	+28646	-7
1.47	0.3576283	+26836	-8	2.05	0.5187001	+28676	-7
1.48	0.3603152	+26869	-8	2.06	0.5215678	+28706	-7
1.49	0.3630053	+26901	-8	2.07	0.5244383	+28735	-7
		+26934	-8				

$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$	$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$
2.08	0.5273119	+28764	-7	2.66	0.6988488	+30382	-7
2.09	0.5301884	+28794	-7	2.67	0.7018870	+30408	-7
2.10	0.5330678	+28824	-7	2.68	0.7049278	+30434	-7
2.11	0.5359502	+28853	-7	2.69	0.7079713	+30460	-6
2.12	0.5388355	+28882	-7	2.70	0.7110173	+30486	-6
2.13	0.5417238	+28912	-7	2.71	0.7140660	+30512	-6
2.14	0.5446149	+28941	-7	2.72	0.7171172	+30538	-6
2.15	0.5475090	+28970	-7	2.73	0.7201710	+30564	-6
2.16	0.5504060	+28999	-7	2.74	0.7232275	+30590	-6
2.17	0.5533059	+29028	-7	2.75	0.7262865	+30616	-6
2.18	0.5562088	+29057	-7	2.76	0.7293480	+30641	-6
2.19	0.5591145	+29086	-7	2.77	0.7324122	+30667	-6
2.20	0.5620231	+29115	-7	2.78	0.7354789	+30693	-6
2.21	0.5649346	+29144	-7	2.79	0.7385481	+30718	-6
2.22	0.5678490	+29173	-7	2.80	0.7416200	+30744	-6
2.23	0.5707662	+29201	-7	2.81	0.7446943	+30769	-6
2.24	0.5736863	+29230	-7	2.82	0.7477712	+30794	-6
2.25	0.5766093	+29258	-7	2.83	0.7508506	+30820	-6
2.26	0.5795352	+29287	-7	2.84	0.7539326	+30845	-6
2.27	0.5824639	+29316	-7	2.85	0.7570171	+30870	-6
2.28	0.5853954	+29344	-7	2.86	0.7601041	+30895	-6
2.29	0.5883298	+29372	-7	2.87	0.7631936	+30920	-6
2.30	0.5912671	+29401	-7	2.88	0.7662856	+30945	-6
2.31	0.5942071	+29429	-7	2.89	0.7693801	+30970	-6
2.32	0.5971500	+29457	-7	2.90	0.7724771	+30995	-6
2.33	0.6000957	+29485	-7	2.91	0.7755766	+31020	-6
2.34	0.6030442	+29513	-7	2.92	0.7786786	+31045	-6
2.35	0.6059956	+29541	-7	2.93	0.7817831	+31069	-6
2.36	0.6089497	+29569	-7	2.94	0.7848900	+31094	-6
2.37	0.6119066	+29597	-7	2.95	0.7879994	+31119	-6
2.38	0.6148663	+29625	-7	2.96	0.7911112	+31143	-6
2.39	0.6178288	+29653	-7	2.97	0.7942255	+31168	-6
2.40	0.6207941	+29681	-7	2.98	0.7973423	+31192	-6
2.41	0.6237622	+29708	-7	2.99	0.8004615	+31216	-6
2.42	0.6267330	+29736	-7	3.00	0.8035831	+31241	-6
2.43	0.6297066	+29764	-7	3.01	0.8067072	+31265	-6
2.44	0.6326830	+29791	-7	3.02	0.8098337	+31289	-6
2.45	0.6356621	+29818	-7	3.03	0.8129626	+31313	-6
2.46	0.6386439	+29846	-7	3.04	0.8160939	+31337	-6
2.47	0.6416285	+29873	-7	3.05	0.8192276	+31361	-6
2.48	0.6446158	+29901	-7	3.06	0.8223638	+31385	-6
2.49	0.6476059	+29928	-7	3.07	0.8255023	+31409	-6
2.50	0.6505987	+29955	-7	3.08	0.8286432	+31433	-6
2.51	0.6535942	+29982	-7	3.09	0.8317865	+31457	-6
2.52	0.6565924	+30009	-7	3.10	0.8349322	+31481	-6
2.53	0.6595933	+30036	-7	3.11	0.8380803	+31504	-6
2.54	0.6625969	+30063	-7	3.12	0.8412307	+31528	-6
2.55	0.6656032	+30090	-7	3.13	0.8443835	+31551	-6
2.56	0.6686122	+30117	-7	3.14	0.8475386	+31575	-6
2.57	0.6716239	+30144	-7	3.15	0.8506961	+31598	-6
2.58	0.6746382	+30170	-7	3.16	0.8538559	+31622	-6
2.59	0.6776553	+30197	-7	3.17	0.8570181	+31645	-6
2.60	0.6806750	+30224	-7	3.18	0.8601826	+31668	-6
2.61	0.6836973	+30250	-7	3.19	0.8633495	+31692	-6
2.62	0.6867223	+30277	-7	3.20	0.8665186	+31715	-6
2.63	0.6897500	+30303	-7	3.21	0.8696901	+31738	-6
2.64	0.6927803	+30329	-7	3.22	0.8728639	+31761	-6
2.65	0.6958132	+30356	-7	3.23	0.8760400	+31784	-6

$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$	$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$
3.24	0.8792184	+31807	-6	3.82	1.0673125	+33044	-5
3.25	0.8823991	+31830	-6	3.83	1.0706169	+33064	-5
3.26	0.8855821	+31853	-6	3.84	1.0739232	+33084	-5
3.27	0.8887674	+31875	-6	3.85	1.0772316	+33103	-5
3.28	0.8919549	+31898	-6	3.86	1.0805419	+33123	-5
3.29	0.8951447	+31921	-6	3.87	1.0838542	+33142	-5
3.30	0.8983368	+31943	-6	3.88	1.0871684	+33162	-5
3.31	0.9015312	+31966	-6	3.89	1.0904846	+33181	-5
3.32	0.9047278	+31989	-6	3.90	1.0938027	+33201	-5
3.33	0.9079266	+32011	-6	3.91	1.0971228	+33220	-5
3.34	0.9111277	+32033	-6	3.92	1.1004448	+33239	-5
3.35	0.9143311	+32056	-6	3.93	1.1037687	+33259	-5
3.36	0.9175366	+32078	-6	3.94	1.1070946	+33278	-5
3.37	0.9207444	+32100	-6	3.95	1.1104223	+33297	-5
3.38	0.9239544	+32122	-6	3.96	1.1137520	+33316	-5
3.39	0.9271667	+32144	-6	3.97	1.1170836	+33335	-5
3.40	0.9303811	+32167	-5	3.98	1.1204171	+33354	-5
3.41	0.9335978	+32189	-5	3.99	1.1237525	+33373	-5
3.42	0.9368166	+32210	-5	4.00	1.1270898	+33392	-5
3.43	0.9400377	+32232	-5	4.01	1.1304290	+33411	-5
3.44	0.9432609	+32254	-5	4.02	1.1337701	+33429	-5
3.45	0.9464863	+32276	-5	4.03	1.1371130	+33448	-5
3.46	0.9497139	+32298	-5	4.04	1.1404578	+33467	-5
3.47	0.9529437	+32319	-5	4.05	1.1438045	+33486	-5
3.48	0.9561756	+32341	-5	4.06	1.1471531	+33504	-5
3.49	0.9594097	+32363	-5	4.07	1.1505035	+33523	-5
3.50	0.9626460	+32384	-5	4.08	1.1538557	+33541	-5
3.51	0.9658844	+32406	-5	4.09	1.1572098	+33560	-5
3.52	0.9691249	+32427	-5	4.10	1.1605658	+33578	-5
3.53	0.9723676	+32448	-5	4.11	1.1639236	+33596	-5
3.54	0.9756125	+32470	-5	4.12	1.1672832	+33615	-5
3.55	0.9788594	+32491	-5	4.13	1.1706447	+33633	-5
3.56	0.9821085	+32512	-5	4.14	1.1740080	+33651	-5
3.57	0.9853597	+32533	-5	4.15	1.1773731	+33669	-5
3.58	0.9886130	+32554	-5	4.16	1.1807400	+33687	-5
3.59	0.9918684	+32575	-5	4.17	1.1841087	+33705	-4
3.60	0.9951259	+32596	-5	4.18	1.1874793	+33723	-4
3.61	0.9983856	+32617	-5	4.19	1.1908516	+33741	-4
3.62	1.0016473	+32638	-5	4.20	1.1942257	+33759	-4
3.63	1.0049111	+32659	-5	4.21	1.1976016	+33777	-4
3.64	1.0081769	+32680	-5	4.22	1.2009793	+33795	-4
3.65	1.0114449	+32700	-5	4.23	1.2043588	+33813	-4
3.66	1.0147149	+32721	-5	4.24	1.2077401	+33830	-4
3.67	1.0179870	+32741	-5	4.25	1.2111231	+33848	-4
3.68	1.0212611	+32762	-5	4.26	1.2145079	+33866	-4
3.69	1.0245373	+32782	-5	4.27	1.2178945	+33883	-4
3.70	1.0278156	+32803	-5	4.28	1.2212828	+33901	-4
3.71	1.0310959	+32823	-5	4.29	1.2246729	+33918	-4
3.72	1.0343782	+32844	-5	4.30	1.2280647	+33936	-4
3.73	1.0376626	+32864	-5	4.31	1.2314583	+33953	-4
3.74	1.0409490	+32884	-5	4.32	1.2348536	+33970	-4
3.75	1.0442374	+32904	-5	4.33	1.2382506	+33988	-4
3.76	1.0475278	+32924	-5	4.34	1.2416494	+34005	-4
3.77	1.0508203	+32944	-5	4.35	1.2450499	+34022	-4
3.78	1.0541147	+32965	-5	4.36	1.2484521	+34039	-4
3.79	1.0574112	+32984	-5	4.37	1.2518560	+34056	-4
3.80	1.0607096	+33004	-5	4.38	1.2552617	+34073	-4
3.81	1.0640100	+33024	-5	4.39	1.2586690	+34090	-4

$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$	$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$
4.40	1.2620781			4.98	1.4625615		
4.41	1.2654888	+34107	-4	4.99	1.4660632	+35017	-4
4.42	1.2689012	+34124	-4	5.00	1.4695662	+35031	-4
4.43	1.2723154	+34141	-4	5.01	1.4730708	+35045	-4
4.44	1.2757312	+34158	-4	5.02	1.4765768	+35060	-4
4.45	1.2791486	+34175	-4	5.03	1.4800842	+35074	-4
4.46	1.2825678	+34192	-4	5.04	1.4835930	+35088	-4
4.47	1.2859886	+34208	-4	5.05	1.4871033	+35103	-4
4.48	1.2894111	+34225	-4	5.06	1.4906149	+35117	-4
4.49	1.2928353	+34242	-4	5.07	1.4941280	+35131	-4
4.50	1.2962611	+34258	-4	5.08	1.4976425	+35145	-4
4.51	1.2996885	+34275	-4	5.09	1.5011585	+35159	-4
4.52	1.3031176	+34291	-4	5.10	1.5046758	+35173	-3
4.53	1.3065484	+34307	-4	5.11	1.5081945	+35187	-3
4.54	1.3099808	+34324	-4	5.12	1.5117146	+35201	-3
4.55	1.3134148	+34340	-4	5.13	1.5152362	+35215	-3
4.56	1.3168504	+34356	-4	5.14	1.5187591	+35229	-3
4.57	1.3202877	+34373	-4	5.15	1.5222834	+35243	-3
4.58	1.3237266	+34389	-4	5.16	1.5258091	+35257	-3
4.59	1.3271671	+34405	-4	5.17	1.5293361	+35271	-3
4.60	1.3306092	+34421	-4	5.18	1.5328645	+35284	-3
4.61	1.3340530	+34437	-4	5.19	1.5363943	+35298	-3
4.62	1.3374983	+34453	-4	5.20	1.5399255	+35312	-3
4.63	1.3409452	+34469	-4	5.21	1.5434581	+35325	-3
4.64	1.3443937	+34485	-4	5.22	1.5469919	+35339	-3
4.65	1.3478439	+34501	-4	5.23	1.5505272	+35353	-3
4.66	1.3512956	+34517	-4	5.24	1.5540638	+35366	-3
4.67	1.3547488	+34533	-4	5.25	1.5576018	+35380	-3
4.68	1.3582037	+34549	-4	5.26	1.5611411	+35393	-3
4.69	1.3616601	+34564	-4	5.27	1.5646817	+35406	-3
4.70	1.3651181	+34580	-4	5.28	1.5682237	+35420	-3
4.71	1.3685777	+34596	-4	5.29	1.5717670	+35433	-3
4.72	1.3720388	+34612	-4	5.30	1.5753116	+35446	-3
4.73	1.3755015	+34627	-4	5.31	1.5788576	+35460	-3
4.74	1.3789657	+34642	-4	5.32	1.5824049	+35473	-3
4.75	1.3824314	+34658	-4	5.33	1.5859535	+35486	-3
4.76	1.3858988	+34673	-4	5.34	1.5895034	+35499	-3
4.77	1.3893676	+34689	-4	5.35	1.5930547	+35512	-3
4.78	1.3928380	+34704	-4	5.36	1.5966072	+35526	-3
4.79	1.3963099	+34719	-4	5.37	1.6001611	+35539	-3
4.80	1.3997834	+34734	-4	5.38	1.6037163	+35552	-3
4.81	1.4032583	+34750	-4	5.39	1.6072727	+35565	-3
4.82	1.4067348	+34765	-4	5.40	1.6108305	+35578	-3
4.83	1.4102128	+34780	-4	5.41	1.6143895	+35590	-3
4.84	1.4136923	+34795	-4	5.42	1.6179499	+35603	-3
4.85	1.4171733	+34810	-4	5.43	1.6215115	+35616	-3
4.86	1.4206558	+34825	-4	5.44	1.6250744	+35629	-3
4.87	1.4241398	+34840	-4	5.45	1.6286385	+35642	-3
4.88	1.4276253	+34855	-4	5.46	1.6322040	+35654	-3
4.89	1.4311123	+34870	-4	5.47	1.6357707	+35667	-3
4.90	1.4346008	+34885	-4	5.48	1.6393387	+35680	-3
4.91	1.4380907	+34900	-4	5.49	1.6429080	+35693	-3
4.92	1.4415822	+34914	-4	5.50	1.6464785	+35705	-3
4.93	1.4450751	+34929	-4	5.51	1.6500502	+35718	-3
4.94	1.4485694	+34944	-4	5.52	1.6536233	+35730	-3
4.95	1.4520653	+34958	-4	5.53	1.6571975	+35743	-3
4.96	1.4555625	+34973	-4	5.54	1.6607730	+35755	-3
4.97	1.4590613	+34988	-4	5.55	1.6643498	+35768	-3
		+35002	-4			+35780	-3

$n$	$\log \frac{e^n - 1}{n}$	$\Delta$	$\Delta'$	$n$	$\log \frac{e^n - 1}{n}$	$\Delta$	$\Delta'$
556	1.6679278	+35792	-3	614	1.8774629	+36455	-3
557	1.6715070	+35805	-3	615	1.8811084	+36466	-3
558	1.6750875	+35817	-3	616	1.8847550	+36476	-3
559	1.6786692	+35829	-3	617	1.8884026	+36487	-3
560	1.6822521	+35841	-3	618	1.8920513	+36497	-3
561	1.6858363	+35854	-3	619	1.8957011	+36508	-3
562	1.6894217	+35866	-3	620	1.8993518	+36518	-3
563	1.6930082	+35878	-3	621	1.9030037	+36529	-3
564	1.6965960	+35890	-3	622	1.9066565	+36539	-3
565	1.7001850	+35902	-3	623	1.9103104	+36549	-3
566	1.7037753	+35914	-3	624	1.9139653	+36560	-3
567	1.7073667	+35926	-3	625	1.9176213	+36570	-3
568	1.7109593	+35938	-3	626	1.9212783	+36580	-3
569	1.7145531	+35950	-3	627	1.9249363	+36590	-3
570	1.7181481	+35962	-3	628	1.9285953	+36601	-3
571	1.7217443	+35974	-3	629	1.9322554	+36611	-3
572	1.7253417	+35986	-3	630	1.9359165	+36621	-3
573	1.7289403	+35998	-3	631	1.9395785	+36631	-3
574	1.7325400	+36009	-3	632	1.9432416	+36641	-3
575	1.7361410	+36021	-3	633	1.9469057	+36651	-3
576	1.7397431	+36033	-3	634	1.9505709	+36661	-2
577	1.7433463	+36044	-3	635	1.9542370	+36671	-2
578	1.7469508	+36056	-3	636	1.9579041	+36681	-2
579	1.7505564	+36068	-3	637	1.9615722	+36691	-2
580	1.7541632	+36079	-3	638	1.9652413	+36701	-2
581	1.7577711	+36091	-3	639	1.9689114	+36711	-2
582	1.7613802	+36102	-3	640	1.9725825	+36721	-2
583	1.7649904	+36114	-3	641	1.9762546	+36731	-2
584	1.7686018	+36125	-3	642	1.9799277	+36740	-2
585	1.7722143	+36137	-3	643	1.9836017	+36750	-2
586	1.7758280	+36148	-3	644	1.9872767	+36760	-2
587	1.7794428	+36160	-3	645	1.9909527	+36770	-2
588	1.7830588	+36171	-3	646	1.9946297	+36780	-2
589	1.7866759	+36182	-3	647	1.9983077	+36789	-2
590	1.7902941	+36193	-3	648	2.0019867	+36799	-2
591	1.7939134	+36205	-3	649	2.0056665	+36809	-2
592	1.7975339	+36216	-3	650	2.0093473	+36818	-2
593	1.8011555	+36227	-3	651	2.0130292	+36828	-2
594	1.8047782	+36238	-3	652	2.0167119	+36837	-2
595	1.8084020	+36249	-3	653	2.0203957	+36847	-2
596	1.8120270	+36261	-3	654	2.0240804	+36856	-2
597	1.8156530	+36272	-3	655	2.0277660	+36866	-2
598	1.8192802	+36283	-3	656	2.0314526	+36875	-2
599	1.8229084	+36294	-3	657	2.0351402	+36885	-2
600	1.8265378	+36305	-3	658	2.0388286	+36894	-2
601	1.8301683	+36316	-3	659	2.0425181	+36904	-2
602	1.8337998	+36326	-3	660	2.0462084	+36913	-2
603	1.8374325	+36337	-3	661	2.0498997	+36922	-2
604	1.8410662	+36348	-3	662	2.0535920	+36932	-2
605	1.8447010	+36359	-3	663	2.0572852	+36941	-2
606	1.8483369	+36370	-3	664	2.0609793	+36950	-2
607	1.8519739	+36381	-3	665	2.0646743	+36960	-2
608	1.8556120	+36391	-3	666	2.0683703	+36969	-2
609	1.8592511	+36402	-3	667	2.0720671	+36978	-2
610	1.8628913	+36413	-3	668	2.0757649	+36987	-2
611	1.8665326	+36424	-3	669	2.0794637	+36996	-2
612	1.8701750	+36434	-3	670	2.0831633	+37006	-2
613	1.8738184	+36445	-3	671	2.0868639	+37015	-2

$x$	$\log \frac{x^x - 1}{x}$	$\Delta$	$\Delta'$	$x$	$\log \frac{x^x - 1}{x}$	$\Delta$	$\Delta'$
6.72	2.0905653	+37024	-2	7.30	2.3067334	+37513	-2
6.73	2.0942677	+37033	-2	7.31	2.3104847	+37521	-2
6.74	2.0979710	+37042	-2	7.32	2.3142369	+37529	-2
6.75	2.1016752	+37051	-2	7.33	2.3179898	+37537	-2
6.76	2.1053802	+37060	-2	7.34	2.3217435	+37545	-2
6.77	2.1090862	+37069	-2	7.35	2.3254979	+37552	-2
6.78	2.1127931	+37078	-2	7.36	2.3292532	+37560	-2
6.79	2.1165009	+37087	-2	7.37	2.3330092	+37568	-2
6.80	2.1202096	+37096	-2	7.38	2.3367660	+37576	-2
6.81	2.1239191	+37105	-2	7.39	2.3405236	+37583	-2
6.82	2.1276296	+37113	-2	7.40	2.3442819	+37591	-2
6.83	2.1313409	+37122	-2	7.41	2.3480410	+37599	-2
6.84	2.1350531	+37131	-2	7.42	2.3518009	+37606	-2
6.85	2.1387663	+37140	-2	7.43	2.3555615	+37614	-2
6.86	2.1424802	+37149	-2	7.44	2.3593229	+37621	-2
6.87	2.1461951	+37157	-2	7.45	2.3630850	+37629	-2
6.88	2.1499108	+37166	-2	7.46	2.3668479	+37637	-2
6.89	2.1536274	+37175	-2	7.47	2.3706116	+37644	-2
6.90	2.1573449	+37183	-2	7.48	2.3743760	+37652	-2
6.91	2.1610633	+37192	-2	7.49	2.3781412	+37659	-2
6.92	2.1647825	+37201	-2	7.50	2.3819071	+37667	-2
6.93	2.1685026	+37209	-2	7.51	2.3856737	+37674	-2
6.94	2.1722235	+37218	-2	7.52	2.3894412	+37682	-2
6.95	2.1759453	+37227	-2	7.53	2.3932093	+37689	-2
6.96	2.1796680	+37235	-2	7.54	2.3969782	+37696	-2
6.97	2.1833915	+37244	-2	7.55	2.4007478	+37704	-2
6.98	2.1871158	+37252	-2	7.56	2.4045182	+37711	-2
6.99	2.1908411	+37261	-2	7.57	2.4082893	+37718	-2
7.00	2.1945671	+37269	-2	7.58	2.4120612	+37726	-2
7.01	2.1982940	+37278	-2	7.59	2.4158338	+37733	-2
7.02	2.2020218	+37286	-2	7.60	2.4196071	+37740	-2
7.03	2.2057504	+37294	-2	7.61	2.4233811	+37748	-2
7.04	2.2094798	+37303	-2	7.62	2.4271559	+37755	-2
7.05	2.2132101	+37311	-2	7.63	2.4309314	+37762	-2
7.06	2.2169412	+37319	-2	7.64	2.4347076	+37769	-2
7.07	2.2206732	+37328	-2	7.65	2.4384846	+37777	-2
7.08	2.2244059	+37336	-2	7.66	2.4422622	+37784	-2
7.09	2.2281396	+37344	-2	7.67	2.4460406	+37791	-2
7.10	2.2318740	+37353	-2	7.68	2.4498197	+37798	-2
7.11	2.2356092	+37361	-2	7.69	2.4535995	+37805	-2
7.12	2.2393453	+37369	-2	7.70	2.4573801	+37812	-2
7.13	2.2430822	+37377	-2	7.71	2.4611613	+37820	-2
7.14	2.2468200	+37385	-2	7.72	2.4649433	+37827	-2
7.15	2.2505585	+37394	-2	7.73	2.4687260	+37834	-2
7.16	2.2542979	+37402	-2	7.74	2.4725093	+37841	-2
7.17	2.2580380	+37410	-2	7.75	2.4762934	+37848	-2
7.18	2.2617790	+37418	-2	7.76	2.4800782	+37855	-2
7.19	2.2655208	+37426	-2	7.77	2.4838637	+37862	-2
7.20	2.2692634	+37434	-2	7.78	2.4876499	+37869	-2
7.21	2.2730068	+37442	-2	7.79	2.4914368	+37876	-2
7.22	2.2767510	+37450	-2	7.80	2.4952244	+37883	-2
7.23	2.2804960	+37458	-2	7.81	2.4990127	+37890	-2
7.24	2.2842418	+37466	-2	7.82	2.5028016	+37897	-2
7.25	2.2879885	+37474	-2	7.83	2.5065913	+37904	-2
7.26	2.2917359	+37482	-2	7.84	2.5103817	+37911	-2
7.27	2.2954841	+37490	-2	7.85	2.5141727	+37917	-2
7.28	2.2992330	+37498	-2	7.86	2.5179645	+37924	-2
7.29	2.3029828	+37506	-2	7.87	2.5217569	+37931	-2

$n$	$\log \frac{e^n - 1}{n}$	$\Delta$	$\Delta'$	$n$	$\log \frac{e^n - 1}{n}$	$\Delta$	$\Delta'$
7.88	2.5255500			8.46	2.7466690	+38308	-1
7.89	2.5293438	+37938	-2	8.47	2.7504998	+38314	-1
7.90	2.5331383	+37945	-2	8.48	2.7543312	+38320	-1
7.91	2.5369334	+37952	-2	8.49	2.7581632	+38326	-1
7.92	2.5407293	+37958	-2	8.50	2.7619958	+38332	-1
7.93	2.5445258	+37965	-2	8.51	2.7658290	+38338	-1
7.94	2.5483230	+37972	-2	8.52	2.7696628	+38344	-1
7.95	2.5521208	+37979	-2	8.53	2.7734971	+38350	-1
7.96	2.5559193	+37985	-2	8.54	2.7773321	+38355	-1
7.97	2.5597185	+37992	-2	8.55	2.7811676	+38361	-1
7.98	2.5635184	+37999	-2	8.56	2.7850038	+38367	-1
7.99	2.5673190	+38005	-2	8.57	2.7888405	+38373	-1
8.00	2.5711202	+38012	-2	8.58	2.7926778	+38379	-1
8.01	2.5749220	+38019	-2	8.59	2.7965157	+38385	-1
8.02	2.5787245	+38025	-2	8.60	2.8003541	+38390	-1
8.03	2.5825277	+38032	-2	8.61	2.8041932	+38396	-1
8.04	2.5863316	+38038	-2	8.62	2.8080328	+38402	-1
8.05	2.5901361	+38045	-2	8.63	2.8118730	+38408	-1
8.06	2.5939413	+38052	-2	8.64	2.8157138	+38413	-1
8.07	2.5977471	+38058	-2	8.65	2.8195551	+38419	-1
8.08	2.6015535	+38065	-2	8.66	2.8233970	+38425	-1
8.09	2.6053607	+38071	-2	8.67	2.8272395	+38431	-1
8.10	2.6091684	+38078	-2	8.68	2.8310826	+38436	-1
8.11	2.6129769	+38084	-2	8.69	2.8349262	+38442	-1
8.12	2.6167859	+38091	-2	8.70	2.8387704	+38448	-1
8.13	2.6205956	+38097	-2	8.71	2.8426151	+38453	-1
8.14	2.6244060	+38104	-2	8.72	2.8464605	+38459	-1
8.15	2.6282170	+38110	-2	8.73	2.8503064	+38465	-1
8.16	2.6320286	+38116	-2	8.74	2.8541528	+38470	-1
8.17	2.6358409	+38123	-2	8.75	2.8579998	+38476	-1
8.18	2.6396539	+38129	-2	8.76	2.8618474	+38481	-1
8.19	2.6434674	+38136	-2	8.77	2.8656956	+38487	-1
8.20	2.6472816	+38142	-2	8.78	2.8695442	+38493	-1
8.21	2.6510964	+38148	-2	8.79	2.8733935	+38498	-1
8.22	2.6549119	+38155	-2	8.80	2.8772433	+38504	-1
8.23	2.6587280	+38161	-2	8.81	2.8810937	+38509	-1
8.24	2.6625447	+38167	-2	8.82	2.8849446	+38515	-1
8.25	2.6663620	+38173	-2	8.83	2.8887960	+38520	-1
8.26	2.6701800	+38180	-2	8.84	2.8926481	+38526	-1
8.27	2.6739986	+38186	-2	8.85	2.8965006	+38531	-1
8.28	2.6778178	+38192	-2	8.86	2.9003537	+38537	-1
8.29	2.6816377	+38198	-2	8.87	2.9042074	+38542	-1
8.30	2.6854582	+38205	-2	8.88	2.9080616	+38548	-1
8.31	2.6892793	+38211	-2	8.89	2.9119164	+38553	-1
8.32	2.6931010	+38217	-2	8.90	2.9157716	+38558	-1
8.33	2.6969233	+38223	-2	8.91	2.9196275	+38564	-1
8.34	2.7007462	+38229	-2	8.92	2.9234839	+38569	-1
8.35	2.7045698	+38236	-2	8.93	2.9273408	+38575	-1
8.36	2.7083939	+38242	-2	8.94	2.9311982	+38580	-1
8.37	2.7122187	+38248	-2	8.95	2.9350562	+38585	-1
8.38	2.7160441	+38254	-2	8.96	2.9389148	+38591	-1
8.39	2.7198701	+38260	-2	8.97	2.9427738	+38596	-1
8.40	2.7236967	+38266	-2	8.98	2.9466334	+38601	-1
8.41	2.7275239	+38272	-2	8.99	2.9504936	+38607	-1
8.42	2.7313517	+38278	-2	9.00	2.9543542	+38612	-1
8.43	2.7351801	+38284	-2	9.01	2.9582154	+38617	-1
8.44	2.7390091	+38290	-1	9.02	2.9620772	+38623	-1
8.45	2.7428388	+38296	-1	9.03	2.9659394	+38628	-1
		+38302	-1				

$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$	$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$
9.04	2.9698022	+38633	-1	9.62	3.1947090	+38920	-1
9.05	2.9736655	+38638	-1	9.63	3.1986010	+38925	-1
9.06	2.9775293	+38644	-1	9.64	3.2024935	+38929	-1
9.07	2.9813937	+38649	-1	9.65	3.2063865	+38934	-1
9.08	2.9852586	+38654	-1	9.66	3.2102799	+38939	-1
9.09	2.9891240	+38659	-1	9.67	3.2141737	+38943	-1
9.10	2.9929899	+38664	-1	9.68	3.2180681	+38948	-1
9.11	2.9968563	+38670	-1	9.69	3.2219629	+38953	-1
9.12	3.0007233	+38675	-1	9.70	3.2258581	+38957	-1
9.13	3.0045908	+38680	-1	9.71	3.2297538	+38962	-1
9.14	3.0084588	+38685	-1	9.72	3.2336500	+38966	-1
9.15	3.0123273	+38690	-1	9.73	3.2375466	+38971	-1
9.16	3.0161963	+38695	-1	9.74	3.2414437	+38975	-1
9.17	3.0200658	+38700	-1	9.75	3.2453413	+38980	-1
9.18	3.0239359	+38706	-1	9.76	3.2492393	+38984	-1
9.19	3.0278065	+38711	-1	9.77	3.2531377	+38989	-1
9.20	3.0316775	+38716	-1	9.78	3.2570366	+38994	-1
9.21	3.0355491	+38721	-1	9.79	3.2609360	+38998	-1
9.22	3.0394212	+38726	-1	9.80	3.2648358	+39003	-1
9.23	3.0432938	+38731	-1	9.81	3.2687360	+39007	-1
9.24	3.0471669	+38736	-1	9.82	3.2726367	+39011	-1
9.25	3.0510405	+38741	-1	9.83	3.2765379	+39016	-1
9.26	3.0549146	+38746	-1	9.84	3.2804395	+39020	-1
9.27	3.0587892	+38751	-1	9.85	3.2843415	+39025	-1
9.28	3.0626643	+38756	-1	9.86	3.2882440	+39029	-1
9.29	3.0665399	+38761	-1	9.87	3.2921469	+39034	-1
9.30	3.0704160	+38766	-1	9.88	3.2960503	+39038	-1
9.31	3.0742926	+38771	-1	9.89	3.2999541	+39043	-1
9.32	3.0781697	+38776	-1	9.90	3.3038584	+39047	-1
9.33	3.0820473	+38781	-1	9.91	3.3077631	+39051	-1
9.34	3.0859254	+38786	-1	9.92	3.3116682	+39056	-1
9.35	3.0898040	+38791	-1	9.93	3.3155738	+39060	-1
9.36	3.0936831	+38796	-1	9.94	3.3194798	+39065	-1
9.37	3.0975627	+38801	-1	9.95	3.3233863	+39069	-1
9.38	3.1014427	+38806	-1	9.96	3.3272932	+39073	-1
9.39	3.1053233	+38810	-1	9.97	3.3312005	+39078	-1
9.40	3.1092043	+38815	-1	9.98	3.3351083	+39082	-1
9.41	3.1130859	+38820	-1	9.99	3.3390165	+39086	-1
9.42	3.1169679	+38825	-1	10.00	3.3429251	+39091	-1
9.43	3.1208504	+38830	-1	10.01	3.3468342	+39095	-1
9.44	3.1247334	+38835	-1	10.02	3.3507437	+39099	-1
9.45	3.1286169	+38840	-1	10.03	3.3546536	+39104	-1
9.46	3.1325008	+38844	-1	10.04	3.3585639	+39108	-1
9.47	3.1363853	+38849	-1	10.05	3.3624747	+39112	-1
9.48	3.1402702	+38854	-1	10.06	3.3663859	+39116	-1
9.49	3.1441556	+38859	-1	10.07	3.3702976	+39121	-1
9.50	3.1480415	+38864	-1	10.08	3.3742096	+39125	-1
9.51	3.1519278	+38868	-1	10.09	3.3781221	+39129	-1
9.52	3.1558147	+38873	-1	10.10	3.3820351	+39133	-1
9.53	3.1597020	+38878	-1	10.11	3.3859484	+39138	-1
9.54	3.1635897	+38883	-1	10.12	3.3898622	+39142	-1
9.55	3.1674780	+38887	-1	10.13	3.3937763	+39146	-1
9.56	3.1713667	+38892	-1	10.14	3.3976910	+39150	-1
9.57	3.1752559	+38897	-1	10.15	3.4016060	+39154	-1
9.58	3.1791456	+38901	-1	10.16	3.4055214	+39159	-1
9.59	3.1830358	+38906	-1	10.17	3.4094373	+39163	-1
9.60	3.1869264	+38911	-1	10.18	3.4133536	+39167	-1
9.61	3.1908175	+38916	-1	10.19	3.4172703	+39171	-1

$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$	$x$	$\log \frac{e^x - 1}{x}$	$\Delta$	$\Delta'$
10-20	3.4211874			10-31	3.4643030		
10-21	3.4251049	+ 39175	- 1	10-32	3.4682250	+ 39221	- 1
10-22	3.4290229	+ 39180	- 1	10-33	3.4721475	+ 39225	- 1
10-23	3.4329413	+ 39184	- 1	10-34	3.4760704	+ 39229	- 1
10-24	3.4368600	+ 39188	- 1	10-35	3.4799936	+ 39233	- 1
10-25	3.4407792	+ 39192	- 1	10-36	3.4839173	+ 39237	- 1
10-26	3.4446988	+ 39196	- 1	10-37	3.4878414	+ 39241	- 1
10-27	3.4486188	+ 39200	- 1	10-38	3.4917659	+ 39245	- 1
10-28	3.4525393	+ 39204	- 1	10-39	3.4956908	+ 39249	- 1
10-29	3.4564601	+ 39208	- 1	10-40	3.4996161	+ 39253	- 1
10-30	3.4603813	+ 39212	- 1				

## HOME AND FOREIGN INTELLIGENCE.

### SCOTTISH PROVINCIAL ASSURANCE COMPANY.

*Established in 1825.*

#### INVESTIGATION REPORT, 1867.

The Directors have now the pleasure of submitting to their constituents the Forty-first Annual Report on the Business of the Company, and the Results of the Investigation into their Affairs for the Quinquennial Period ending the 31st January, 1867.

\* \* \* \* \*

With reference to the Investigation into the state of the Company's affairs, the Board now beg to submit the following statement, premising that the Valuations and relative calculations, have been conducted on the same principles as were adopted on last occasion, as then recommended by Mr. James John Downes, the late eminent Actuary of the Economic Life Assurance Society. [From the Report of Mr. Downes (in 1862) it appears that the particulars of each Policy were set forth in schedules, and the respective values of the sums assured, and the *nett* values *only* of the future premiums, computed to the nearest month. The difference of these would represent the value of each Policy at the date of valuation. That in all cases the age was taken as that for which the premium was charged at the date of entry.

With regard to the law of Mortality, and the rate of Interest employed, that they were the same as were adopted in the construction of the Company's Tables of Assurance Premiums, and that, as the difference between the Participating and Non-participating scales arises only from the difference in the amounts of loadings (the value of the risk being the same in both cases), an uniform system was adopted, and the whole Valuation based on the Carlisle Tables, at  $3\frac{1}{2}$  per cent., except for Immediate Annuities.]

The details, as shown in the Valuation States, now laid on the Table, have been carried out exactly in the same manner. Each Policy has been

separately valued in duplicate, and the calculations afterwards carefully compared, so that the same confidence may be accorded by both Shareholders and Policy-holders to their correctness and soundness as was done at the Investigation in 1862.

From the Memorandum of Result of Investigation appended hereto, it will be seen that the Total available Surplus amounts to the sum of £67,381 19s. 3d.

Out of this sum the Directors recommend that £55,527 7s. 10d. be set apart to declare a retrospective Bonus at the rate of £1 7s. 6d., per cent. per annum, for the last Five Years, on the original sums Assured, on all Policies on the Participation Branch, not on an ascending scale of Premiums, in force at 31st January last, for their number of full years standing on the Books since 31st January, 1862.

Further, out of said Surplus, the Directors propose that the sum of £5,000 be set apart to pay a Dividend at the increased rate of Ten per cent. on the Capital of £50,000, and that the balance of £6,854 11s. 5d. be carried to the credit of the Reserve Fund, which will then stand at £18,363 6s. 5d.

During the last Five Years, the Office has issued 4,845 Policies, Assuring £2,092,973, and yielding an Annual Revenue of £62,452 2s. 2d.

(In the five years 1858 to 1862, the corresponding figures were 3,209 Policies, Assuring £1,219,697; New Annual Premiums, £37,400 15s. 10d.)

The claims by Death during the Quinquennial period amount to £172,963 8s.

The total number of Life Policies in force at 31st January last, was 7,662, assuring the sum of £3,250,000—yielding in Annual Premiums £95,263 11s. 11d.

#### BALANCE SHEET,

AS AT 31ST JANUARY, 1867.

Dr.	
Invested on Redeemable Annuity, .....	£37,484 10 11
Do. on Heritable and Real Estate, .....	57,590 4 8
Do. on Government Stocks, Consols, .....	8,950 0 0
Do. on Government Annuities, .....	1,009 0 0
Do. on Railway and other Debentures, .....	73,864 9 8
Do. on Colonial Government Bonds, .....	62,141 12 3
Amount of Stock Investments, .....	107,651 8 8
Do. of Reversionary do., .....	8,095 17 0
Advanced on Personal Security including Loans on Surrender Value of Company's Policies, .....	78,182 10 8
Deposits in Banks for fixed periods, .....	56,318 16 9
Sundry other Investments, .....	776 3 4
Company's Offices, .....	8,834 16 6
Cash at Bankers, .....	23,860 12 1
Amount due by Company's Branches, Agents, and other Companies, .....	38,816 17 6
Life Premiums on Credit, .....	8,159 12 3
Outstanding Premiums, Interests, and Remittances not yet due, .....	10,403 15 6
Stamps in hand, .....	193 2 4
Cash in hand, .....	37 0 10
	<hr/>
	£582,370 10 11

Or.		
Capital Stock, .....		£50,000 0 0
General Reserve Fund, .....	£18,363 6 5	
Fire Do., .....	20,000 0 0	
		38,363 6 5
Fire Premiums to meet unexpired Risks, .....		5,223 14 10
Participation Life Fund, .....	229,413 12 0	
Bonus Fund, .....	95,356 13 4	
Non-Participation Life Fund, .....	86,537 9 0	
Total Life Funds, .....		411,307 14 4
Annuitants' Fund, .....		36,470 3 10
Dividends uncalled for, .....		399 16 4
Government Duty, .....		3,365 5 6
Amount consigned with the Company, .....		613 12 9
Sum required to meet Outstanding Claims, .....		31,626 16 11
Dividend, at Ten per cent., .....		5,000 0 0
		<u>£582,370 10 11</u>

**MEMORANDUM OF RESULT OF INVESTIGATION,  
SHOWING THE AVAILABLE SURPLUS FUND AT 31ST JANUARY, 1867.**

I.—PARTICIPATION LIFE FUND, .....	£260,789 16 4	
Value of Policies, as per Abstract of Valuation States, .....	229,413 12 0	
Surplus, .....		£31,376 4 4
II.—BONUS FUND, .....	42,228 17 2	
Value of Bonuses, as per said Abstract, .....	39,829 5 6	
Surplus, .....		2,399 11 8
III.—NON-PARTICIPATION LIFE FUND, .....	91,896 7 11	
Value of Policies, as per said Abstract, ..	86,537 9 0	
Surplus, .....		5,358 18 11
IV.—ANNUITANTS' FUND, .....	37,048 14 1	
Value of Policies, as per Valuation States, ..	36,470 3 10	
Surplus, .....		578 10 3
V.—GOVERNMENT ANNUITY INVESTMENT, as per Valuation, .....	1,009 0 0	
Ledger Account, .....	676 7 7	
Surplus, .....		332 12 5
VI.—SURPLUS INTERESTS ON ACCUMULATED LIFE FUNDS, during Quinquennial Period, .....		19,486 2 8
VII.—SURPLUS ON PAST YEAR, .....		7,849 19 0
AVAILABLE SURPLUS, .....		<u>£67,381 19 3</u>

NOTE.—In the Valuation of Policies, the pure Premiums only having been used, a Surplus of Annual Revenue, amounting to £18,433 12s. 3d., remains unvalued, to meet future charges, and to form a source of profit for next Investigation.

**DECLARATION OF BONUS.**

THE VESTED ADDITIONS TO POLICIES FOR £1000 ARE AS UNDER:—

Policies opened before Annual Balance in	Bonus vested at 10th February, 1847, at £1 5s. per cent.	Bonus vested at 10th February, 1852, at £1 5s. per cent.	Bonus vested at 31st January, 1857, at £1 7s. 6d. per cent.	Bonus vested at 31st January, 1862, at £1 7s. 6d. per cent.	Bonus vested at 31st January, 1867, at £1 7s. 6d. per cent.	TOTAL VESTED BONUSES IN 1867.
£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
1841	87 10 0	67 19 5	79 8 9	84 18 0	68 15 0	388 11 2
1847	12 10 0	63 5 8	73 19 2	79 0 11	63 15 0	297 10 9
1852	..	12 10 0	69 12 2	74 7 11	68 15 0	225 5 1
1857	..	..	13 15 0	69 13 11	68 15 0	152 3 11
1862	..	..	..	13 15 0	68 15 0	82 10 0
1867	..	..	..	..	13 15 0	13 15 0

A further Bonus, at the rate of £1 per cent. for every additional year for which Premiums may be received, will also be payable on Policies to which vested additions have been made, and that may become claims by death before next Investigation.

## CORRESPONDENCE.

## "ON THE OBJECTIONS TO THE NET-PREMIUM MODE OF VALUATION."

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—Since the announcement of Mr. Sprague's recent paper "On the proper method of estimating the liability of an Assurance Company," I have looked forward with much interest to learn the arguments by which the objections which I, in common I believe with many actuaries of higher reputation, feel very strongly to attach to the net-premium method, would be met; and having attentively perused the observations of the members present at the reading of Mr. Sprague's paper, I shall be glad now to be allowed to avail myself of the pages of your influential *Journal* to contribute in my own way my modest quota to the discussion of this important question.

The extravagant language so liberally indulged in by a few extreme adherents of the net-premium school—who seem to have taken silence for consent, and to have supposed their views were approved because not controverted—had convinced me that the time had arrived to make a stand against an abuse which, owing to recent events, was threatening to become mischievous; and I am glad that so eminent and able a man as Mr. Sprague has forestalled me in a task which, however unqualified for its performance, I should certainly, in the absence of a better champion, have felt myself called upon to undertake.

It forms no part of my present design to discuss the merits of the particular method which Mr. Sprague has suggested as a more efficient substitute (for general purposes) for that which forms the object of his attack; and although, in my opinion, there exist other and perhaps scarcely less fatal objections to the net-premium method, I shall confine myself on this occasion to a recapitulation of those already brought forward by Mr. Sprague, and to an examination of the arguments used in opposition.

The first in Mr. Sprague's list of objections is that the net-premium method is in fact open to the charge which proved fatal to the old "re-insurance" plan, as exemplified in the mode of valuation adopted by Offices using the Northampton Table. This objection is that the method fails to accomplish the first and principal object for which a valuation is supposed to be made, viz., to exhibit the actual financial position of the Office to which it is applied.

In illustration of this assertion I will take the case of two Offices, A and B, doing business upon the proprietary or non-participating system,—Office B having started exactly two years after Office A. For simplicity suppose that in Office A the lives all enter at the age of 30 at an annual premium of £2. 7s.,\* while in Office B they all enter at age 32 at an annual premium of £2. 5s.\* Suppose further that after the expiration of 12 years a "net-premium" actuary is required to say what fund, according to his opinion, Office A should have in hand to enable it to resist an application for a winding-up order in the Court of Chancery. Assume that the valuation is to be made by the old Experience Table at 4 per cent,

\* These are actual rates—taken from the table at the end of Jones's work,—and are by no means extreme cases.

which gives a net premium at age 30 of £1·697 per cent. Under these circumstances the Estimated Liability will be £13·409 per cent.

Now let us further suppose that by a remarkable coincidence the same actuary is called upon at the same time to investigate the condition of Office B. In this case he finds his net-premium (age 32) is £1·804, and the Estimated Liability consequently £11·738 per cent.

So that, putting this and that together:

Office.	Present Age.	Annual Premium payable (per cent).	Estimated Liability (per cent).
A	42	2 7 0	13·409
B	42	2 5 0	11·738

Hence it appears that, to meet precisely the same risk, Office B will be declared solvent if it have in hand £11·738 per cent, while Office A under the same circumstances with the same fund in hand must be pronounced insolvent (if the Actuary has faith in his principles), notwithstanding that in aid of that fund it has to receive in future a higher premium than Office B.

To this it will perhaps be answered that Office A may reasonably be expected to have in hand a larger fund, seeing that it has been receiving a larger premium and for a longer term. This, however, is quite a distinct question,—into which it is not necessary here to enter.\* The assertion is, "That the method in question fails to exhibit the *actual* financial position "of the Office to which it is applied," and the adherents of the net-premium school must either admit the charge, or they must assert that, under the circumstances supposed, Office B is better able to meet the claims upon it than Office A. It is quite immaterial to their opponents which horn of the dilemma they may prefer.

I am quite aware that the more moderate of the net-premium school would not venture to declare the Office A in a state of insolvency under the circumstances supposed,—altho' I am by no means sure that others would not do so. But why should they hesitate to carry their reasoning to its only legitimate conclusion? Evidently because they know that altho', *conventionally*, their method is supposed to give a true view of the respective conditions of the two Offices, yet, *as a matter of fact*, it wholly fails to do anything of the kind.

So much then in illustration of Mr. Sprague's first objection, to which apparently not the slightest allusion, direct or indirect, was made in the course of the discussion that followed the reading of the paper.

Mr. Sprague's second objection is that if the valuation be made by a different table from that by which the premiums have been calculated, the use of the formula  $1 - \frac{1 + a_{x+n}}{1 + a_x}$  (which is what is invariably understood by a net-premium valuation in such a case) may have the effect of giving credit for a higher premium than is actually receivable.

The only reply made to this objection would appear to be the opinion

\* The deficiency might arise in many ways. Say, for instance, that a large portion of the fund had been sunk in investments of too speculative a character.

expressed in one instance that the same table ought always to be used in the valuation as in computing the premium. The adherents of the net-premium school must elect which ground they prefer to take up. As regards the case supposed by Mr. Sprague it is obvious that if the premium payable should happen to be less than the net premium required by the table used in the valuation (which is quite possible when different tables are used) we shall be committing, without knowing it, the inconceivable absurdity of valuing a higher premium than we are actually entitled to receive. Altho' I think it unlikely that the advocates of the net-premium system, generally, will feel disposed to take up the ground that in selecting your table you take it for better or worse, and must never after change it for another; yet it is evident, I think, that such a step would at least render their position more logically defensible.

If the number of deaths in a given time, among a limited number of individuals, were a fixed and determinate quantity, instead of being a fluctuating and uncertain one, depending upon imperfectly known and varying causes, the system of Life Assurance finance might have taken quite a different shape from that in which it has actually developed itself. The whole matter would then have resolved itself into a question of ordinary book-keeping; and the duties of the actuary would have been entirely superseded by those of the accountant. It is a tendency to contemplate the subject from what I may term the accountant's point of view that has given rise to that extraordinary idea which is occasionally ventilated under the denomination of the "retrospective" method; suggested apparently by those popular expositions of the practical working of a Life Assurance Society given by Dr. Farr and others, in which the variation of the amount of the fund from year to year is shown by assuming that the deaths fall out exactly as predicted by the Mortality Table. But absurd as it is, and opposed to the very fundamental principles of the doctrine of probabilities, a "retrospective" method in some form or shape affords, I believe, the one firm footing for those who have abandoned the only true prospective method, which takes into account the facts and probabilities of the *actual* state of affairs,—regarding the *past* only as it tends to throw light on the *future*,—which method Mr. Sprague properly insists upon as the very basis of true actuarial science.

I come now to Mr. Sprague's third objection, viz., that when a Company has been at great expense to procure business and has spent more than the loading of the premiums received, the net-premium method brings out a deficiency, notwithstanding that the expense may have been perfectly wise and justifiable,—and when (I presume Mr. Sprague means) a valuation of the premiums actually payable after making a sufficient reserve for future expenses and omitting negative values would show a surplus sufficient for the safety of the Office.

The answers made to this objection seem to resolve themselves into a denial of the possibility of the conditions with which (somewhat superfluously, I think) Mr. Sprague has qualified his objection. The more moderate of his opponents assert that such an expenditure is under all circumstances unwise, while the more extreme maintain that it is unjustifiable.

That it is unwise to pay for the acquisition of business more than is absolutely necessary to obtain it, or more than it is worth when obtained; and that it is better that the expense of obtaining it should be spread as

much as possible over the future premiums; are propositions to which I readily subscribe. But that the entire expense of obtaining an assurance—including, of course, under that head the medical fees, together with a due proportion of the current general expenses of the Office,—cannot consistently with prudence exceed the loading of the first year's premium, is an assertion which cannot be accepted without proof. To those who are in the habit of making their calculations upon true data the determination of the limit of a wise and profitable expenditure for the acquisition of business offers no difficulty whatever. When, therefore, those who tell us that it is unwise to exceed the loading in question can prove by calculation that the Office is better without the assurance than with it if obtained upon such terms, they will not indeed have answered Mr. Sprague's third objection, but they will at all events have convicted that gentleman of having rashly sanctioned an imprudent rate of expenditure.

The assertion that the expenditure is unjustifiable,—that there is in fact an "implied contract" with the assured that the loading only on the premiums received shall be available for expenditure—is easily disposed of. If it mean that there is an implied contract to this effect because it is the proper course, it merely amounts to begging the question at issue—while if it mean more than this it is simply and obviously untrue. It is generally known that the Deeds of Settlement of many Societies lay down rules of procedure quite incompatible with the net-premium mode of valuation. But indeed it is evident that the only possible "implied contract" is, that a sufficient fund shall at all times be maintained for the safety of the Office, and that the profits shall be equitably distributed among those entitled to them. In what manner these important conditions are to be fulfilled the public wisely leaves to the decision of the responsible actuary,—being sufficiently alive to the fact that the most obvious view of a scientific question is frequently the very reverse of the true one, to distrust its own opinion upon a matter in reference to which the most eminent authorities are by no means unanimous. If at the same time it were as careful to ascertain the title that officer has to its confidence, it would seldom have reason to regret having left him unfettered in his judgment.

I hope at some future time to submit to the Institute a description of the method of valuation which seems to me to be the best adapted to accomplish the end in view; and I shall then endeavour to show that, quite independently of the fatal objections urged by Mr. Sprague, the net-premium method, so far from possessing that title to preference which its admirers so unreasonably claim for it, is one which must inevitably disappear with that advance of true actuarial science, which the foundation of the Institute of Actuaries was designed to aid.

I remain, Sir,

Your very obedient servant,

London, 1st June, 1870.

W. M. MAKEHAM.

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ON HERR WILHELM LAZARUS'S PAPER "ON SOME PROBLEMS  
IN THE THEORY OF PROBABILITIES."

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—In the paper by Herr Wilhelm Lazarus, in the January number of the *Journal*, which treats of an important branch of the theory of proba-

bilities in a very lucid manner, there is one passage which seems to me capable of being expressed a little more clearly. The obscurity has no doubt arisen from the great difficulty attendant upon the translation of a paper in German, upon a very abstruse mathematical subject, into a form intelligible to English readers. I hope, therefore, that an attempt to render the paper more easily intelligible will be of some service to your English readers.

The passage to which I allude is at the bottom of page 256, and is as follows:—"Making  $z=u$ , expanding  $\Omega_1$  and  $\Omega_2$  under this supposition, "and adding  $\Omega_0$ , which may be done in the simplest manner as we expand "equation (28) or (29) while calculating the most probable case—making "also  $\Omega_0 + \Omega_1 + \Omega_2 = \Omega$ , we find in place of equation (8),

$$" \Omega = \frac{1}{\sqrt{\pi}} \int_0^{k_2} e^{-t^2} dt + \frac{1}{\sqrt{\pi}} \int_0^{k_3} e^{-t^2} dt + \frac{B_2}{A_2 \sqrt{\pi}} e^{-k_2^2} - \frac{B_3}{A_3 \sqrt{\pi}} e^{-k_3^2} "$$

Now I imagine that this passage will be to most readers by no means self-evident, and that some amplification of it may be acceptable to them.

Herr Lazarus shows that

$$\Omega_1 = \frac{\int_0^p x^m (1-x)^{n-1} dx}{\int_0^1 x^m (1-x)^{n-1} dx} - \frac{\int_0^p x^{m+z} (1-x)^{n-z-1} dx}{\int_0^1 x^{m+z} (1-x)^{n-z-1} dx}$$

$$\Omega_2 = \frac{\int_0^p x^{m-z-1} (1-x)^{n+z} dx}{\int_0^1 x^{m-z-1} (1-x)^{n+z} dx} - \frac{\int_0^p x^{m-1} (1-x)^n dx}{\int_0^1 x^{m-1} (1-x)^n dx}$$

These are simply equations (28) and (29) after making  $u=z$ ; adding,

$$\Omega_1 + \Omega_2 = \frac{\int_0^p x^{m-z-1} (1-x)^{n+z} dx}{\int_0^1 x^{m-z-1} (1-x)^{n+z} dx} - \frac{\int_0^p x^{m+z} (1-x)^{n-z-1} dx}{\int_0^1 x^{m+z} (1-x)^{n-z-1} dx}$$

$$- \left\{ \frac{\int_0^p x^{m-1} (1-x)^n dx}{\int_0^1 x^{m-1} (1-x)^n dx} - \frac{\int_0^p x^m (1-x)^{n-1} dx}{\int_0^1 x^m (1-x)^{n-1} dx} \right\}$$

Now the expression between brackets, on the right hand side of this equation, using the notation adopted by Herr Lazarus, is equal to  $\sum_{\mu}^{m+1} (p^{\mu}) - \sum_{\mu} (p^{\mu})$ , which is nothing but  $\Omega_0$ , or the probability of the occurrence of the most probable combination, i.e., exactly  $m$  E's, and  $\mu - m$  F's. Thus

$$\Omega_1 + \Omega_2 = \frac{\int_0^p x^{m-z-1} (1-x)^{n+z} dx}{\int_0^1 x^{m-z-1} (1-x)^{n+z} dx} - \frac{\int_0^p x^{m+z} (1-x)^{n-z-1} dx}{\int_0^1 x^{m+z} (1-x)^{n-z-1} dx} - \Omega_0$$

or

$$\begin{aligned}
 \Omega_0 + \Omega_1 + \Omega_2 &= \frac{\int_0^p x^{m-z-1}(1-x)^{n+z} dx}{\int_0^1 x^{m-z-1}(1-x)^{n+z} dx} - \frac{\int_0^p x^{m+z}(1-x)^{n-z-1} dx}{\int_0^1 x^{m+z}(1-x)^{n-z-1} dx} \dots (I) \\
 &= \frac{1}{2} \mp \frac{1}{\sqrt{\pi}} \int_0^{k_2} e^{-t^2} dt \left( \mp \text{as } \frac{m-z-1}{\mu-1} > \text{or } < p \right) - \frac{B_2}{A_2 \sqrt{\pi}} e^{-k_2^2} \\
 &\quad - \left\{ \frac{1}{2} \mp \int_0^{k_2} e^{-t^2} dt \left( \mp \text{as } \frac{m+z}{\mu-1} > \text{or } < p \right) - \frac{B_2}{A_2 \sqrt{\pi}} e^{-k_2^2} \right\} \\
 &= \pm \int_0^{k_2} e^{-t^2} dt \left( \pm \text{as } \frac{m+z}{\mu-1} > \text{or } < p \right) \\
 &\quad \mp \int_0^{k_2} e^{-t^2} dt \left( \mp \text{as } \frac{m-z-1}{\mu-1} > \text{or } < p \right) \\
 &\quad + \frac{B_2}{A_2 \sqrt{\pi}} e^{-k_2^2} - \frac{B_2}{A_2 \sqrt{\pi}} e^{-k_2^2}
 \end{aligned}$$

Herr Lazarus appears to have dropped the distinction with regard to the use of the signs preceding the first two terms of this expression; but I do not see his reason for so doing.

The expression for  $\Omega$  might have been obtained in the same form directly; only then, as Herr Lazarus points out, the elements of which it is composed would have been lost sight of.

We have

$$\Omega = \sum_{\mu}^{m-z} (p^{\mu}) - \sum_{\mu}^{m+z+1} (p^{\mu})$$

and using the equation (25), viz.,

$$\sum_{\mu}^p (p^{\mu}) = \frac{\int_0^p x^{p-1}(1-x)^r dx}{\int_0^1 x^{p-1}(1-x)^r dx} (r+p=\mu)$$

we have

$$\Omega = \frac{\int_0^p x^{m-z-1}(1-x)^{n+z} dx}{\int_0^1 x^{m-z-1}(1-x)^{n+z} dx} - \frac{\int_0^p x^{m+z}(1-x)^{n-z-1} dx}{\int_0^1 x^{m+z}(1-x)^{n-z-1} dx}$$

which is the equation (I) already obtained.

Herr Lazarus refers to Laplace and Poisson as his guides in the method by which he obtains the value of the integral  $\int x^a(1-x)^b dx$  between the limits 0 and  $p$ , and 0 and 1. Laplace's demonstration of this method will be found in his "*Théorie Analytique des Probabilités*,"

Seconde Partie, Chapitre Premier; and an excellent English version of this is given in the article on Probabilities (sections 62-69) by Professor de Morgan, in the "*Encyclopædia Metropolitana*."

I am, Sir,

Your obedient servant,

June 7th, 1870.

18, Lincoln's Inn Fields.

WILLIAM SUTTON.

P.S. The last two terms of equation (49) in Herr Lazarus's paper are printed incorrectly.

$$\text{They ought to be } + \frac{B_4}{A_4\sqrt{\pi}}e^{-k_4^2} - \frac{B_3}{A_3\sqrt{\pi}}e^{-k_3^2}.$$

## INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.—SESSION 1869-70.

*First Ordinary Meeting, Monday, 29th November, 1869.*

The President in the Chair.

Read and confirmed the minutes of the anniversary meeting, held on the 5th June, 1869.

The following gentlemen were elected Associates, viz.:—

William Walter Wainwright.  
Robert George Hann.  
Thomas Gans Ackland.  
Edward Bellingham Trew.  
Theodore Henry Adey.  
Josiah Owen.  
George Robert Storrow.  
David Francis Park.  
Francis Glanville Richards.

William Thomas Gray.  
Henry Gentry.  
Charles James Harvey.  
William Steadman Aldis.  
Bion Reynolds, B.A.  
James Pringle.  
David John Surenne.  
George Todd.  
William Richardson, Jun.

Mr. Bumsted read a Translation of a paper by Herr Hopf, entitled "Suggestions for a Law to regulate the Calculation and Investment of the Reserve in Life Assurance Companies."

Thanks having been voted to Mr. Bumsted and Herr Hopf, the meeting adjourned to Monday, 20th December, 1869.

*Second Ordinary Meeting, Monday, 20th December, 1869.*

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected members, viz.:—

*Fellows.*

Cornelius Walford.  
Joseph John Dymond.

*Associates.*

Edward T. Sims, Jun.  
John Hill Elder.  
Jonas Ashton, M.A.  
James Dalby Hobson.

Henry Jeula.  
Ainslie Talon.  
Joseph Burne.  
Alfred Charles Waters.

Mr. T. B. Sprague, M.A., read a paper "On the Rate of Mortality prevailing among Assured Lives as influenced by the length of time for which they have been assured."

Thanks having been voted to Mr. Sprague, the meeting adjourned to Monday, 31st January, 1870.

*Third Ordinary Meeting, Monday, 31st January, 1870.*

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz. :—

Duncan McLauchlan Slater.  
Josiah Samuel Parker.

The following was announced to be the result of the Examinations for 1869 :—

**MATRICULATION EXAMINATION.**

Twenty-three gentlemen appeared for this Examination, of whom two withdrew, and twelve passed in the following order of merit :—

1. B. Reynolds.
2. A. C. Waters.
3. G. Todd.
4. J. Pringle.
5. D. F. Park.
6. { C. H. Oldham.
- { T. E. Young.
8. Clarence Smith.
9. W. T. Gray.
10. J. Burne.
11. E. T. Sims, Jun.
12. T. H. Adey.

**SECOND YEAR'S EXAMINATION.**

Three gentlemen presented themselves for this Examination, and two passed in the following order of merit, viz. :—

1. T. N. Toller, M.A.
2. J. C. Hopkinson, B.A.

**THIRD YEAR'S EXAMINATION.**

Two gentlemen presented themselves for this Examination, and passed in the following order of merit, viz. :—

1. F. Addiscott.
2. W. Hughes.

The best thanks of the meeting were given to the Examiners for their recent services.

Professor Oppermann read a paper "On the Graduation of Life Tables with special application to the Rate of Mortality in Infancy and Childhood."

Mr. W. S. B. Woolhouse read a paper "On a New Method of adjusting Mortality Tables."

Thanks having been voted to Professor Oppermann and Mr. Woolhouse, the meeting adjourned to Monday, 28th February, 1870.

*Fourth Ordinary Meeting, Monday, 28th February, 1870.*

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz. :—

Wilson Emery Stark.  
Charles Bright.

Mr. W. M. Makeham read a paper "On the Proper Method of Loading the Premiums required for the Grant of Life Annuities and Assurances."

Thanks having been voted to Mr. Makeham, the meeting adjourned to Monday, 28th March, 1870.

*Fifth Ordinary Meeting, Monday, March 28th, 1870.*

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz. :—

Charles Ernest Costerton.  
George Samuel Fennell.  
Léon Pierre Adrien de Montluc.

Mr. W. P. Pattison read a paper "On the Existing Legislation affecting Friendly Societies, with Suggestions for its Amendment and Extension."

Thanks having been voted to Mr. Pattison, the meeting adjourned to Monday, 25th April, 1870.

*Sixth Ordinary Meeting, Monday, 25th April, 1870.*

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected members:—

*Fellow.*

John Brookes Johnston.

*Associates.*

John Michael Cramsie Johnston.  
Septimus Joyce.  
William King.

Mr. T. B. Sprague, M.A., read a paper "On the Proper Method of estimating the Liabilities of a Life Assurance Company."

Thanks having been voted to Mr. Sprague, the meeting adjourned to Monday, 28th November, 1870.

*The Twenty-third Annual General Meeting, Saturday, 4th June, 1870.*

SAMUEL BROWN, Esq., the President, in the Chair.

Mr. ARCHIBALD DAY (Hon. Secretary) read the notice convening the meeting, the minutes of the preceding ordinary meeting, and the Report of the Council and Statement of Accounts, which were as follows:—

"The number of members of the Institute on the 31st March last was 255, 101 Fellows and 154 Associates, showing an increase of 21 in the year. The increase is entirely in the class of Associates, the number of Fellows being one less.

"By the accompanying accounts, which have been examined by the Auditors, it will be observed that the finances of the Institute are in a very satisfactory condition. The income from Annual Subscriptions is £523. 19s., while the ordinary expenditure is £458. 8s. 10d.; so that the anticipations expressed in the last report that the usual outgoings would not again exceed the income have been realised. The total funds now amount to £1,720. 7s. 2d., of which £877. 13s. 8d. stands to the credit of the General Fund. The latter sum includes £220. 10s. from compounded subscriptions.

"The Council have great pleasure in announcing that the results of the

Mortality Experience, on which they have so long been engaged, are now published; and the volume is obtainable by any member of the Institute at a reduced price. The Council intend at some future time to publish a second volume, containing monetary tables for practical use, deduced from these materials.

"The following papers have been read during the Session and have appeared or will appear in the *Journal*:—

- 'Suggestions for a Law to regulate the Calculation and Investment of the Reserve in Life Assurance Companies.' By Herr Hopf. Translation by Mr. D. A. Bumsted.
- 'On the Rate of Mortality prevailing among Assured Lives, as influenced by the length of time for which they have been assured.' By Mr. T. B. Sprague, M.A.
- 'On the Graduation of Life Tables with Special Application to the Rate of Mortality in Infancy and Childhood.' By Professor Oppermann.
- 'On a new Method of Adjusting Mortality Tables.' By Mr. W. S. B. Woolhouse.
- On the Proper Method of Loading the Premiums required for the grant of Life Annuities and Assurances.' By Mr. Makeham.
- 'On the existing legislation affecting Friendly Societies, with suggestions for its amendment and extension.' By Mr. W. P. Pattison.
- 'On the Proper Method of estimating the Liabilities of a Life Assurance Company.' By Mr. T. B. Sprague, M.A.

"Mr. Caye has again introduced into the House of Commons a Bill to amend the law relating to Life Assurance Companies. This Bill the Council have watched with interest, and made some suggestions for its improvement, which they have reason to believe will meet with proper consideration. Among many notices of amendment that have been given, one by Mr. Bowring proposes to enact that the Actuaries responsible for the periodical investigations of Life Assurance Societies shall be members of this Institute. The Council think it right to say that, so far as they are aware, this notice was not given at the suggestion, direct or indirect, of any member of the Institute, but was entirely spontaneous on the part of Mr. Bowring; and whatever the result may be, the Council cannot but regard this notice as a gratifying proof of the progress which the Institute has made in public estimation, and a recognition of its efforts to raise the status of the Actuarial Profession."

The PRESIDENT said—"Gentlemen, in rising to propose a resolution in regard to the report which we have just heard read, I will not detain you with many observations; but there are two or three points which have occurred during the last year to which I shall be glad to have the opportunity of drawing your attention, and there are also one or two other points to be noticed which do not occur on every anniversary of our meeting. I think I may commence by a few words regarding the present state of the legislation as proposed for Life Assurance Companies. It is evident that for some time past a very uneasy feeling has pervaded the public mind in consequence of one unfortunate failure, and has suggested what in my opinion is a very unnecessary interference with Assurance Companies in general; but we must submit to this condition of things and do the best we can to meet it. Therefore I am prepared, in deference to public opinion, to accept any legislation which will not interfere directly with the action of the Companies, and is not likely to injure their responsible management of their own concerns. (Hear, hear.) So far as the public can be secured by law against either the consequences of mismanagement, or what appears much more frequently the case, the consequences of their own folly in rashly entering ill-managed Companies, I should be very glad to assist the Government in any way possible in perfecting this new attempt at legislation. That I think is the general view of the subject which has been taken up by this Institute. To a certain extent other gentle-

men may differ from me in regard to these views as to non-interference which I entertain, but we have thought it necessary to make a few suggestions in which we could agree towards amending the Bill now passing through Parliament. I will now venture to draw your attention to the great importance and very interesting nature of the papers which have been read during the last year. They are not only practical, but they are of the very highest scientific importance to us. (Hear, hear.) The question of mortality as deduced from the experience of the Offices has also been taken up with effect. The labour we have undergone in collecting and preparing this experience is well worthy of this Institute, and it is a subject the discussion on which can be renewed again and again with great interest to us all—(hear)—and I am sure that those papers which Mr. Sprague, Mr. Woolhouse, and Mr. Makeham have given us, and the way in which they are assisting us in deducing laws from that experience, which will enable us to correct the tables in general use, deserve the warmest thanks of every member of this Institute. (Hear, hear.) I may specially allude to Mr. Woolhouse and Mr. Makeham, who have graduated tables for us and placed the Committee in a position now to begin, and, I trust, very soon finish, the great work we have in hand. (Hear, hear.) There is one subject which I think strikes us as especially worthy of notice at this time, and that is the progress that is being made in every country, not only with regard to improvement in the regulations and in the general practice of insurance, but in the formation of scientific bodies like our own, which are calculated to place it upon the surest and best foundation. The German Institute seems to be progressing most energetically; and the journal of their proceedings, one number of which I have had the honour of laying before the Council, will in the course of a short time constitute a volume of great interest. Four parts have already appeared, containing both scientific and practical papers of the greatest importance. (Hear, hear.) Gentlemen, there is yet another subject,—one which is personal to myself. You will in a few minutes have to decide upon a successor to me in this chair. I beg to thank you most sincerely for the honour you have done in placing me for these three years at the head of this Institute. I assure you I feel deeply the kindness which first allotted me this great honour, and more especially do I gratefully acknowledge the kind support and cordial good feeling which I have met with on all sides in endeavouring to maintain the dignity and utility of the position. It is not owing so much to any efforts of my own that I can look back with pleasure to the last three years as having been a period of what I trust will be considered increasing success to the Institute and continued progress in the public mind. It is to those gentlemen around me who devote themselves so earnestly to the maintenance of our high character, that I owe the fortunate position of having been able to assist so earnestly in the movement. I shall now have to resign into your hands this trust, which will, I feel sure, devolve upon one whom you are justly proud to place in this chair, who has done so much that is worthy of his high reasoning powers, his great ability, and the general character for earnestness of purpose and public spirit which he has earned. There is not one on whom can more worthily devolve than on my friend Mr. Hodge, the honour you have in store for him. (Hear, hear.) In thinking over how I could advance in some slight degree the further progress of the Institute, a little plan has occurred to me which, though it may appear to be leading the members beyond our ordinary professional subjects, will, I believe, tend to raise the character of every member of the Institute as a public man and as a man of intellect,—I mean by endeavouring to found a prize for the purpose of engaging the attention and abilities of the members in discussing some new questions to which the doctrine of probabilities may be applied beyond those of the mere practice of insurance, in which we are more especially engaged. You all know that the doctrine of probabilities enters most largely into a multitude of questions of political and social economy. Let us take the subject of population, for instance. We have in our profession our own experience to collect, but we

are confined within a small compass of facts, and within certain limits, both as to the mode of collection and as to the practical application of the tables when formed. But when we examine the large and important question of population tables for this kingdom, or for various other countries, we then find numerous problems arising of the greatest interest to the public, which require men thoroughly versed in the mode of collecting the facts by the most perfect methods, and in the power of drawing from them precise and logical deductions; and the application of the scientific skill of an actuary to such subjects is likely, I think, to conduce greatly to the public benefit. (Hear, hear.) We have, further, the question of emigration. This subject has been very little considered, either as to the way in which it affects this country, in which it is so marked a feature, with respect to the effects it produces upon capital, labour, and the wages of those persons who are left behind in this country, or with respect to the effect it produces upon the country to which it is directed, where young and active men, in the prime of life, arrive and make up at once a hard-working and wealth-producing population, instead of a young and growing and dependent one, which may form a large proportion in the country they have left. That is an important question which an actuary by the direction of his studies, is, better than any other person, competent to understand and reason upon, and it is a question which deeply involves the interest of this country and its dependencies and some other of the European and transatlantic countries. You have, then, in commercial matters, the question of the law of the recurrence of bankruptcies; the regularity with which they follow certain conditions of credit, or certain phases of commercial or trading speculations. In the commercial world, again, Marine and Fire Insurance must, to a great extent, depend on the constant recurrence of events which must be regulated by definite laws, which, if not so precise as those of human mortality, should be found to vary only within certain limits. I speak of the application of scientific methods to the collection and observation of facts in such subjects, so as to trace causes and the regular recurrence of certain events which may depend upon them. I do not say that we, as actuaries, should have any right or wish to intermeddle in that capacity with the practice or methods of conducting business by any other class of gentlemen who thoroughly understand by experience the subject to which they may have long devoted themselves. I am speaking merely of an actuary bringing his trained intellect to bear upon these and similar inquiries in political and social economy, which affect the public welfare in such a way that we may deduce the greatest amount of precise and scientific information from any collection of the facts which may be made. There are also the questions of the effect of various hazardous employments upon health, the growth, and strength of man, the regularity of action in the human will, and similar topics, which that eminent writer, M. Quetelet, has so ably discussed. These are but a few subjects thrown out as indicating the direction in which, I think, the intellectual training of the members of this Institute might be usefully employed for the benefit of the public. (Hear, hear.) But there are many other questions in political economy beyond the range of our purely professional studies which might also be enumerated. I propose, therefore, to place at the disposal of the Council of the Institute a very small sum; but sufficient, perhaps, by the accumulation of the interest, in the course of every two or three consecutive years, to yield an adequate prize for an essay to be written upon one of these or upon some similar topic. I do not pretend to dictate how this little sum shall be used; I leave it entirely at the disposal of the Council; all I suggest is that the prize should be given for those subjects which, though beyond the actual necessities of our own profession, will enable the intellectual power and training which this Institute is designed to cultivate to be brought out at frequent intervals, to the effective service of the public, by the scientific elucidation of some new and important question of political and social economy. I will, if you please, present the Treasurer with this little slip of paper [handing a cheque to Mr. Cutcliffe] and leave it at

the service and, I trust, for the benefit of the Institute. (Cheers.) Although I now retire from this chair, I will still endeavour to do whatever is in my power to promote the success of the Institute in which I have always taken so warm and active an interest. (Applause.) I trust still to preserve in the position I shall hold amongst the members, that cordial good feeling and those true friendships of my many professional friends around me, which form such charming reminiscences from the time I first had the honour of belonging to this Institute, the reputation of which I shall do my best to maintain as long as life and strength are spared me. (Cheers.) I now beg to move, 'That the report of the Council, the abstract of income and expenditure, and the balance sheet be adopted, entered on the minutes, and printed in the *Journal*.' (Cheers.)

Mr. HODGE seconded the motion.

Mr. CUTCLIFFE—"I ought to announce to the meeting that the sum given by Mr. Brown is the very liberal one of £200." (Cheers.)

Mr. JOHN COLES—"I should like to suggest whether, as our President has presented so handsome a sum, it may not presently occur to the Council to designate the prize as the 'Brown Prize,' or by some title which shall serve to identify it from year to year with the name of the founder." (Hear, hear.)

The report was then unanimously adopted.

Mr. Y. R. ECCLES and Mr. BUMSTED having been appointed scrutineers, a ballot was taken for the election of President, Vice-Presidents, Council, and Officers for the ensuing year, and it was reported that the following gentlemen were duly elected:—

#### *President.*

WILLIAM BARWICK HODGE.

#### *Vice Presidents.*

CHARLES JOHN BUNYON, M.A.  
ALEXANDER GLEN FINLAISON.

THOMAS BOND SPRAGUE, M.A.  
J. HILL WILLIAMS.

#### *Council.*

MARCUS N. ADLER, M.A.  
ANDREW BADEN.  
ARTHUR H. BAILEY.  
SAMUEL BROWN.  
CHARLES JOHN BUNYON, M.A.  
\*EDWARD CUTBUSH.  
GEORGE CUTCLIFFE.  
ARCHIBALD DAY.  
HENRY DEVEREUX DAVENPORT.  
ALEXANDER GLEN FINLAISON.  
ALEXANDER PEARSON FLETCHER.  
WILLIAM JOHN HANCOCK.  
\*RALPH PRICE HARDY.  
AUGUSTUS HENDRIKS.  
WILLIAM BARWICK HODGE.

CHARLES JELlicoe.  
WILLIAM MATTHEW MAKEHAM.  
\*HENRY MARSHAL.  
JAMES MEIKLE.  
BENJAMIN NEWBATT.  
\*WILLIAM LEWIN NEWMAN.  
EDWARD A. NEWTON, M.A.  
WILLIAM P. PATTISON.  
HENRY WILLIAM PORTER, B.A.  
HENRY AMBROSE SMITH.  
COL. JOHN THOMAS SMITH.  
THOMAS BOND SPRAGUE, M.A.  
JOHN STOTT.  
ROBERT TUCKER.  
JOHN HILL WILLIAMS.

#### *Treasurer.*

GEORGE CUTCLIFFE.

#### *Honorary Secretaries.*

ARTHUR H. BAILEY.

ARCHIBALD DAY.

On the motion of the CHAIRMAN, seconded by Mr. CUTCLIFFE, a cordial vote of thanks was presented to the scrutineers.

\* New Members.

Messrs. Emmens, Hopkinson, and Manly were reappointed as auditors.

Mr. B. NEWBATT—"As this concludes the business, I wish, gentlemen, with your permission, to say a few words. If it is at all times, as I think it must be, meet and right, we shall feel it to-day our especial and bounden duty to return our thanks to the retiring President for the services he has rendered during his occupancy of the presidential chair—services which I think we shall all agree have worthily crowned a long and brilliant career of devotion to this Institute. (Cheers.) In your presence it is not necessary that I should speak of these services. It is enough to remind you, as has been already done in the report of the Council, that during Mr. Brown's reign the work of collecting the mortality experience of various Assurance Offices has been brought to a conclusion. The value of that work cannot well be overrated. (Hear, hear.) Its benefit belongs no doubt first and most to us, but its advantages will extend far beyond us. It is an addition to the sum of human knowledge. If I may be allowed a metaphor, 'Its sound has gone out into all lands,' and as the echoes travel back to us, they cannot fail to bring fame to this Institute, and dignity to the profession of which it is the worthy and unselfish exponent. (Hear, hear.) Mr. Brown's munificent donation to our funds for a specific purpose entitles him—I will not say to more thanks, because I am sure he does not wish to purchase our thanks—but at all events to the warmest expression of our gratitude, equally with the services to which I have alluded. (Hear, hear.) It may perhaps be right to mention that Mr. Brown retires in willing obedience to an understanding which was arrived at some time ago, and which it is desired should have hereafter the force of custom, that the term of occupancy of the presidential chair should be limited. That is not, of course, intended to supersede the rule under which all the officers are yearly submitted for your approval, but in order that a maximum term may be fixed, as is the case with most kindred institutions. Three years ago Mr. Jellicoe put aside an honour which he had not ceased to appreciate, in order that such a rule might be established; and in doing so, he showed himself more farsighted than some of us, who rather deprecated the step he took. He doubtless felt that in order to perpetuate an association like this it was necessary that its highest honour, equally with its subsidiary ones, should be open to all its members. (Hear, hear.) It is, I think, a satisfactory effect of this rule that on its first operation it has been the means of placing in the chair one whom we so much honour and respect as our friend Mr. Hodge, one who brings such large general culture, such intimate acquaintance with our special studies, and so genial and generous a nature, and who, I am sure, will reflect back upon the Institute the honour which he no doubt feels your suffrages have to-day conferred upon him. (Hear, hear.) I beg to move, 'That the hearty thanks of this meeting be offered to the retiring President, as well for his distinguished services to the Institute, especially during his occupancy of the presidential chair, as for the munificent donation of £200 which he has this day placed in the hands of the Council for the purpose of founding a special prize.'" (Cheers.)

Mr. HODGE—"Perhaps, gentlemen, you will allow me to second this resolution, which I do with most cordial feelings. At the same time I will take this opportunity of thanking you for the high and distinguished position in which you have done me the honour of placing me—a position to which I should never thought of aspiring. My elevation to the presidential chair of this Institute is due to the fact that I have been longer engaged in the active practice of the profession than any other member rather than to any special qualification I may possess for the office. Following as I do two such distinguished gentlemen as Mr. Jellicoe and Mr. Brown, who by their great energy, extensive experience, and earnest endeavours to further the interests of the Institute, have accustomed the members to a high degree of efficiency, I cannot but feel that I may, in comparison, occupy a disadvantageous position. But I should never have dared to accept the office which you have been kind enough to tender me, if I did not rely upon the cordial and constant co-operation of

the many talented, able, and accomplished members of the Institute whom I have the pleasure and honour of meeting at the Council. With regard to our President—our *late* President I must say now, and I say it with some regret,—we are indebted to him not only for what he has done as President, but for what he has invariably done as a member of the Institute; and it must, I think, be very gratifying to him to consider that that great work—the ascertainment of the true law of mortality—in which he took so deep an interest, and which he forwarded so much by his exertions, should have been accomplished during the term of his Presidency. (Hear, hear.) If the Institute of Actuaries had conferred no other advantage on the profession and the public, that work alone would be quite sufficient for its justification and estimation, and quite sufficient proof of its usefulness. (Hear, hear.) The noble gift which the retiring President has bestowed upon the Institute to-day will, if possible, endear him more than ever to the feelings of the members. It is not every one who has the means or the opportunity to indulge in such liberality, or to bestow it in such a handsome manner; and, unfortunately, those who have the means have not always the inclination; but both the means and the inclination are united in our President. I have no doubt the donation which he has presented to-day will be a source of additional efficiency to the Institute. (Hear, hear.) I beg leave to thank you for myself for the honour you have done me to-day, and also to second most cordially the vote of thanks to Mr. Brown.

The resolution was at once carried unanimously.

The PRESIDENT—"Gentlemen, the very kind and flattering manner in which Mr. Newbatt and Mr. Hodge have alluded to my services deserves my most grateful acknowledgments. I assure you I feel deeply the kindness of every member of this Institute, and the cordial support which has been given me in my endeavours to promote its success. I have always felt from the first a deep interest in its progress, because I felt satisfied it was raising the status of our profession very high indeed in the opinion of the public; and from the time it once began to the present day it has never gone back but has continued progressing in utility and advancing in public estimation. (Hear, hear.) It has been rather a piece of good fortune that during my short term of office, I should have had the pleasure of assisting in bringing to a conclusion the great work of collecting the mortality experience of assured lives. I believe in the utility of this work as far as it is completed; but there is still much more to be done, which will afford our succeeding President the pleasure of going on with it and rendering you still more effective service in connection with that very important work. (Hear, hear.) For myself I need not add more, except that I feel very glad indeed to have been allowed the honour of representing you on one or two occasions before the public, out-of-doors, and I feel deeply grateful for the kindness you have shown me both on many previous occasions and now." (Applause.)

Mr. W. P. CLIREHUGH—"A motion has been placed in my hand, which I have very great pleasure in proposing. It is, 'That the best thanks of the meeting be given to the Vice-Presidents, Council, and other officers of the Institute, for their services during the past year.' I am sure I speak the feelings of the members when I say they highly appreciate the way in which the affairs of the Institute are administered by the Council, a strong evidence of which is, I think, afforded by the small attendance here to-day." (Hear, hear.)

Mr. Y. R. ECCLES seconded the resolution, and it was passed unanimously.

The CHAIRMAN announced that the library would be closed as usual during the month of September.\*

\* The above report of the proceedings is extracted from the *Insurance Record*.

# INSTITUTE OF ACTUARIES.

*Income and Expenditure for the Year ended 31st March, 1870.*

Gr.

	£	s.	d.	£	s.	d.
Amount of Funds on 31st March, 1869, viz.:—						
Mortality Experience Fund	51	12	9			
Messenger Legacy	226	12	10			
Hardy Memorial	194	15	5			
General	755	16	2			
			—1228 17 2			
Annual Subscriptions, viz.:—						
68 Town Fellows	214	4	0			
26 Country Fellows	54	12	0			
91 Town Associates	191	2	0			
61 Country Associates	64	1	0			
			—523 19 0			
Subscriptions compounded						
Certificate Fees			31 10 0			
Mortality Experience Contributions			10 10 0			
" Sales at Office			594 10 0			
			—20 10 0			
Dividends, viz.:—						
Messenger Legacy Fund	6	4	0			
Hardy Memorial	5	17	6			
General	14	7	4			
			—26 8 10			
				£2,436	5	0

## Balance Sheet, 31st March, 1870.

Mortality Experience Fund	409	3	9
Messenger Legacy Fund, viz.:—			
£211. 1s. 10d. Consols, cost	203	17	8
Unappropriated dividends	28	19	2
			—292 16 10
Hardy Memorial Fund, viz.:—			
£200 Consols, cost	179	14	6
Unappropriated dividends	20	18	5
			—200 12 11
General Fund	877	13	8
			—£1,720 7 2
			£2,436 5 0
Union Bank Deposit Account	600	0	0
Three per Cent Consols, £900 Stock	823	19	3
Cash, viz.:—			
London and Westminster Bank	268	9	8
Petty Cash	6	18	3
			—275 7 11
Arrears of Subscriptions	21	0	0
			—£1,720 7 2

*Examined and found correct, 22nd April, 1870.*

(Signed) { STEPHEN H. EMMENS.

{ H. W. MARLY.

{ J. CLIFFORD HOPKINSON. } Auditors.

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